RESEARCH ARTICLE

Heat Transfer and Modified Darcy's Principle in Peristaltic Motion with Hartmann Boundary Layer

Shahid Farooq[1,*] **and Aiman Sana**[2]

[1] Department of Mathematics and Statistics, Riphah International University I-14, Islamabad, 44000, Pakistan
[2] Department of Mathematics, National University of Modern Languages (NUML), Islamabad 44000, Pakistan

Abstract

This research contains the Hartmann boundary layer effectiveness in peristaltic flow of non-Newtonian viscoelastic fluids through asymmetric channel walls. Due to the Hartmann boundary layer, Hartmann number is considered very large. Porosity effects are included in view of modified Darcy's principle. Energy equation is modelled in the presence of viscous dissipation and Joule heating features. No slip condition for fluid velocity is considered at both channel walls. Large wavelength and dominating viscous forces implementation reduce the PDEs into ODEs. The resulting velocity ODE is solved analytically through perturbation and matched asymptotic expansion techniques for large magnetic field effects, while the temperature equation is solved numerically. The obtained solutions are utilized to study the behaviour of velocity and temperature profiles against the involved parameters through graphs.

Keywords: jeffrey fluid, modified darcy law, hartmann boundary layer, joule heating, viscous dissipation, asymmetric channel.

1 Introduction

The consideration of magnetohydrodynamic (MHD) in moving fluids grabs the attention of researchers due to its attractive and significant usages in physiology and technical industry. Externally applied magnetic field plays an important role in drug delivery to targeted points. Flow due to periodic waves propagations becomes more significant in bio medical industry when magnetic field is applied. It is important during bleeding preventions, hyperthermia, ulcer, inflammation etc. treatment. Motivated by the above-mentioned usages of MHD, researchers have investigated peristaltic motion under various aspects. Peristaltically driven flow in the presence of magnetic field and heat transfer of tiny particles is managed by Bhatti et al. [1]. Asghar et al. [2] established the existence of a Hartmann boundary layer in the peristaltic flow of Sisko fluid under strong magnetic field using matched asymptotic expansion. Ali et al. [3] extended the Hartmann boundary layer analysis to peristaltic flow through a curved channel under a strong radial magnetic field, obtaining asymptotic solutions via matched expansion technique. Hayat et al. [4] investigated MHD peristaltic transport of

Submitted: 09 May 2025
Revised: 23 May 2025
Accepted: 26 May 2025
Published: 27 June 2025

Vol. 1, **No.** 1, 2025.
10.62762/JAM.2025.463651

***Corresponding author:**
✉ Shahid Farooq
farooq.fmg89@yahoo.com

Citation
Farooq, S., & Sana, A. (2025). Heat Transfer and Modified Darcy's Principle in Peristaltic Motion with Hartmann Boundary Layer. *ICCK Journal of Applied Mathematics*, 1(1), 32–40.

© 2025 by the Authors. Published by Institute of Central Computation and Knowledge. This is an open access article under the CC BY license (https://creativecommons.org/licenses/by/4.0/).

Jeffrey fluid through a channel with compliant walls and porous space, establishing foundational results for non-Newtonian fluid flow under magnetic and Darcy effects. Ali et al. [5] demonstrated the existence of the Hartmann boundary layer in peristaltic flow of a viscoelastic fluid.

Various clinical and engineering procedures operate via peristaltic mechanisms. It is an inherent property of various physiological organs due to sinusoidal movements of waves e.g. intestine, ureter, lungs, digestive tract, fallopian tube, bile ducts, arteries. Earthworm locomotion also obeys the peristaltic mechanism. Ceramics and sanitary liquid transport are the industrial applications of peristalsis. Various bio-medical gadgets such as dialysis, heart-lung, blood pressure, nebulizer machines and different clinical injections, coronary bypass, hose pumps are working through mechanism of peristalsis. In light of these applications Ali et al. [6] analyzed heat transfer in peristaltic flow of a viscous fluid through a curved channel under long wavelength and low Reynolds number assumptions, examining the role of the Brinkman number on the temperature field, while Shapiro et al. [7] investigated peristalsis firstly for large wavelength numerically and theoretically. Further Fung and Yih [8] analyzed peristaltic transport in a two-dimensional channel, whereas Yin and Fung [9] discussed the peristaltic movement through circular cylindrical ducts. Subsequent studies have advanced peristaltic flow analysis through diverse approaches: Hussain et al. [10] developed a shooting technique for curved radiative flows, while Vajravelu et al. [11] analyzed the nonlinear peristaltic transport of a Jeffrey fluid through a finite vertical porous channel with viscous and Darcy dissipation effects on heat transfer. Numerical innovations include Gnaneswara Reddy et al.'s [12] analysis of MHD peristaltic rotating flow of a Jeffrey fluid in an asymmetric channel with heat transfer and Salman and Ali's [13] approximate treatment of MHD peristaltic transport of Jeffrey fluid in a tapered asymmetric channel with heat transfer and porous medium, complemented by Yasmeen et al.'s [14] quantitative and qualitative analysis of the Hartmann boundary layer in peristaltic flow of Jeffrey fluid and Abd-Alla et al.'s [15] investigation of MHD peristaltic flow of a Jeffrey fluid model through a non-uniform porous channel with heat transfer under modified Darcy's law.

The mechanism of heat transfer from lower to higher temperature places is most important and has crucial utilizations in the procedure of haemodialysis and oxygenation. Heat conduction in biological tissues, environmental heat exchangers, cooling system of industrial and mechanical devices, hyperthermia etc. are few more significant examples of heat transfer. Several researchers analyse the characteristics of heat transfer in various physiological liquids transportation. Peristaltic motion with heat transfer was investigated by Vajravelu et al. [16] in a vertical porous annulus under the long wave approximation. Riaz et al. [17] presented a thermal analysis of peristaltic flow of nanosized particles within a curved channel in the presence of second-order partial slip and a porous medium. The compliant wall properties and heat transfer under large wavelength and low Reynolds number approximations were studied by Hayat et al. [18], who examined peristaltic transport of viscous fluid in a curved channel with compliant walls. Hayat et al. [19] further studied the effects of thermal radiation and radial magnetic field on peristaltic flow of Sutterby liquid in a curved channel with wall properties. Heat transfer analysis of MHD peristaltic motion in a Jeffrey fluid with compliant walls is theoretically examined by Javed and Hayat [20]. Another important impulsive resistive situation based on the production of heat transfer when electric current passes through electrically conductive liquid is Joule heating. In this direction, Hayat et al. [21] incorporated the Joule heating aspects in radiative peristaltic flow of Jeffrey nanofluid with slip conditions, while Hussain et al. [10] quantified the Joule heating contribution to the temperature field of Jeffrey fluid under peristalsis. Recent advances in MHD peristaltic transport of Jeffrey material through porous media have significantly expanded the theoretical and computational frontiers: Vaidya et al. [22] investigated the influence of transport properties on peristaltic MHD Jeffrey fluid flow through a porous asymmetric tapered channel, while Kothandapani and Srinivas [23] examined the peristaltic transport of a Jeffrey fluid under the effect of magnetic field in an asymmetric channel. Selvi et al. [24] examined the effect of heat transfer on peristaltic flow of Jeffrey fluid in an inclined porous stratum, and Hussain et al. [25] studied the peristaltic transport of hydromagnetic Jeffrey fluid with temperature-dependent viscosity and thermal conductivity, complemented by Hayat et al. [26] who examined the effectiveness of entropy generation and energy transfer on peristaltic flow of Jeffrey material with Darcy resistance, and Abd-Alla et al. [27] who investigated magneto-hydrodynamic peristaltic flow of a Jeffrey fluid through a porous asymmetric channel with heat transfer. Nonlinear radiative effects

were advanced by Hayat et al. [28] incorporating activation energy and modified Darcy's law for Jeffrey nanofluid peristaltic flow, synergizing with Rehman et al.'s [29] Jeffrey fluid benchmarks on heat sink/source effects and Srinivas and Muthuraj's [30] analysis of MHD mixed convective peristaltic flow with chemical reaction and space porosity in a vertical asymmetric channel.

2 Formulation

Here we consider peristalsis of an incompressible and highly electrically conducting Jeffrey material through an asymmetric channel configuration. Fluid is flowing because of asymmetric propagation of waves with speed c (see Figure 1). Channel is of width $d_1 + d_2$. Rectangular coordinates $\overline{X}$ and $\overline{Y}$ are respectively taken along and normal to the asymmetric walls. Magnetic field of large intensity is executed normal to the flow. Permeability effects are encountered through modified Darcy's principle and energy equation comprises the viscous dissipation and Joule heating features.

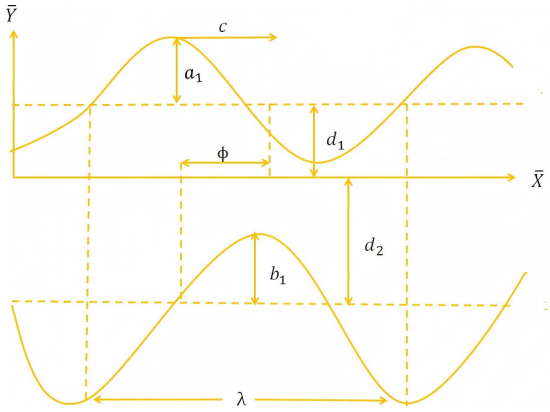

Figure 1. Flow mechanism.

Asymmetric channel walls mathematically are expressed as given below:

$$\overline{H}_1(\overline{X},\bar{t}) = d_1 + a_1\cos\left[\frac{2\pi}{\lambda}\left(\overline{X}-c\bar{t}\right)\right] \tag{1}$$

$$\overline{H}_2(\overline{X},\bar{t}) = -d_2 - b_1\cos\left[\frac{2\pi}{\lambda}\left(\overline{X}-c\bar{t}\right)+\varphi\right] \tag{2}$$

Equations which govern the Jeffery fluid flow in compact form are:

$$\mathrm{div}\overline{\mathbf{V}} = 0 \tag{3}$$

$$\rho\frac{d\overline{\mathbf{V}}}{d\bar{t}} = -\mathrm{grad}\overline{P} + \mathrm{div}\overline{\mathbf{S}} - \sigma B_0^2\overline{\mathbf{V}} - \frac{\mu}{(1+\lambda_1)}K_1\left(1+\lambda_2\frac{d}{d\bar{t}}\right)\overline{\mathbf{V}} \tag{4}$$

$$\rho C_p\frac{dT}{d\bar{t}} = \kappa\nabla^2 T + \mathrm{tr}(\overline{\mathbf{S}}\cdot\overline{\mathbf{L}}) + \overline{\mathbf{J}}\cdot\overline{\mathbf{J}} \tag{5}$$

where $\mathbf{V} = \left(\overline{U}(\overline{X},\overline{Y},\bar{t}),\overline{V}(\overline{X},\overline{Y},\bar{t}),0\right)$, $\frac{d}{d\bar{t}}$, $\overline{P}$, ρ, and $B_0, \mathbf{J}, \mathbf{L}, \mathbf{K}_1, C_p$ denote the velocity field, the material time derivative, the pressure, the fluid density, and the magnetic field strength, current density, velocity gradient, the permeability, and the specific heat capacity respectively. The extra stress tensor $\mathbf{S}$ for Jeffrey fluid is given as follows:

$$\overline{\mathbf{S}} = \frac{\mu}{1+\lambda_1}\left(\overline{\mathbf{A}}_1 + \lambda_2\frac{d\overline{\mathbf{A}}_1}{d\bar{t}}\right) \tag{6}$$

Eqs. (3) - (5) in expanded form yields:

$$\frac{\partial\overline{U}}{\partial\overline{X}} + \frac{\partial\overline{V}}{\partial\overline{Y}} = 0 \tag{7}$$

$$\rho\left[\frac{\partial\overline{U}}{\partial\bar{t}} + \overline{U}\frac{\partial\overline{U}}{\partial\overline{X}} + \overline{V}\frac{\partial\overline{U}}{\partial\overline{Y}}\right] = -\frac{\partial\overline{P}}{\partial\overline{X}} + \frac{\partial\overline{S}_{\overline{XX}}}{\partial\overline{X}} + \frac{\partial\overline{S}_{\overline{XY}}}{\partial\overline{Y}} - \sigma B_0^2\overline{U} - \frac{\mu}{(1+\lambda_1)}K_1\left(1+\lambda_2\frac{d}{d\bar{t}}\right)\overline{U} \tag{8}$$

$$\rho\left[\frac{\partial\overline{V}}{\partial\bar{t}} + \overline{U}\frac{\partial\overline{V}}{\partial\overline{X}} + \overline{V}\frac{\partial\overline{V}}{\partial\overline{Y}}\right] = -\frac{\partial\overline{P}}{\partial\overline{Y}} + \frac{\partial\overline{S}_{\overline{XY}}}{\partial\overline{X}} + \frac{\partial\overline{S}_{\overline{YY}}}{\partial\overline{Y}} - \sigma B_0^2\overline{V} - \frac{\mu}{(1+\lambda_1)K_1}\left(1+\lambda_2\frac{d}{d\bar{t}}\right)\overline{V} \tag{9}$$

$$\rho C_p\frac{dT}{d\bar{t}} = K\left(\frac{\partial^2 T}{\partial\overline{X}^2} + \frac{\partial^2 T}{\partial\overline{Y}^2}\right) + S_{\overline{XX}}\frac{\partial\overline{U}}{\partial\overline{X}} + \overline{S}_{\overline{YY}}\frac{\partial\overline{V}}{\partial\overline{Y}} + S_{\overline{XY}}\left(\frac{\partial\overline{U}}{\partial\overline{Y}} + \frac{\partial\overline{V}}{\partial\overline{X}}\right) + \sigma B_0^2\overline{U}^2 \tag{10}$$

Conditions at boundaries are taken in the form at

$$\overline{U} = 0, \quad T = T_1 \quad \text{at} \quad \overline{Y} = \overline{H}_1 \tag{11}$$

$$\overline{U} = 0, \quad T = T_0 \quad \text{at} \quad \overline{Y} = \overline{H}_2 \tag{12}$$

Stress components are

$$S_{\overline{XX}} = \frac{2\mu}{1+\lambda_1}\left[\lambda_2\left(\frac{\partial}{\partial\bar{t}} + \overline{U}\frac{\partial}{\partial\overline{X}} + \overline{V}\frac{\partial}{\partial\overline{Y}}\right) + 1\right]\frac{\partial\overline{U}}{\partial\overline{X}} \tag{13}$$

$$S_{\overline{YY}} = \frac{2\mu}{1+\lambda_1}\left[1 + \lambda_2\left(\frac{\partial}{\partial\bar{t}} + \overline{U}\frac{\partial}{\partial\overline{X}} + \overline{V}\frac{\partial}{\partial\overline{Y}}\right)\right]\frac{\partial\overline{V}}{\partial\overline{Y}} \tag{14}$$

$$S_{\overline{XY}} = \frac{2\mu}{1+\lambda_1}\left[\lambda_2\left(\frac{\partial}{\partial\bar{t}} + \overline{U}\frac{\partial}{\partial\overline{X}} + \overline{V}\frac{\partial}{\partial\overline{Y}}\right) + 1\right]\left(\frac{\partial\overline{U}}{\partial\overline{Y}} + \frac{\partial\overline{V}}{\partial\overline{X}}\right) \tag{15}$$

To transform the system (7-15), the following relation between fixed and moving frames are utilized.

$$\bar{x} = \overline{X} - c\bar{t}, \quad \bar{y} = \overline{Y}, \quad u(\bar{x},\bar{y}) = \overline{U}(\overline{X},\overline{Y},\bar{t}) - c, \\ v(\bar{x},\bar{y}) = \overline{V}(\overline{X},\overline{Y},\bar{t}), \quad p(\bar{x},\bar{y}) = \overline{P}(\overline{X},\overline{Y},\bar{t}) \tag{16}$$

Variables in dimensionless are defined as:

$$x = \frac{2\pi\bar{x}}{\lambda}, \quad y = \frac{\bar{y}}{d_1}, \quad u = \frac{\bar{u}}{c}, \quad v = \frac{\bar{v}}{c}, \\ \theta = \frac{T - T_0}{T_1 - T_0}, \quad S_{ij} = \frac{d_1}{c\mu}\overline{S_{ij}} \tag{17}$$

Relation between dimensionless velocity component and stream function is

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\delta\frac{\partial\psi}{\partial x}. \tag{18}$$

Using fixed to wave frame transformations (16), dimensionless variables (17) and velocities in terms of stream suction (18) Eqs. (7) satisfied identically and the other expressions takes the following form:

$$\delta\,\mathrm{Re}\left[\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial\psi}{\partial x}\right)\right] + \frac{\partial p}{\partial x} = \delta\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial x} \\ - M^2\left(\frac{\partial\psi}{\partial y} + 1\right) - \frac{1}{D_a K_1(1+\lambda_1)}\left[\left(\frac{\partial\psi}{\partial y} + 1\right) + \right. \\ \left. c\frac{\lambda_2}{d_1}\left(\delta\frac{\partial^2\psi}{\partial y^2}\frac{\partial}{\partial x} - \delta\frac{\partial^2\psi}{\partial x^2}\frac{\partial}{\partial y}\right)\right]. \tag{19}$$

$$-\delta^3\,\mathrm{Re}\left[\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial\psi}{\partial x}\right)\right] + \frac{\partial p}{\partial y} = \delta^2\frac{\partial S_{xy}}{\partial x} \\ + \delta\frac{\partial S_{yy}}{\partial y} + M^2\delta\frac{\partial\psi}{\partial x} - \frac{\delta}{D_a K_1(1+\lambda_1)}\left[-\delta\frac{\partial\psi}{\partial x}\right. \\ \left. + \frac{\lambda_2}{d_1}\left(-\delta^2\frac{\partial^2\psi}{\partial x\partial y} - \delta\frac{\partial^2\psi}{\partial y^2}\right)\right] \tag{20}$$

$$\mathrm{Re}\,\delta\left[\left(\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}\right)\right] = \frac{1}{Pr}\left(\delta\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) \\ + \delta E_c S_{xx}\frac{\partial^2\psi}{\partial x\partial y}, -\delta E_c S_{xy}\frac{\partial\psi}{\partial x} + E_c M^2\left(\frac{\partial\psi}{\partial y} + 1\right)^2 + \delta\frac{\partial u}{\partial x} \\ + \delta\frac{\partial v}{\partial x} + E_c S_{xy}\left(\frac{\partial u}{\partial y} + \delta\frac{\partial v}{\partial x}\right) \tag{21}$$

$$S_{xx} = \frac{2\delta}{1+\lambda_1}\left[1 + \delta\lambda_2\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\right] \\ \times\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) \tag{22}$$

$$S_{xy} = \frac{1}{1+\lambda_1}\left[1 + \delta\lambda_2\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\right] \\ \times\left(\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right) - \delta^3\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)\right) \tag{23}$$

$$S_{yy} = \frac{-2\delta}{1+\lambda_1}\left[1 + \lambda_2\delta\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\right] \\ \times\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right). \tag{24}$$

Dimensionless numbers arises in Eqs. (19-24) are defined as

$$p = \frac{2\pi d_1^2\bar{p}}{c\lambda\mu}, \quad h_1 = \frac{\bar{h_1}}{d_1}, \quad h_2 = \frac{\bar{h_2}}{d_2}, \quad \delta = \frac{2\pi d_1}{\lambda}, \\ Re = \frac{\rho c d_1}{\mu}, \quad M^2 = \frac{\sigma B_0^2 d_1^2}{\mu}, \\ a = \frac{a_1}{d_1}, \quad b = \frac{b_1}{d_1}, \quad d = \frac{d_2}{d_1}, \quad E_c = \frac{c^2}{(T_1 - T_0)c_p}, \\ P_r = \frac{\mu c_p}{K}, \quad D_a = \frac{d_1^2}{K_1}. \tag{25}$$

Boundary conditions becomes

$$\psi = \frac{q}{2}, \quad \psi' = -1, \quad \theta = 1, \quad \text{at} \quad y = h_1. \tag{26}$$

$$\psi = -\frac{q}{2}, \quad \psi' = -1, \quad \theta = 0, \quad \text{at} \quad y = h_2. \tag{27}$$

3 The solution methodology

Eqs. (19–24) yields after employing dominating large viscous forces (i.e., Re $\ll 1$) and wavelength (i.e., $\delta \ll 1$) conditions:

$$\frac{\partial p}{\partial x} = \frac{\partial S_{xy}}{\partial y} - \left(M^2 + \frac{1}{D_a(1+\lambda_1)}\right)\left(\frac{\partial\psi}{\partial y} + 1\right) \tag{28}$$

$$\frac{\partial p}{\partial y} = 0 \tag{29}$$

$$\frac{1}{P_r}\left(\frac{\partial^2\theta}{\partial y^2}\right) + E_c\left[S_{xy}\frac{\partial u}{\partial y} + M^2\left(\frac{\partial\psi}{\partial y} + 1\right)^2\right] = 0 \tag{30}$$

$$S_{xy} = \frac{1}{1+\lambda_1}\left(\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right)\right) \tag{31}$$

where S_{xy} is given in Eq. (31). After using the cross differentiation technique, Eqs. (28) and (29) yield:

$$\frac{\partial^4\psi}{\partial y^4} - \left[M^2(1+\lambda_1) + \frac{1}{D_a}\right]\frac{\partial^2\psi}{\partial y^2} = 0 \tag{32}$$

In this section we follow the asymptotic analysis and higher order matching methodology developed

in [2, 3, 5, 14] to evaluate Eq. (32) with the respective boundary conditions. Using this solution Eq. (30) with respective boundary condition is solved numerically through ND Solve command in Mathematica 9.0.

3.1 Asymptotic approximation for large magnetic field(M)

For large M approximate solution is obtained away and adjacent to the channel boundaries and then match it with solution at the edges of boundaries. After that a uniform composite solution is obtained.

3.2 The outer solution

Express Eq.(32) in terms of Eq.(33) to evaluate asymptotic solution for very large M.

$$\psi(y) = \psi_0 + \epsilon\psi_1 \tag{33}$$

where $M = \frac{1}{\epsilon}$ and $\frac{1}{D_a} = \frac{1}{\epsilon}$ where $\epsilon \ll 1$. Then the leading order equation of Eq. (32) and its solution is:

$$(2+\lambda_1)\frac{\partial^2\psi_0}{\partial y^2} = 0 \tag{34}$$

$$\psi_0(y) = a_0 y + b_0 \tag{35}$$

Similarly, the first order equation and its solution is:

$$\frac{\partial^4\psi_0}{\partial y^4} - (2+\lambda_1)\frac{\partial^2\psi_1}{\partial y^2} = 0 \tag{36}$$

$$\psi_1(y) = a_1 y + b_1 \tag{37}$$

Combining Eqs. (34 and 36), the asymptotic solution up to order ϵ is:

$$\psi^{\text{out}}(y) = a_0 y + b_0 + \epsilon\,(a_1 y + b_1) \tag{38}$$

where a_0, b_0, a_1 and b_1 can be calculated through comparing the higher order term of inner and outer solutions. Here we evaluate two inner solutions to attain a valid and uniform solution at two boundaries because outer solution is not valid at $o(\epsilon)$.

3.3 The inner solutions

(i) For inner solution at $y = h_1$, Eq. (32), the solution near to the boundary at $y = h_1(x)$, the stretched variable is in the form $\eta = \frac{h_1 - y}{\epsilon^p}$ yields:

$$\epsilon^{2-4p}\frac{\partial^4\psi^{in}}{\partial\eta^4} - (1+\lambda_1)\epsilon^{-2p}\frac{\partial^2\psi^{in}}{\partial\eta^2} - \epsilon^{-2p}\frac{\partial^2\psi^{in}}{\partial\eta^2} = 0 \tag{39}$$

Taking $p = 1$ in view of least degeneracy condition to balance the solution of Eq. (39) because viscosity effects must be present in inner solution.

$$\frac{\partial^4\psi^{in}}{\partial\eta^4} - k\frac{\partial^2\psi^{in}}{\partial\eta^2} = 0 \tag{40}$$

in which Eq. (40), $K = 2 + \lambda_1$. The condition at $y = h_1$ boundary becomes:

$$\psi^{in}(\eta) = \frac{q}{2} \quad \text{and} \quad \frac{\partial\psi^{in}}{\partial\eta}(\eta) = \epsilon \quad \text{at} \quad \eta = 0 \tag{41}$$

The two term solution of Eq.(40) is assumed as

$$\psi^{in}(\eta) = \psi_0^{in}(\eta) + \epsilon\psi_1^{in}(\eta) \tag{42}$$

where Zeroth order system and its solution is of the form:

$$\frac{\partial^4\psi_0^{in}}{\partial\eta^4} - k\frac{\partial^2\psi_0^{in}}{\partial\eta^2} = 0 \tag{43}$$

$$\psi_0^{in}(0) = \frac{q}{2}, \quad \frac{\partial\psi_0^{in}}{\partial\eta}(0) = 0 \tag{44}$$

$$\psi_0^{in} = \frac{q}{2} + D_0\left(-1 + \sqrt{k\eta} + e^{-\sqrt{k\eta}}\right) \tag{45}$$

The two term inner solution at $y = h_1$ is obtained by putting Eqs. (45 and 48) in $\psi^{in}(\eta) = \psi_0^{in}(\eta) + \epsilon\psi_1^{in}(\eta)$, we get:

$$\begin{aligned}\psi_2^{in} = {} & \frac{q}{2} + D_0\left(-1 + \sqrt{k\eta} + e^{-\sqrt{k\eta}}\right) \\ & + \epsilon\left[\eta + D_1\left(-1 + \sqrt{k\eta} + e^{-\sqrt{k\eta}}\right)\right]\end{aligned} \tag{46}$$

(ii) In similar manner the two term inner solution is found at $y = h_2$ by using $\zeta = \frac{y - h_2}{\epsilon}$ in Eq. (39).

$$\begin{aligned}\tilde{\psi_2}^{in}(\zeta) = {} & -\frac{q}{2} + H_0\left(-1 + \sqrt{k}\zeta + e^{-\sqrt{k}\zeta}\right) \\ & + \epsilon\left[-\zeta + H_1\left(-1 + \sqrt{k}\zeta + e^{-\sqrt{k}\zeta}\right)\right]\end{aligned} \tag{47}$$

The values of all constants appearing in Eqs.(49 and 50) is found by comparing inner and outer solutions at both walls.

3.4 The higher order mode matching

To employ higher order mode of matching procedure use an intermediate variable t given in Eq. (50) at $y = h_1$ in the intermediate location $o(\epsilon^\alpha)$. Two terms inner and outer solution in the form of variable t at $y = h_1$ by neglecting the terms higher than $\epsilon^{\alpha+1}$ and

greater after converting the outer and inner solutions into intermediate parameter t at lower plate yields:

$$\left(\psi^{\text{out}}\right)^{\text{int}} = a_0 h_1 + b_0 - a_0 t(\epsilon)^{\alpha} + \epsilon\left(a_1 h_1 + b_1\right) + o(\epsilon^{\alpha+1}) \tag{51}$$

$$\left(\psi^{\text{in}}\right)^{\text{int}} = \tfrac{q}{2} + D_0\left(-1 + kt(\epsilon)^{\alpha-1}\right) + t(\epsilon)^{\alpha}\left(1 + kD_1\right) - D_1\epsilon \tag{48}$$

implies:

$$\begin{aligned} &a_0 h_1 + b_0 = \frac{q}{2} - D_0, \quad D_0 = 0, \\ &1 + \sqrt{k} D_1 = -a_0, \quad a_1 h_1 + b_1 = -D_1 \end{aligned} \tag{49}$$

Similarly at upper plate $y = h_2(x)$, matching of outer and inner solutions is:

$$\begin{aligned} &a_0 h_2 + b_0 = -\frac{q}{2} - H_0, \quad H_0 = 0, \\ &-1 + \sqrt{k} H_1 = a_0, \quad a_1 h_2 + b_1 = -H_1 \end{aligned} \tag{50}$$

Eq.(53) and Eq.(54) are evaluated simultaneously to obtain the unknown constants:

$$\begin{aligned} a_0 &= \frac{q}{h_1 - h_2} \\ a_1 &= \frac{2}{\sqrt{k}(h_1 - h_2)}\left(1 + \frac{q}{h_1 - h_2}\right) \\ b_0 &= -\frac{q}{2}\left(\frac{h_1 + h_2}{h_1 - h_2}\right) \\ b_1 &= -\frac{1}{\sqrt{k}}\left(\frac{h_1 + h_2}{h_1 - h_2}\right)\left(1 + \frac{q}{h_1 - h_2}\right) \\ H_1 &= \frac{1}{\sqrt{k}}\left(1 + \frac{q}{h_1 - h_2}\right) \end{aligned} \tag{51}$$

The composite solution can be expressed as:

$$\psi_{\text{composite}} = \psi^{\text{out}} + \psi^{\text{in}}(\eta) - \left(\psi^{\text{int}}\right)^{\text{in}} + \psi^{\sim\text{int}}(\zeta) - \left(\psi^{\sim\text{int}}\right) \tag{52}$$

Substituting Eqs. (49), (50) and the matching results (53), (54) into Eq. (56), the composite solution is finally given by:

$$\begin{aligned} \psi_{\text{composite}} = {} & \frac{qy}{h_1 - h_2} - \frac{q}{2}\left(\frac{h_1 + h_2}{h_1 - h_2}\right) \\ & + \frac{\epsilon}{\sqrt{k}}\left(1 + \frac{q}{h_1 - h_2}\right) \times \\ & \left[\frac{2y}{h_1 - h_2} - \frac{h_1 + h_2}{h_1 - h_2} e^{\sqrt{k}\frac{h_1 - y}{\epsilon}} + e^{-\sqrt{k}\frac{y - h_2}{\epsilon}}\right] \end{aligned} \tag{53}$$

The composite solution presented in Eq. (57) is completely valid for all independent variable values which fulfills the accounted boundary conditions.

4 Discussion

The goal of this section is to study the graphical behaviour of velocity and temperature profiles against Darcy number Da, Hartmann number (M), Jeffery fluid parameters λ_1 and Brinkman number Br.

4.1 Velocity Profile

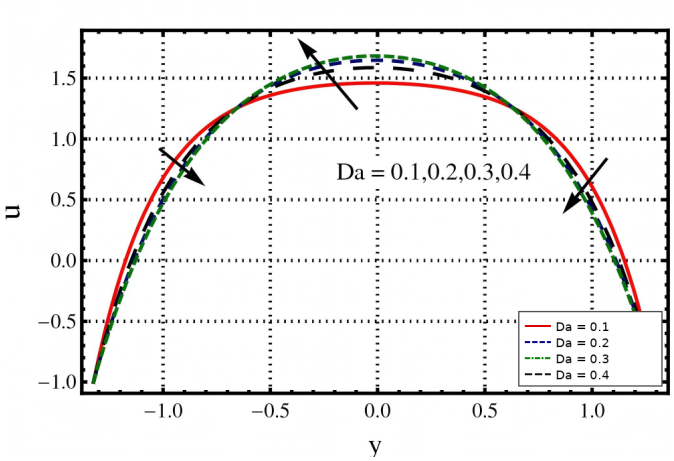

Figure 2. u of Da for $Br = 0.2$; $x = 0$; $M = 1.0$; $\lambda_1 = 0.5$; $\alpha = 0.02$; $\beta = 0.02$.

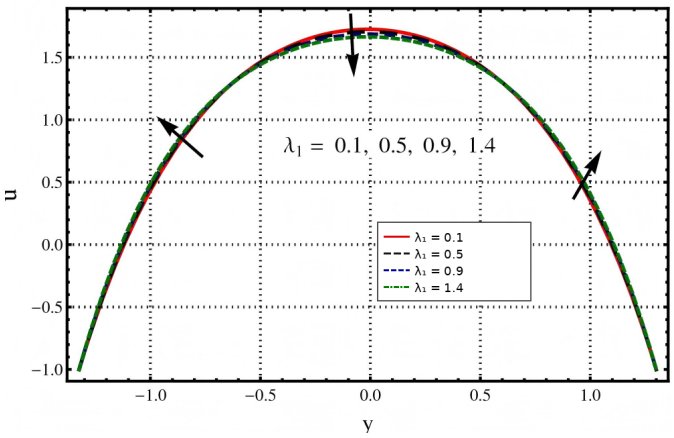

Figure 3. u of λ_1 for $Br = 0.2$; $x = 0$; $M = 1.0$; $Da = 0.5$; $\alpha = 0.02$; $\beta = 0.02$.

The impact of Da, M and λ_1 on velocity profile are captured in Figures 2, 3 and 4. The velocity of flowing fluid increases with an increase in Darcy number (Da) as graphically presented in Figure 2. When studying peristaltic flow through veins and arteries, the porosity of walls must be considered. Because an increase in porosity causes a reduction in drag force gives a resulting decrease in flow resistance. The decrease in velocity towards the channel boundaries is also observed in this Figure 3 depicts that an increase in the value of Jeffery fluid parameter (λ_1) causes a reduction in velocity. It is because by increasing λ_1 the viscous effects dominate at the centre of the channel.

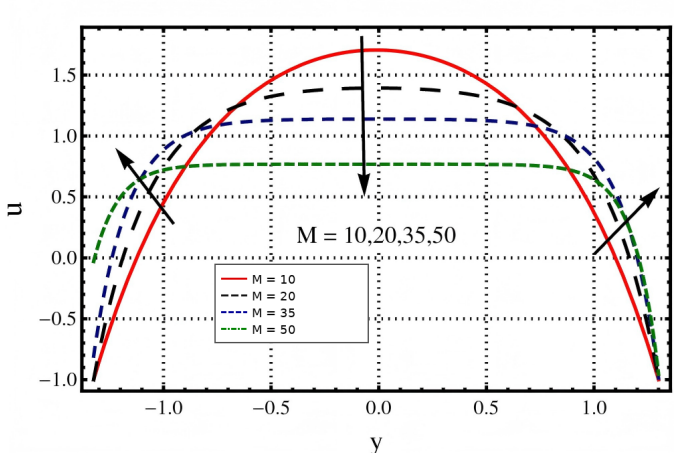

Figure 4. u of M for Br=0.2; x=0; Da=0.5; λ_1=0.5; α=0.01; β=0.02.

Due to the classical effect of Hartmann number, it is obvious that an increase in the value of M causes a decrease in velocity profile as illustrated in Figure 4. As Lorentz force (a resistive force) is present in MHD phenomena, thus by increasing M the Lorentz force produce more resistance to the fluid flow due to which velocity decreases.

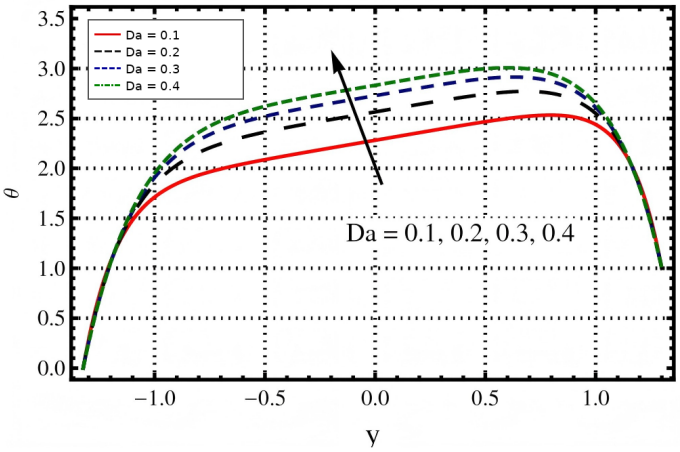

Figure 5. Temperature profile of Da for Br=0.2; x=0; M=1.0; λ_1=0.5; α=0.02; β=0.02.

4.2 Temperature Profile

The impact of Da, M, λ_1, Br on temperature profile (a resistive force) through Figures 5, 6, 7 and 8. In this section it is observed from Figure 5 that the profile for temperature distribution increases with an increase in the value of Darcy number (Da). Figure 6 depicts that an increase in the value of Jeffery fluid parameter (λ_1) causes a reduction in temperature. Figure 7 depicts that higher increase in the value of Hartmann number (M) causes an increase in temperature. Figure 8 depicts that an increase in the

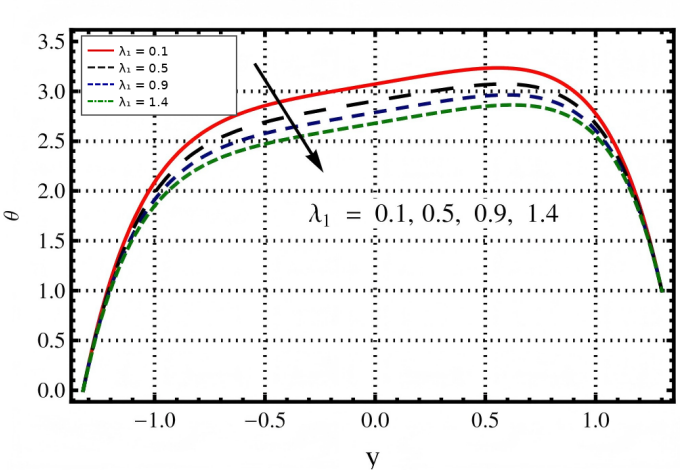

Figure 6. Temperature profile of λ_1 for Br=0.2; x=0; M=1.0; Da=0.5; α=0.01; β=0.02.

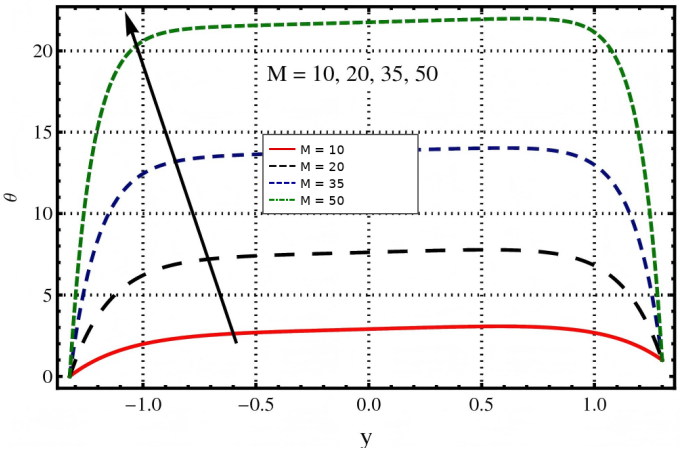

Figure 7. Temperature profile of M for Br=0.2; x=0; Da=0.5; λ_1=0.5; α=0.01; β=0.02.

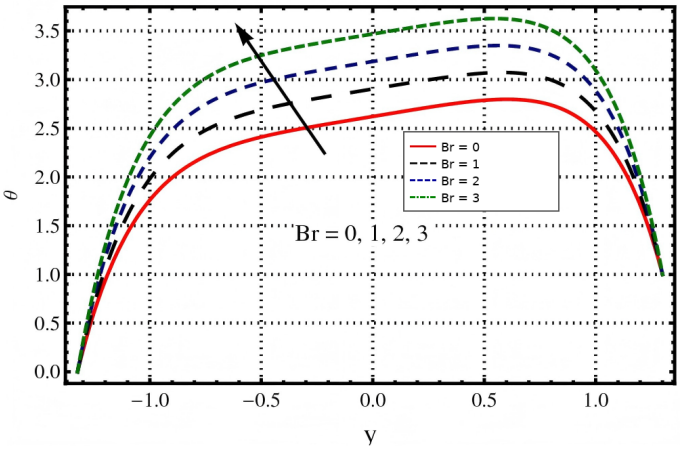

Figure 8. Temperature profile of Br for x=0; M=1.0; Da=0.5; λ_1=0.5; α=0.01; β=0.02.

value of Brinkman number (Br) causes an increase in temperature. It is observed through these Figures 5, 6, 7 and 8 that temperature increases because of resistive

characteristics of Da, M, and Br parameters.

5 Concluding remarks

The key findings of this study are: 1. The velocity changes for modified Darcy parameter. 2. The Hartmann boundary layer thickness decreases for increasing M. 3. For higher λ_1 temperature decays. 4. Both Magnetic and Darcy parameter enhances the temperature.

Data Availability Statement

Data will be made available on request.

Funding

This work was supported without any funding.

Conflicts of Interest

The authors declare no conflicts of interest.

Ethical Approval and Consent to Participate

Not applicable.

References

[1] Bhatti, M. M., Zeeshan, A., Ijaz, N., & Ellahi, R. (2017). Heat transfer and inclined magnetic field analysis on peristaltically induced motion of small particles. *Journal of the Brazilian Society of Mechanical Sciences and Engineering, 39*(8), 3259–3267. [CrossRef]

[2] Asghar, S., Minhas, T., & Ali, A. (2014). Existence of a Hartmann layer in the peristalsis of Sisko fluid. *Chinese Physics B, 23*(5), 054702. [CrossRef]

[3] Ali, N., Javid, K., Sajid, M., & Hayat, T. (2016). New concept about existence of Hartmann boundary layer in peristalsis through curved channel-asymptotic solution. *Meccanica, 51*(8), 1783-1795. [CrossRef]

[4] Hayat, T., Javed, M., & Ali, N. (2008). MHD peristaltic transport of a Jeffery fluid in a channel with compliant walls and porous space. *Transport in Porous Media, 74*(3), 259-274. [CrossRef]

[5] Ali, A., Awais, M., Zubaidi, A., Saleem, S., & Marwat, D. N. K. (2022). Hartmann boundary layer in peristaltic flow for viscoelastic fluid: Existence. *Ain Shams Engineering Journal, 13*(2), 101555. [CrossRef]

[6] Ali, N., Sajid, M., Javed, T., & Abbas, Z. (2010). Heat transfer analysis of peristaltic flow in a curved channel. *International Journal of Heat and Mass Transfer, 53*(15–16), 3319–3325. [CrossRef]

[7] Shapiro, A. H., Jaffrin, M. Y., & Weinberg, S. L. (1969). Peristaltic pumping with long wavelengths at low Reynolds number. *Journal of Fluid Mechanics, 37*(4), 799–825. [CrossRef]

[8] Fung, Y. C., & Yih, C. S. (1968). Peristaltic transport. *ASME Journal of Applied Mechanics, 35*(4), 669–675. [CrossRef]

[9] Yin, F., & Fung, Y. C. (1969). Peristaltic waves in circular cylindrical tubes. *ASME Journal of Applied Mechanics, 36*(4), 579–587. [CrossRef]

[10] Hussain, Z., Khaled, K., Sharif, U., Abbasi, A., Khan, S. U., Farooq, W., ... & Malik, M. Y. (2022). A mathematical model for radiative peristaltic flow of Jeffrey fluid in curved channel with Joule heating and different walls: Shooting technique analysis. *Ain Shams Engineering Journal, 13*(5), 101685. [CrossRef]

[11] Vajravelu, K., Sreenadh, S., Lakshminarayana, P., & Sucharitha, G. (2016). The effect of heat transfer on the nonlinear peristaltic transport of a Jeffrey fluid through a finite vertical porous channel. *International journal of biomathematics, 9*(02), 1650023. [CrossRef]

[12] Gnaneswara Reddy, M., Venugopal Reddy, K., & Makinde, O. D. (2017). Heat transfer on MHD peristaltic rotating flow of a Jeffrey fluid in an asymmetric channel. *International Journal of Applied and Computational Mathematics, 3*(4), 3201-3227. [CrossRef]

[13] Salman, M. R., & Ali, H. A. (2020). Approximate treatment for the MHD peristaltic transport of jeffrey fluid in inclined tapered asymmetric channel with effects of heat transfer and porous medium. *Iraqi Journal of Science*, 3342-3354. [CrossRef]

[14] Yasmeen, S., Asghar, S., Anjum, H. J., & Ehsan, T. (2019). Analysis of Hartmann boundary layer peristaltic flow of Jeffrey fluid: Quantitative and qualitative approaches. *Communications in Nonlinear Science and Numerical Simulation, 76*, 51–65. [CrossRef]

[15] Abd-Alla, A. M., Thabet, E. N., Bayones, F. S., & Alsharif, A. M. (2023). Heat transfer in a non-uniform channel on MHD peristaltic flow of a fractional Jeffrey model via porous medium. *Indian Journal of Physics, 97*(6), 1799-1809. [CrossRef]

[16] Vajravelu, K., Radhakrishnamacharya, G., & Radhakrishnamurty, V. (2007). Peristaltic flow and heat transfer in a vertical porous annulus, with long wave approximation. *International Journal of Non-Linear Mechanics, 42*(5), 754–759. [CrossRef]

[17] Riaz, A., Khan, S. U. D., Zeeshan, A., Khan, S. U., Hassan, M., & Muhammad, T. (2021). Thermal analysis of peristaltic flow of nanosized particles within a curved channel with second-order partial slip and porous medium. *Journal of Thermal Analysis and Calorimetry, 143*(3), 1997–2009. [CrossRef]

[18] Hayat, T., Javed, M., & Hendi, A. A. (2011). Peristaltic transport of viscous fluid in a curved channel with compliant walls. *International Journal of Heat and Mass Transfer, 54*(7–8), 1615–1621. [CrossRef]

[19] Hayat, T., Quratulain, Alsaadi, F., Rafiq, M., & Ahmad, B. (2017). On effects of thermal radiation and radial magnetic field for peristalsis of Sutterby liquid in a curved channel with wall properties. *Chinese Journal*

of Physics, 55(5), 2005–2024. [CrossRef]

[20] Javed, M., & Hayat, T. (2020). Heat transfer analysis of MHD peristaltic motion in a Jeffrey fluid with compliant walls. *Journal of Porous Media, 23*(12). [CrossRef]

[21] Hayat, T., Shafique, M., Tanveer, A., & Alsaedi, A. (2016). Radiative peristaltic flow of Jeffrey nanofluid with slip conditions and Joule heating. *PloS one, 11*(2), e0148002. [CrossRef]

[22] Vaidya, H., Rajashekhar, C., Divya, B. B., Manjunatha, G., Prasad, K. V., & Animasaun, I. L. (2020). Influence of transport properties on the peristaltic MHD Jeffrey fluid flow through a porous asymmetric tapered channel. *Results in Physics, 18*, 103295. [CrossRef]

[23] Kothandapani, M., & Srinivas, S. (2008). Peristaltic transport of a Jeffrey fluid under the effect of magnetic field in an asymmetric channel. *International Journal of Non-Linear Mechanics, 43*(9), 915-924. [CrossRef]

[24] Selvi, C. K., Haseena, C., Srinivas, A. N. S., & Sreenadh, S. (2017, November). The effect of heat transfer on peristaltic flow of Jeffrey fluid in an inclined porous stratum. In *IOP conference series: materials science and engineering* (Vol. 263, No. 6, p. 062027). IOP Publishing. [CrossRef]

[25] Hussain, Q., Asghar, S., Hayat, T., & Alsaedi, A. (2016). Peristaltic transport of hydromagnetic Jeffrey fluid with temperature-dependent viscosity and thermal conductivity. *International Journal of Biomathematics, 9*(02), 1650029. [CrossRef]

[26] Hayat, T., Farooq, S., Ahmad, B., & Alsaedi, A. (2017). Effectiveness of entropy generation and energy transfer on peristaltic flow of Jeffrey material with Darcy resistance. *International Journal of Heat and Mass Transfer, 106*, 244-252. [CrossRef]

[27] Abd-Alla, A. M., Abo-Dahab, S. M., Salah, D. M., Bayones, F. S., & Abdelhafez, M. A. (2023). Magneto-hydrodynamic peristaltic flow of a Jeffery fluid in the presence of heat transfer through a porous medium in an asymmetric channel. *Scientific Reports, 13*(1), 21088. [CrossRef]

[28] Hayat, T., Bibi, F., Farooq, S., & Khan, A. A. (2019). Nonlinear radiative peristaltic flow of Jeffrey nanofluid with activation energy and modified Darcy's law. *Journal of the Brazilian Society of Mechanical Sciences and Engineering, 41*(7), 296. [CrossRef]

[29] Rehman, M., Noreen, S., Haider, A., & Azam, H. (2015). Effect of heat sink/source on peristaltic flow of Jeffrey fluid through a symmetric channel. Alexandria Engineering Journal, 54(3), 733-743. [CrossRef]

[30] Srinivas, S., & Muthuraj, R. (2011). Effects of chemical reaction and space porosity on MHD mixed convective flow in a vertical asymmetric channel with peristalsis. *Mathematical and Computer Modelling, 54*(5-6), 1213-1227. [CrossRef]



Dr. Shahid Farooq is working as an Assistant Professor in the Department of mathematics Riphah International University, I-14 Islamabad, Pakistan. He is working mainly in the field of Newtonian, non-Newtonian fluid mechanics, Porous Media, Heat and mass transfer, stretching sheet, stretching cylinders, Peristaltic transport, Nanoparticles, Entropy generation, curved surfaces etc. (Email: farooq.fmg89@yahoo.com)



Ms. Aiman Sana is a Ph.D scholar in Department of Mathematics, National University of Modern Languages. (Email: aimansana96@gmail.com)