RESEARCH ARTICLE

Bornological Semi Continuous Maps

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Abstract

In the current study, a new approach had been constructed to define new maps using the concept of bornological semi open and bornological semi closed sets, which includes sequential bornological semi continuous maps, bornological semi closed (open) maps, bornological strongly semi closed (open) maps, and bornological semi-irresolute closed (open) maps. We investigate and study the properties of these concepts.

Keywords: bornological semi open map, bornological semi closed map, bornological semi continuous map.

1 Introduction

Bornology on a set and convergence of sequences in bornological vector space have been studied by H. Hogbe-Nlend after he introduced b-closed set (b-open set) [1]. In topological spaces, Noiri introduced semi-closed mappings, a class strictly containing the closed mappings, and obtained several characterizations of them [2]; the bornological semi closed maps studied here are their counterparts in bornological vector spaces. In [3], Al-Basri studied the concepts of semi-convergence of nets and semi-complete bornological vector spaces in convex (bvs) space. In [4], bornological convergence of nets in convex (bvs) and its relationship with topological convergence were investigated; this bornological convergence underlies the semi-convergence $x_n \xrightarrow{s} x$ on which all the bornological semi notions of this paper are based. Semi-open, strongly semi-open, and semi-irresolute open sets play an important role in the study of continuity generalizations in topological spaces. By using these sets, many authors introduced and investigated various types of modifications to continuity. In 1963, Levine [5] introduced the notions of semi-open sets and semi-continuity in topological spaces. Crossley and Hildebrand introduced irresolute functions, under which inverse images of semi-open sets are semi-open, together with semi-homeomorphisms and semi-topological properties [7]; these notions motivate the bornological semi-irresolute maps considered in this paper. The relationship between Levine's semi-continuity and other generalizations of continuity, in particular quasi-continuity, has been investigated in [6], showing that semi-continuity is a natural and robust weakening of continuity. Generalized continuous maps, characterized by the property that inverse images of closed sets are generalized closed, are studied in [8]; an analogous inverse-image property of seq bs-con maps is given in Theorem 5 of this paper, and semi-generalized continuous and semi-generalized irresolute maps in topological spaces are introduced in [9].

Submitted: 18 June 2025
Revised: 02 July 2025
Accepted: 11 July 2025
Published: 04 August 2025

Vol. 1, **No.** 2, 2025.
10.62762/JAM.2025.997630

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Citation
AL-Basri, F. K. (2025). Bornological Semi Continuous Maps. *ICCK Journal of Applied Mathematics*, 1(2), 62–65.

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In this work, new types of maps in convex bornological vector space "cbvs" within bornological vector space E will be introduced. Further properties about sequentially bornological semi continuous "seq bs-con" maps, bornological semi closed "bs-cl" maps, bornological strongly semi closed "bss-cl" maps, bornological semi irresolute closed "bsi-cl" maps, bornological semi open "bs-op" maps, bornological strongly semi open "bss-op" maps, and bornological semi irresolute open "bsi-op" maps have been introduced with their relationships.

2 Preliminaries

Definition 1 *Consider E as a bornological vector space (bvs). A subset A of E is said to be bornological semi-open (bs-op, in short) if for every $x \in A$ and every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ with $x_n \xrightarrow{s} x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in A$ for all $n > n_0$.*

Definition 2 *Consider E as a bornological vector space (bvs). A subset A contained within E is termed bornological semi-closed or abbreviated as bs-cl if the following condition is satisfied: If you have a sequence $\{x_n\}$ indexed by natural numbers $\mathbb{N}$, where all the elements are part of A and these elements x_n tend to approach a point x in E according to the bornological semi convergence, then it must also be the case that x belongs to the set A.*

Remark 1 *In a (bvs) every b-op (b-cl) set is bs-op (bs-cl), but the converse is not true in general.*

Proposition 1 *Take A be a b-op (b-cl) set in a (bvs) E. Then:*

(i) If B is bs-op (bs-cl) set in E, then $B \bigcap A$ is bs-op (bs-cl) set in E.

(ii) If B is bs-op (bs-cl) set in E then $B \bigcap A$ is bs-open (bs-cl) set in A.

Proof 1 *(i) Take A to be a b-op and take $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ semi converging bornological to a point $x \in A \cap B$, then $x \in A$ and $x \in B$. If $x \in A$ and by Remark 1 A is bs-op then $x_n \in A \; \forall \; n > n_o$. If $x \in B$ and B is bs-op then $x_n \in B \; \forall \; n > n_o$, we have $x_n \in A \cap B \; \forall \; n > n_o$ then $A \cap B$ is bs-op.*

Now take A be a b-cl $\{x_n\}_{n\in\mathbb{N}} \subset A \cap B$ and $x_n \xrightarrow{s} x$ in E then $\{x_n\}_{n\in\mathbb{N}} \subseteq B$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq A$. If $\{x_n\}_{n\in\mathbb{N}} \subseteq B$ and since B is bs-cl we have $x_n \xrightarrow{s} x$ then $x \in B$. If $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ and since A is b-cl then by Remark 1 A is bs-cl we have $x_n \xrightarrow{s} x$ then $x \in A$, then $x \in A \cap B$.

(ii) Same approach as above proof.

Proposition 2 *Take E be a (bvs) and $B \subseteq A \subseteq E$. Then:*

(i) If B is bs-op (bs-cl) set in A and A bs-op (bs-cl) set in E then B is bs-op (bs-cl) set in E.

(ii) If B is bs-op (bs-cl) set in E then B is bs-op (bs-cl) in A.

Proof 2 *(i) Let $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ and $x_n \xrightarrow{s} x$ such that $x \in B$, thus $x \in B \subseteq A$ we have $x \in A$, since A is bs-op set in E and $\{x_n\}_{n\in\mathbb{N}} \subseteq A$, $x_n \xrightarrow{s} x$ such that $x \in B$, since B is bs-op in A then $\{x_n\}_{n\in\mathbb{N}} \subseteq B$, we have B is bs-op in E.*

Definition 3 *If we take E to be a bornological vector space (bvs), then the bornological semi-closure of a subset A in E, denoted as bs-clr(A), is the intersection of all bs-cl subsets of E that contain the set A.*

3 Some Types of Bornological Semi Maps

This section includes some basic properties about sequentially bornological semi continuous map is denoted by seq bs-con map, bs-cl(bs-op) map, bss-cl(bss-op) map, bsi-cl(op) map and relationships among their maps are investigated.

Definition 4 *Let E, F are (bvs) and let f be a mapping from E into F. We say that f is a seq bs-con map at a point x if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in E, $x_n \xrightarrow{s} x$ then $f(x_n) \xrightarrow{s} f(x)$ in F. If f is a seq bs-con map at every point x in E, then f is called seq bs-con map. This notion is the semi analogue of the sequentially bornological continuous mappings introduced in [10].*

Example 1 *Let A be a non-empty subset of a bornological vector space (bvs) E with the relative bornology. Then the inclusion map $i : A \to E$ is a seq bs-con map, since $x_n \xrightarrow{s} x$ in A implies $x_n \xrightarrow{s} x$ in E.*

Definition 5 *Let E and F are (bvs)s, a map $f : E \to F$ is called:*

(i) bs-cl map if $\forall$ b-cl set A in E, $f(A)$ is bs-cl set in F

(ii) bss-cl map if $\forall$ bs-cl set A in E, $f(A)$ is b-cl set in F

(iii) bsi-cl map if $\forall$ bs-cl set A in E, $f(A)$ is bs-cl set in F

Definition 6 *Let E and F are bornological vector spaces (bvs)s, a map $f : E \to F$ is called:*

(i) bs-op map if $\forall$ b-op set A in E, $f(A)$ is bs-op set in F

(ii) bss-op map if $\forall$ bs-op set A in E, $f(A)$ b-op set in F

(iii) bsi-op map if $\forall$ bs-op set A in E, $f(A)$ is bs-op set in F

Remark 2 *Every b-cl (b-op) map is bs-cl (bs-op) map; this follows directly from Remark 1.*

Theorem 1 *Let E, F and H are (bvs), $f : E \to F$, $g : F \to H$, then, if f is b-cl map and g is bs-cl map, then $g \circ f$ is bs-cl.*

Proof 3 *Let $A \subseteq E$ is a b-cl set, since f b-cl map then $f(A)$ b-cl set in F, by g bs-cl map we have $g(f(A))$ is bs-cl set in H, then $g \circ f$ is bs-cl map.*

Theorem 2 *Let E, F and H are (bvs)s, $f : E \to F$, $g : F \to H$*

(i) If f is bss-cl map and g is bss-cl map then $g \circ f$ is bss-cl map.

(ii) If f is bsi-cl map and g is bsi-cl map then $g \circ f$ is bsi-cl map.

Proof 4 *(i) Let $A \subseteq E$ is a bs-cl set, since f is a bss-cl map, $f(A)$ is a b-cl set in F, and by Remark 1 $f(A)$ is bs-cl in F, by g bss-cl map we have $g(f(A))$ is b-cl set in H, then $g \circ f$ is bss-cl map.*

(ii) Let $A \subseteq E$ be a bs-cl set. Since f is a bsi-cl map, $f(A)$ is a bs-cl set in F. Since g is a bsi-cl map, $g(f(A))$ is a bs-cl set in H. Hence $g \circ f$ is a bsi-cl map.

Example 2 *Let A be subset of a (bvs) E, then the inclusion map $i : A \to E$ is b-cl (bs-cl) iff A is b-cl set in E.*

Theorem 3 *Let E, F are (bvs)s. Assuming $f : E \to F$ and $g : E \to F$ seq bs-con map then:*

(i) $cf : E \to F$ is a seq bs-con map for every $c \in K$, where K is the scalar field of F

(ii) $f + g$ seq bs-con map

(iii) $f \cdot g$ is a seq bs-con map

Proof 5 *(i) Let $x \in E$ and $\{x_n\}$ be a sequence in (bvs) E, such that $x_n \xrightarrow{s} x$. Since f is a seq bs-con at x, then $f(x_n) \xrightarrow{s} f(x)$ in F implies $cf(x_n) \xrightarrow{s} cf(x)\ \forall\ c \in K$. Then $\forall\ x$ we have cf seq bs-con.*

(ii) Let $x \in E$ and $\{x_n\}$ be a sequence in (bvs) E, such that $x_n \xrightarrow{s} x$. Since f, g seq bs-con map at x then $f(x_n) \xrightarrow{s} f(x)$ in F and $g(x_n) \xrightarrow{s} g(x)$ in F, then $f(x_n) + g(x_n) \xrightarrow{s} f(x) + g(x)$ in F. We have $(f + g)(x_n) \xrightarrow{s} (f + g)(x)$ in F, then $f + g$ is a seq bs-con map at every point x in E. Hence $f + g$ is a seq bs-con map.

(iii) Let $x \in E$ and $\{x_n\}$ be a sequence in (bvs) E, such that $x_n \xrightarrow{s} x$. Since f, g seq bs-con map at x then $f(x_n) \xrightarrow{s} f(x)$ in F and $g(x_n) \xrightarrow{s} g(x)$ in F then $f(x_n) \cdot g(x_n) \xrightarrow{s} f(x) \cdot g(x)$ in F, hence $(f \cdot g)(x_n) \xrightarrow{s} (fg)(x)$ in F at every x in E then $f \cdot g$ is seq bs-con map.

Theorem 4 *Let E and F are (bvs)s and A a non-empty subset of E if $f : E \xrightarrow{s} F$ is a seq bs-con map then the restriction f_A is a seq bs-con map, where A has the relative bornology B_A.*

Proof 6 *Let $x \in A$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ such that $x_n \xrightarrow{s} x$ in A. Since $A \subseteq E$ then $x_n \xrightarrow{s} x$ in E. Since $f : E \to F$ is a seq bs-con map then if $x_n \xrightarrow{s} x$ we have $f(x_n) \xrightarrow{s} f(x)$ hence f_A is a seq bs-con map.*

Theorem 5 *Let E and F be (bvs)s. If $f : E \to F$ is a seq bs-con map, then for every bs-cl set B in F, $f^{-1}(B)$ is a bs-cl set in E.*

Proof 7 *Let B is bs-cl set in F, if $f^{-1}(B) = \phi$ and thus the proof is complete. If $f^{-1}(B) \neq \phi$. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq f^{-1}(B)$ such that $x_n \xrightarrow{s} x$ in E, we have $f(x_n) \in B$. Since f is seq bs-con map then $f(x_n) \xrightarrow{s} f(x)$. Since B is bs-cl set then $f(x) \in B$. We have $x \in f^{-1}(B)$ then $f^{-1}(B)$ is bs-cl set in E.*

Theorem 6 *Let E and F are (bvs)s, if $f : E \to F$ is seq bs-con map then:*

(i) $f(\text{bs-clr } A) \subseteq \text{ bs-clr } f(A)$ for every $A \subseteq E$

(ii) bs-clr $f^{-1}(B) \subseteq f^{-1}(\text{bs-clr } B)$ for every $B \subseteq F$

Proof 8 *(i) Let $A \subseteq E$. Since bs-clr $f(A)$ is a bs-cl set in F, by Theorem 5 $f^{-1}(\text{bs-clr } f(A))$ is a bs-cl set in E. Since $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\text{bs-clr } f(A))$ and bs-clr A is the intersection of all bs-cl sets containing A, we have bs-clr $A \subseteq f^{-1}(\text{bs-clr } f(A))$. Hence $f(\text{bs-clr } A) \subseteq$ bs-clr $f(A)$.*

(ii) Let $B \subseteq F$. Since bs-clr B is a bs-cl set in F, by Theorem 5 $f^{-1}(\text{bs-clr } B)$ is a bs-cl set in E. Since $B \subseteq$ bs-clr B, we have $f^{-1}(B) \subseteq f^{-1}(\text{bs-clr } B)$. Hence, by the definition of the bornological semi-closure, bs-clr $f^{-1}(B) \subseteq f^{-1}(\text{bs-clr } B)$.

Theorem 7 *Let E, F and H be (bvs)s, let $f : E \to F$ be a seq bs-con map at a point x, and let $g : F \to H$ be seq bs-con at $f(x)$. Then $g \circ f : E \to H$ is a seq bs-con map at x.*

Proof 9 *Let $\{x_n\}$ be a sequence in E such that $x_n \xrightarrow{s} x$ since f is a seq bs-con map at x then $f(x_n) \xrightarrow{s} f(x)$. Since $\{f(x_n)\}$ a sequence in F. Since g is a seq bs-con map at $f(x)$ then $g(f(x_n)) \xrightarrow{s} g(f(x))$. Then $(g \circ f)(x_n) \xrightarrow{s} (g \circ f)(x)$. Hence $g \circ f$ is a seq bs-con map at x.*

4 Conclusion

In this study, we introduced and explored a novel framework for defining new classes of maps grounded in the notions of bornological semi open and bornological semi closed sets. Through this approach, we established and analyzed several types

of maps, including sequential bornological semi continuous maps, bornological semi closed (open) maps, bornological strongly semi closed (open) maps, and bornological semi-irresolute closed (open) maps. Our investigation provided insight into the structural properties and interrelationships among these mappings, offering a foundation for further development in the field of bornological topology and its applications in general topological structures.

Data Availability Statement

Not applicable.

Funding

This work was supported without any funding.

Conflicts of Interest

The author declares no conflicts of interest.

Ethical Approval and Consent to Participate

Not applicable.

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