ICCK Journal of Applied Mathematics
http://dx.doi.org/10.62762/JAM.2025.698605

RESEARCH ARTICLE

# Electrostatic Freak Waves in Pair-Ion and Pair-Ion-Electron Plasmas

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## Abstract

**This work investigates the formation and dynamics of electrostatic freak waves in pair-ion (PI) and pair-ion–electron (PIE) plasmas. The analysis begins with the derivation of the Korteweg–de Vries (KdV) equations for both plasma configurations, from which the corresponding nonlinear and dispersive coefficients are obtained. By employing the wave superposition principle, the KdV equations are systematically reduced to the nonlinear Schrödinger equation (NLSE), enabling the exploration of modulation instability and rogue wave generation. Analytical solutions of the NLSE are utilized to construct parametric plots that elucidate the evolution of freak waves in PI and PIE plasmas. Comparative analysis reveals pronounced differences in the amplitude, localization, and structural properties of the freak waves in the two plasma environments, highlighting the critical role of electron contributions in shaping nonlinear wave phenomena.**

**Keywords**: wave phenomena, freak waves, Pair-Ion plasmas, Pair-Ion–Electron plasmas, NLS equation, reductive perturbation method.

**Submitted:** 13 August 2025
**Accepted:** 29 September 2025
**Published:** 29 October 2025

**Vol.** 1, **No.** 3, 2025.
10.62762/JAM.2025.698605

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| Symbol | Abbreviations |
|---|---|
| PI | Pair Ion |
| EPI | Electron Pair Ion |
| Kdv | Korteweg-de Vries |
| NLSE | Non linear schoridenger wave equations |

## 1 Introduction

Freak waves, also referred to as rogue waves, are extreme wave events characterized by amplitudes significantly exceeding those of the surrounding waves. Although rare, their occurrence can be highly destructive, posing severe risks to ships, offshore structures, and other installations. Such waves have been observed in diverse physical environments, including oceans, lakes, and even in plasma systems [1]. In plasma systems, freak waves are generally attributed to the combined influence of nonlinear and dispersive effects [2]. Nonlinear effects refer to phenomena in which the wave amplitude influences its propagation characteristics, whereas dispersive effects arise when different frequency components of the wave travel at different velocities. Pair-ion (PI) and pair-ion–electron (PIE) plasmas represent two classes of plasma systems that are of particular

**Citation**
Khan, M. Y., & Ahmed, M. W. (2025). Electrostatic Freak Waves in Pair-Ion and Pair-Ion-Electron Plasmas. *ICCK Journal of Applied Mathematics*, 1(3), 120–128.

interest in the study of freak wave dynamics [3]. Pair-ion (PI) plasmas consist solely of positive and negative ion species, whereas pair-ion–electron (PIE) plasmas contain an additional electron population. Freak waves in PI and PIE plasmas were first identified through numerical simulations in the early 2000s. Since then, numerous investigations have explored their occurrence in these systems. These studies have demonstrated that freak waves can arise under a wide range of conditions, encompassing variations in plasma parameters as well as different forms of external perturbations [4, 5]. The characteristics of freak waves in PI and PIE plasmas depend on many factors, including the plasma parameters and the type of external perturbation. However, some general trends have been observed [6]. Freak waves in PI and PIE plasmas are typically highly localized in both space and time, and they can possess considerable energy, with amplitudes several times greater than those of the surrounding wave field. Although the study of freak waves in these plasma systems remains in its early stages, ongoing research holds significant potential for advancing our understanding of their underlying mechanisms and for developing strategies to mitigate the risks associated with their occurrence [7]. This research may contribute to the development of advanced early warning systems for detecting freak waves in plasmas. As noted earlier, the precise mechanisms responsible for the generation of freak waves in PI and PIE plasmas are not yet fully understood. Nevertheless, several possible mechanisms have been proposed, among which nonlinear wave focusing is a prominent candidate [8]. Wave focusing leads to an increase in amplitude, and when the focusing is sufficiently strong, it can result in the formation of a freak wave. Another plausible mechanism for their generation is the modulation instability of waves [9]. Modulational instability is a process in which a wave can break up into a series of smaller waves. These smaller waves can then interact with each other to form a freak wave. In addition to these nonlinear mechanisms, it is also thought that dispersive effects can play a role in the generation of freak waves [10]. If a plasma wave packet undergoes sufficient dispersion, it may evolve into a freak wave. Although freak waves in PI and PIE plasmas have been extensively reported in numerical simulations, experimental observations remain comparatively scarce. This scarcity is largely due to the challenges associated with reproducing the requisite plasma conditions in laboratory environments. Nonetheless, a limited number of successful experiments have been conducted. In one such study, a PI plasma was generated using a laser, and the subsequent formation of freak waves within the plasma was observed [11]. In a separate experiment, PIE plasma was produced using a microwave discharge, after which the emergence of freak waves within the plasma was recorded [12]. These experimental findings further substantiate the occurrence of freak waves in both PI and PIE plasmas. Investigations in this area hold considerable promise for practical applications. For instance, such studies could facilitate the development of advanced early warning systems for detecting freak waves in plasmas, thereby enhancing the protection of spacecraft and other critical structures from potential damage. Moreover, this line of research may enable the creation of novel techniques for controlling freak waves in plasma environments, which could prove valuable in various fields, including plasma fusion and plasma-based material processing [13–15].

In this study, the ion-to-electron density ratio was chosen to capture both laboratory-accessible and astrophysical plasma regimes. Higher electron concentrations highlight dispersive effects, while lower fractions emphasize nonlinear ion dynamics. The selected parameters which strongly influence freak wave formation. Cooler ions enhance localization and amplitude growth, whereas warmer ions broaden velocity distributions and reduce extreme wave events. These choices ensure that our parameter space remains both physically relevant and broadly applicable.

Our work is positioned within the broader context of plasma freak wave research, which has drawn increasing attention in connection with turbulence, nonlinear instabilities, and energy localization. Recent studies in space and astrophysical plasmas further highlight the relevance of extreme wave events to naturally occurring environments. By linking our findings to these developments, we extend the understanding of freak wave dynamics in both laboratory and astrophysical plasma systems.

## 2 Freak waves by conversion of KDV to NLS Equation

The soliton solutions for PI and PIE plasmas are reported in [16]. In the present work, the analysis is extended to freak waves by transforming the KdV equation into the NLSE. The general form of the KdV equation, along with its associated coefficients, is taken from [17, 18]. An infinite series expansion, based on the superposition principle of waves, is then employed to generate the freak wave solutions as described below:

$$\frac{\partial \phi}{\partial \tau}+A\phi\frac{\partial \phi}{\partial \xi}+B\frac{\partial^3 \phi}{\partial \xi^3}=0 \tag{1}$$

Stretched Variables X and T are given as:

$$X=\epsilon\left(\xi+\mu\tau\right), \quad T=\epsilon^2\tau \tag{2}$$

Find the relevant derivatives by Using stretched variables form (2) we get

$$\frac{\partial X}{\partial \xi}=\epsilon, \quad \frac{\partial X}{\partial \tau}=\epsilon\mu, \quad \frac{\partial T}{\partial \xi}=0, \quad \frac{\partial T}{\partial \tau}=\epsilon^2$$

$$\frac{\partial}{\partial \xi}=\frac{\partial}{\partial \xi}+\epsilon\frac{\partial}{\partial X} \quad \& \quad \frac{\partial}{\partial \tau}=\frac{\partial}{\partial \tau}+\epsilon\mu\frac{\partial}{\partial X}+\epsilon^2\frac{\partial}{\partial T} \tag{3}$$

Now let the infinite transformation series [15]

$$\phi\left(\xi,\tau\right)=\sum_{n=1}^{\infty}\epsilon^n\sum_{l=-\infty}^{\infty}\phi_l^n\left(X,T\right)e^{il(k\xi-\omega\tau)} \tag{4}$$

By putting Equation (3) results to the given KdV implies

$$\left(\frac{\partial}{\partial \tau}+\epsilon\mu\frac{\partial}{\partial X}+\epsilon^2\frac{\partial}{\partial T}\right)\phi+A\phi\left(\frac{\partial}{\partial \xi}+\epsilon\frac{\partial}{\partial X}\right)\phi+B\frac{\partial^3\phi}{\partial \xi^3}=0 \tag{5}$$

Using Equation(4) into Equation(5)

$$\begin{aligned}&\frac{\partial \phi}{\partial \tau}+A\phi\frac{\partial \phi}{\partial \xi}+B\frac{\partial^3 \phi}{\partial \xi^3}=\\&\left(\frac{\partial}{\partial \tau}+\epsilon\mu\frac{\partial}{\partial X}+\epsilon^2\frac{\partial}{\partial T}\right)\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)+\\&A\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\left(\frac{\partial}{\partial \xi}+\epsilon\frac{\partial}{\partial X}\right)\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\\&+B\frac{\partial^3}{\partial \xi^3}\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)=0\end{aligned} \tag{6}$$

a. Calculate

$\frac{\partial\phi}{\partial\tau}=\left(\frac{\partial}{\partial\tau}+\epsilon\mu\frac{\partial}{\partial X}+\epsilon^2\frac{\partial}{\partial T}\right)\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)$:

$$\begin{aligned}\frac{\partial \phi}{\partial \tau}&=\left(\frac{\partial}{\partial \tau}+\epsilon\mu\frac{\partial}{\partial X}+\epsilon^2\frac{\partial}{\partial T}\right)\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\\&=\frac{\partial}{\partial \tau}\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)+\epsilon\mu\frac{\partial}{\partial X}\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\\&+\epsilon^2\frac{\partial}{\partial T}\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\end{aligned} \tag{7}$$

$$\begin{aligned}\Rightarrow\frac{\partial \phi}{\partial \tau}&=-il\omega\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}+\mu\epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)}\\&+\epsilon^{n+2}\frac{\partial\phi_l^n}{\partial T}e^{il(k\xi-\omega\tau)}\end{aligned} \tag{8}$$

b. Calculate

$A\phi\frac{\partial\phi}{\partial\xi}=A\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\left(\frac{\partial}{\partial\xi}+\epsilon\frac{\partial}{\partial X}\right)\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)$:

$$\begin{aligned}&A\phi\frac{\partial \phi}{\partial \xi}=\\&A\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\left(\frac{\partial}{\partial \xi}+\epsilon\frac{\partial}{\partial X}\right)\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\\&=A\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\left[\frac{\partial}{\partial \xi}\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\right.\\&\left.+\epsilon\frac{\partial}{\partial X}\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\right]\end{aligned} \tag{9}$$

$$\begin{aligned}&\Rightarrow A\phi\frac{\partial \phi}{\partial \xi}=A\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\left[ilk\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right.\\&\left.+\epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)}\right]\end{aligned} \tag{10}$$

c. Calculate: $B\frac{\partial^3\phi}{\partial\xi^3}=B\frac{\partial^3}{\partial\xi^3}\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)$:

Now apply $\frac{\partial}{\partial\xi}$ on $\left[ilk\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}+\epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)}\right]$ in Equation(10) we get

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \phi}{\partial \xi}\right)=\frac{\partial}{\partial \xi}\left[\left(ilk\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}+\epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)}\right)\right]$$

As we know that $\frac{\partial}{\partial\xi}=\frac{\partial}{\partial\xi}+\epsilon\frac{\partial}{\partial X}$

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\phi}{\partial\xi}\right) = \left(\frac{\partial}{\partial\xi} + \epsilon\frac{\partial}{\partial X}\right)\left[\left(ilk\epsilon^{n}\phi_l^n e^{il(k\xi-\omega\tau)} + \epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)}\right)\right] \tag{11}$$

Second order derivative is given below:

$$\frac{\partial^2\phi_l^n}{\partial\xi^2} = \left[\frac{\partial}{\partial\xi}\left(ilk\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right) + \frac{\partial}{\partial\xi}\left(\epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)}\right) + \epsilon\frac{\partial}{\partial X}\left(ilk\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right) + \epsilon\frac{\partial}{\partial X}\left(\epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)}\right)\right] \tag{12}$$

Now again derive Equation (12) and using the value of $\frac{\partial}{\partial\xi}$ from Equation (3) we get

$$\frac{\partial^3\phi_l^n}{\partial\xi^3} = \left[\left(\frac{\partial}{\partial\xi} + \epsilon\frac{\partial}{\partial X}\right)\left(-l^2k^2\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)} + 2ilk\,\epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)} + \epsilon^{n+2}\frac{\partial^2\phi_l^n}{\partial X^2}e^{il(k\xi-\omega\tau)}\right)\right]$$

$$\Rightarrow B\frac{\partial^3\phi_l^n}{\partial\xi^3} = B\left[-il^3k^3\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)} - 3l^2k^2\,\epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)} + 3ilk\epsilon^{n+2}\frac{\partial^2\phi_l^n}{\partial X^2}e^{il(k\xi-\omega\tau)} + \epsilon^{n+3}\frac{\partial^3\phi_l^n}{\partial X^3}e^{il(k\xi-\omega\tau)}\right] \tag{13}$$

Use all Equations from (8) to (13) into Equation (6)

$$\begin{aligned}\frac{\partial\phi}{\partial\tau} + A\phi\frac{\partial\phi}{\partial\xi} + B\frac{\partial^3\phi}{\partial\xi^3} =& - il\omega\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)} + \mu\epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)} + \epsilon^{n+2}\frac{\partial\phi_l^n}{\partial T}e^{il(k\xi-\omega\tau)}\\ &+ A\left(\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)}\right)\left[ilk\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)} + \epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)}\right]\\ &+ B\left[-il^3k^3\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)} - 3l^2k^2\,\epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)} + 3ilk\epsilon^{n+2}\frac{\partial^2\phi_l^n}{\partial X^2}e^{il(k\xi-\omega\tau)} + \epsilon^{n+3}\frac{\partial^3\phi_l^n}{\partial X^3}e^{il(k\xi-\omega\tau)}\right] = 0\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial\phi}{\partial\tau} + A\phi\frac{\partial\phi}{\partial\xi} + B\frac{\partial^3\phi}{\partial\xi^3} =& - il\omega\epsilon^n\phi_l^n + \mu\epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X} + \epsilon^{n+2}\frac{\partial\phi_l^n}{\partial T}\\ &+ A\epsilon^{2n}\phi_l^n\left(ilk\epsilon^n\phi_l^n e^{il(k\xi-\omega\tau)} + \epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X}e^{il(k\xi-\omega\tau)}\right)\\ &- B\left[il^3k^3\epsilon^n\phi_l^n + 3l^2k^2\,\epsilon^{n+1}\frac{\partial\phi_l^n}{\partial X} - 3ilk\epsilon^{n+2}\frac{\partial^2\phi_l^n}{\partial X^2} - \epsilon^{n+3}\frac{\partial^3\phi_l^n}{\partial X^3}\right] = 0\end{aligned}$$

Above Equation with summation is given below:

$$\begin{aligned}\frac{\partial\phi}{\partial\tau} + A\phi\frac{\partial\phi}{\partial\xi} + B\frac{\partial^3\phi}{\partial\xi^3} =& - il\omega\phi_l^n + \mu\epsilon\frac{\partial\phi_l^{n-1}}{\partial X} + \epsilon^2\frac{\partial\phi_l^{n-2}}{\partial T}\\ &+ A\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(ilk\phi_l^n\phi_{l-l'}^{n-n'} + \epsilon\phi_{l-l'}^{n-n'-1}\frac{\partial\phi_l^{n-1}}{\partial X}\right)\\ &- B\left(il^3k^3\phi_l^n + 3\epsilon l^2k^2\frac{\partial\phi_l^{n-1}}{\partial X} - 3ilk\epsilon^2\frac{\partial^2\phi_l^{n-2}}{\partial X^2} - \epsilon^3\frac{\partial^3\phi_l^{n-3}}{\partial X^3}\right) = 0\end{aligned} \tag{15}$$

**For First Order Approximation let l = n = 1 yield:**

$$-i\omega\phi_1^1 - iBk^3\phi_1^1 = 0 \tag{16}$$

$$\Rightarrow \omega = -Bk^3 \tag{17}$$

**For First Harmonic of Second Order Approximation let l = 1, n = 2 yields**:

$$\mu \frac{\partial \phi_1^1}{\partial X} - 3Bk^2 \frac{\partial \phi_1^1}{\partial X} = 0$$

$$\Rightarrow \mu = 3Bk^2 \tag{18}$$

**For Second Harmonic model let l = n = 2 with coefficients of$\epsilon^0$**:

$$-i2\omega \phi_2^{(2)} + Aik\phi_1^{(1)2} - 8Bik^3 \phi_2^{(2)} = 0$$

$$-2\left(\omega + 4Bk^3\right) \phi_2^{(2)} + Ak\phi_1^{(1)2} = 0$$

Put value of $\omega = -Bk^3$:

$$-2\left(-Bk^3 + 4Bk^3\right) \phi_2^{(2)} + Ak\phi_1^{(1)2} = 0$$

$$-2\left(3Bk^3\right) \phi_2^{(2)} + Ak\phi_1^{(1)2} = 0$$

$$\phi_2^{(2)} = \frac{A}{6Bk^3} \phi_1^{(1)2} \tag{19}$$

**The zero order harmonics for *l=0* is**:

$$\phi_0^{(2)} = |\frac{-A}{\lambda} \left|\phi_1^{(1)}\right|^2 \tag{20}$$

By solving the 1st harmonic equations ($l$= 1) in the 3rd-order approximation (n = 3), we get an explicit compatibility condition, and thus the NLS equation, with little effort:

$$i\frac{\partial \phi_1^1}{\partial T} + \frac{1}{2} P \frac{\partial^2 \phi_1^1}{\partial X^2} + Q\phi_1^{(1)2}\phi_1^1 = 0$$

For simplify we let $\phi_1^1 = \Psi$

$$i\frac{\partial \Psi}{\partial T} + \frac{1}{2} P \frac{\partial^2 \Psi}{\partial X^2} + Q|\Psi|^2\Psi = 0 \tag{21}$$

The Equation (21) called the Nonlinear Schrodinger Equation and describe the nonlinear evolution of an amplitude modulated IA wave carrier. The nonlinear coefficient Q and the dispersion coefficient P are provided by:

$$\Rightarrow P = 6Bk \text{ and } Q = \frac{A^2}{6Bk} \tag{22}$$

By extending as described in Taniuti and Yajima (1969), Asano et al. (1969), Shimizu and Ichikawa (1972), and El-Labany et al. (2007), we view the solution of Equation (21) as a weakly modulated sinusoidal wave.

$$\Psi = \frac{1}{Q}\left[\frac{4(1+2iT)}{1+4T^2+\frac{4}{P}X^2} - 1\right] e^{iT} \tag{23}$$

Putting values of Equation(22) into Equation(23) we get

$$\Psi = \frac{1}{\frac{A^2}{6Bk}}\left[\frac{4(1+2iT)}{1+4T^2+\frac{4}{6Bk}X^2} - 1\right] e^{iT}$$

At end we get the calculation

$$\Psi = \frac{6Bk}{A^2}\left[\frac{6Bk\left(3+8iT-4T^2\right)-4X^2}{6Bk\left(1+4T^2\right)+4X^2}\right] e^{iT} \tag{24}$$

Solution of Equation (23) specifically demonstrates that the DIAW energy is focused in a tiny amount of space. This aspect of the nonlinear solution may characterize the DIA rogue wave in plasmas. The wavelength of most rogue waves is shorter than the wavelength of the center portion of the envelope generated around a carrier wave.

### 2.1 CASE-1. Pair Ion (PI) Plasma

A set of nonlinear equations for pure PI plasma When we consider a pure PI plasma, we do not consider magnetization. The electrostatic waves of equally massed positively and negatively charged ions heated to the same temperature have been examined using the two-fluid theory. where A and B specify the nonlinear and dispersive coefficients, respectively (Phys. Scr. 80 (2009) 035502):

$$A = \left[\frac{\left[\left(3\lambda^2-1\right)\left(\lambda^2-\mu\right)^3 - \left(3\lambda^2-\mu\right)\left(\lambda^2-1\right)^3\right]}{2\lambda\left(\lambda^2-1\right)\left(\lambda^2-\mu\right)\left[\left(\lambda^2-\mu\right)^3+\left(\lambda^2-1\right)^2\right]}\right] \tag{25}$$

and

$$B = \frac{\left(\lambda^2 - 1\right)^2 \left(\lambda^2 - \mu\right)^2}{2\lambda\left[\left(\lambda^2 - \mu\right)^2 + \left(\lambda^2 - 1\right)^2\right]} \tag{26}$$

The parameter $\lambda$ in term of $\mu$ is defined here:

$$\lambda = \sqrt{\frac{1+\mu}{2}} \tag{27}$$

All these parameters of PI plasma are defined as:

- $P, Q\&A, B$ = The Dispersion Coefficients
- $\mu$ = Temperature Ratio = $\frac{T_-}{T_+}$
- $\lambda$ = Linear Dispersion Relation = $\frac{Linear\ Phase\ Velocity}{Thermal\ Velocity\ Of\ Positive\ Ion}$
- $k$ = Carrier Wavenumber

**Note:**

The Pure PI depends of $\mu, \lambda\ \&\ K$ values

### 2.1.1 *Calculation of solution for pure PI plasma*

Putting values of Equations (25-26) into Equation (24) we get

### 2.1.2 *Graphical Analysis of PI Plasma*

**parameters values for plot:**

- The solution is not valid for K $= 0\&\lambda = \mu = 1$
- Solution will hold for $\mu > 0\&K < 0$ and $K > 0$

## 2.2 CASE-2. Pair Ion-Electron (PIE) Plasma

In unmagnetized plasmas of pure PI, electrons are now thought of as an impurity. For PIE plasmas with electrostatic waves perpendicular to the magnetic field, we give a description of the corresponding nonlinear set of equations. When electrons are introduced to pure PI plasmas, the dynamics shift from slow to fast. Ions are considered to be heated adiabatically, but electrons are supposed to be isothermal, because ions have greater inertia than electrons. Electrons are expected to be Boltzmann distributed and to have no inertia or temperature dependence, but ions of the same mass and temperature are thought to be dynamic. Landau damping effects are disregarded in the model, which has been developed using a multi fluid technique (Phys. Scr. 80 (2009) 035502):

$$A = \left[\frac{3p\lambda^2 + 3\beta - \left(\lambda^2 p - 3\beta\right)^3}{2p\lambda(2p-1)(\lambda^2 p - 3\beta)}\right] \tag{29}$$

$$B = \left[\frac{\left(\lambda^2 p - 3\beta\right)^2}{2p\lambda(2p-1)}\right] \tag{30}$$

The parameter $\lambda$ in term of $\mu$ is defined here:

$$\lambda = \sqrt{\frac{3\beta + 2p - 1}{p}} \tag{31}$$

All these parameters of PIE plasma are defined as:

- $P, Q\&A, B$ = The Dispersion Coefficients
- $\beta = \frac{T}{T_c} = \frac{Total temperature}{temperature of electron}$
- $p$ = Linear Dispersion Relation = $\frac{n_{0+}}{n_{0e}} = \frac{perturb density}{equilibrium density}$
- $\lambda$ = Linear Dispersion Relation
- $k$ = Carrier Wavenumber

The PIE depends of $\beta, \lambda, p\&K$ values

### 2.2.1 *Calculation of solution for PIE plasma*

Putting values of Equations (29) and (30) into Equation (24), we get

$$\Psi = \frac{6\left[\frac{\left(\lambda^2 p - 3\beta\right)^2}{2p\lambda(2p-1)}\right]k}{\left[\frac{3p\lambda^2 + 3\beta - \left(\lambda^2 p - 3\beta\right)^3}{2p\lambda(2p-1)(\lambda^2 p - 3\beta)}\right]^2} \times \left[\frac{6\left[\frac{\left(\lambda^2 p - 3\beta\right)^2}{2p\lambda(2p-1)}\right]k\left(3 + 8iT - 4T^2\right) - 4X^2}{6\left[\frac{(\lambda^2 p - 3\beta)^2}{2p\lambda(2p-1)}\right]k\left(1 + 4T^2\right) + 4X^2}\right]e^{iT} \tag{32}$$

$$\Psi = \frac{6\left[\frac{\left(\lambda^2 p - 3\beta\right)^2}{2p\lambda(2p-1)}\right]k}{\left[\frac{3p\lambda^2 + 3\beta - \left(\lambda^2 p - 3\beta\right)^3}{2p\lambda(2p-1)(\lambda^2 p - 3\beta)}\right]^2} \times \left[\frac{\frac{6k\left(\lambda^2 p - 3\beta\right)^2(3+8iT-4T^2) - 4X^2(2p\lambda(2p-1))}{2p\lambda(2p-1)}}{\frac{6k(\lambda^2 p - 3\beta)^2(1+4T^2) + 4X^2(2p\lambda(2p-1))}{2p\lambda(2p-1)}}\right]e^{iT} \tag{33}$$

$$\Psi = \frac{3k\left[2p\lambda(2p-1)\left(\lambda^2 p - 3\beta\right)\right]^2\left(\lambda^2 p - 3\beta\right)^2}{p\lambda(2p-1)\left[3p\lambda^2 + 3\beta - \left(\lambda^2 p - 3\beta\right)^3\right]^2} \times \left[\frac{3k\left(\lambda^2 p - 3\beta\right)^2\left(3 + 8iT - 4T^2\right) - 4X^2 p\lambda(2p-1)}{3k\left(\lambda^2 p - 3\beta\right)^2\left(1 + 4T^2\right) + 4X^2 p\lambda(2p-1)}\right]e^{iT} \tag{34}$$

$$\Psi = \frac{6\left[\frac{(\lambda^2-1)^2(\lambda^2-\mu)^2}{2\lambda[(\lambda^2-\mu)^2+(\lambda^2-1)^2]}\right]k}{\left[\frac{[(3\lambda^2-1)(\lambda^2-\mu)^3-(3\lambda^2-\mu)(\lambda^2-1)^3]}{2\lambda(\lambda^2-1)(\lambda^2-\mu)[(\lambda^2-\mu)^3+(\lambda^2-1)^2]}\right]^2}\left[\frac{\left[6k\frac{(\lambda^2-1)^2(\lambda^2-\mu)^2}{2\lambda[(\lambda^2-\mu)^2+(\lambda^2-1)^2]}(3+8iT-4T^2)\right]-4X^2}{\left[6\left(\frac{(\lambda^2-1)^2(\lambda^2-\mu)^2}{2\lambda[(\lambda^2-\mu)^2+(\lambda^2-1)^2]}\right)k(1+4T^2)\right]+4X^2}\right]e^{iT}$$

$$\Psi = \frac{6k\left[\left(2\lambda\left(\lambda^2-1\right)\left(\lambda^2-\mu\right)\left[\left(\lambda^2-\mu\right)^3+\left(\lambda^2-1\right)^2\right]\right)^2 * \left[\left(\lambda^2-1\right)^2\left(\lambda^2-\mu\right)^2\right]\right]}{2\lambda\left[(\lambda^2-\mu)^2+(\lambda^2-1)^2\right] * \left((3\lambda^2-1)(\lambda^2-\mu)^3-(3\lambda^2-\mu)(\lambda^2-1)^3\right)^2}\left[\frac{\frac{6k\left(\lambda^2-1\right)^2\left(\lambda^2-\mu\right)^2 * \left(3+8iT-4T^2\right)-4X^2.\ 2\lambda\left[\left(\lambda^2-\mu\right)^2+\left(\lambda^2-1\right)^2\right]}{2\lambda\left[\left(\lambda^2-\mu\right)^2+\left(\lambda^2-1\right)^2\right]}}{\frac{6k\left(1+4T^2\right)\left(\lambda^2-1\right)^2\left(\lambda^2-\mu\right)^2+4X^2.\ 2\lambda\left[\left(\lambda^2-\mu\right)^2+\left(\lambda^2-1\right)^2\right]}{2\lambda\left[\left(\lambda^2-\mu\right)^2+\left(\lambda^2-1\right)^2\right]}}\right]e^{iT}$$

$$\Psi = \frac{12\lambda k\left[\left(\left(\lambda^2-1\right)\left(\lambda^2-\mu\right)\left[\left(\lambda^2-\mu\right)^3+\left(\lambda^2-1\right)^2\right]\right)^2 * \left[\left(\lambda^2-1\right)^2\left(\lambda^2-\mu\right)^2\right]\right]}{\left[(\lambda^2-\mu)^2+(\lambda^2-1)^2\right] * \left((3\lambda^2-1)(\lambda^2-\mu)^3-(3\lambda^2-\mu)(\lambda^2-1)^3\right)^2}$$
$$* \left[\frac{6k\left(\lambda^2-1\right)^2\left(\lambda^2-\mu\right)^2 * \left(3+8iT-4T^2\right)-8\lambda X^2\left[\left(\lambda^2-\mu\right)^2+\left(\lambda^2-1\right)^2\right]}{6k\left(1+4T^2\right)(\lambda^2-1)^2(\lambda^2-\mu)^2+8\lambda X^2\left[(\lambda^2-\mu)^2+(\lambda^2-1)^2\right]}\right]e^{iT}$$

The final PI solution including with the dispersion coefficients A & B are here below:

$$\Psi = \left[\frac{12\lambda k\left(\left(\lambda^2-1\right)\left(\lambda^2-\mu\right)\left[\left(\lambda^2-\mu\right)^3+\left(\lambda^2-1\right)^2\right]\right)^2\left[\left(\lambda^2-1\right)^2\left(\lambda^2-\mu\right)^2\right] * 6k\left(\lambda^2-1\right)^2\left(\lambda^2-\mu\right)^2 * \left(3+8iT-4T^2\right)-8\lambda X^2\left[\left(\lambda^2-\mu\right)^2+\left(\lambda^2-1\right)^2\right]}{\left[(\lambda^2-\mu)^2+(\lambda^2-1)^2\right]\left((3\lambda^2-1)(\lambda^2-\mu)^3-(3\lambda^2-\mu)(\lambda^2-1)^3\right)^2 * 6k\left(1+4T^2\right)(\lambda^2-1)^2(\lambda^2-\mu)^2+8\lambda X^2\left[(\lambda^2-\mu)^2+(\lambda^2-1)^2\right]}\right]e^{iT} \tag{28}$$

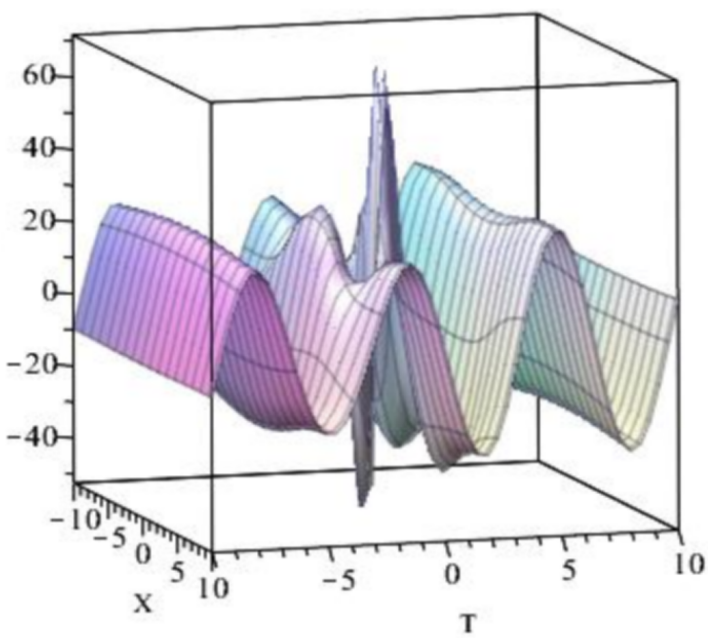


**Figure 1.** Oscillations with downward sharp freakwave generated with negative wavenumber $K < 0$ and temperature ratio $\mu = 0.5$.

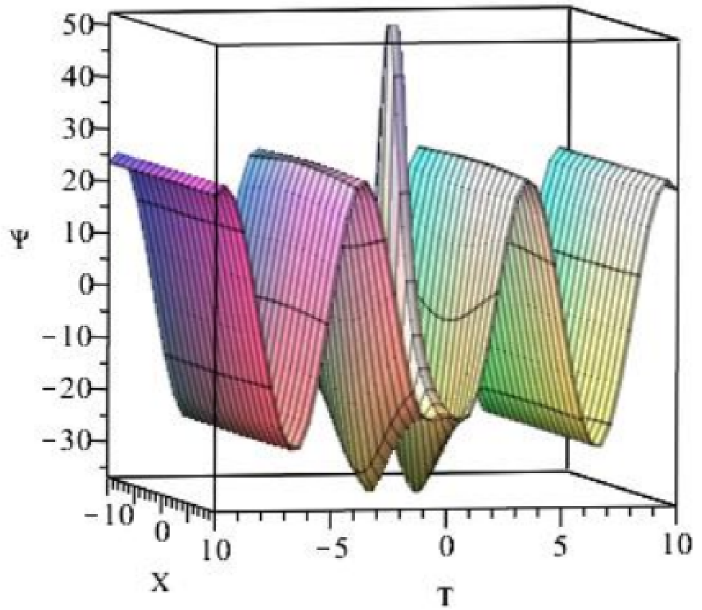


**Figure 2.** Oscillations with upward sharp freakwave generated with positive wavenumber $K > 0$ and temperature ratio $\mu = 0.5$.

Final result alongwith coefficients are given here below

$$\Psi = \frac{3k[2p\lambda(2p-1)(\lambda^2 p-3\beta)]^2(\lambda^2 p-3\beta)^2}{p\lambda(2p-1)[3p\lambda^2+3\beta-(\lambda^2 p-3\beta)^3]^2} \times \left[\frac{3k(\lambda^2 p-3\beta)^2(3+8iT-4T^2)-4X^2 p\lambda(2p-1)}{3k(\lambda^2 p-3\beta)^2(1+4T^2)+4X^2 p\lambda(2p-1)}\right]e^{iT} \tag{35}$$

### 2.2.2 *Graphical Analysis of PIE Plasma*

For parameters values for plot:

- The solution is not valid for $k$=0&$p = 0$
- Solution will hold for $p > 0, k > 0 \& \beta > 0$

Figures 1 and 2 illustrate the freak wave structures

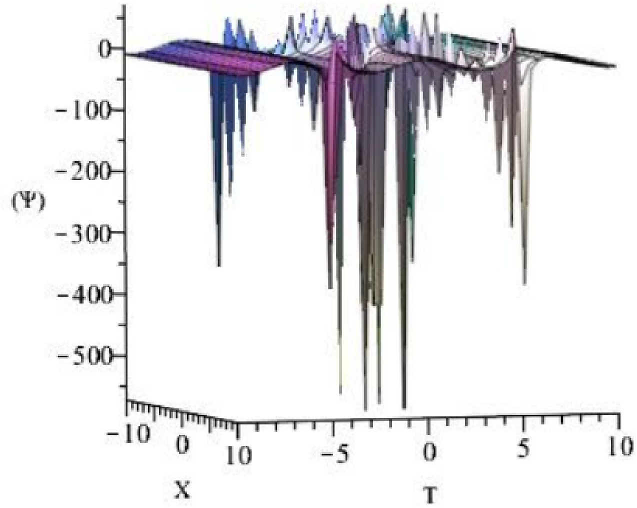


**Figure 3.** PIE for $k < 0$ , $\beta = 0.5$ and $p > 0$ here for the negative value of wavenumber $K$ and interval values of $\beta$&$p$, provide multiple Sharpe downward breather freak waves are generated.

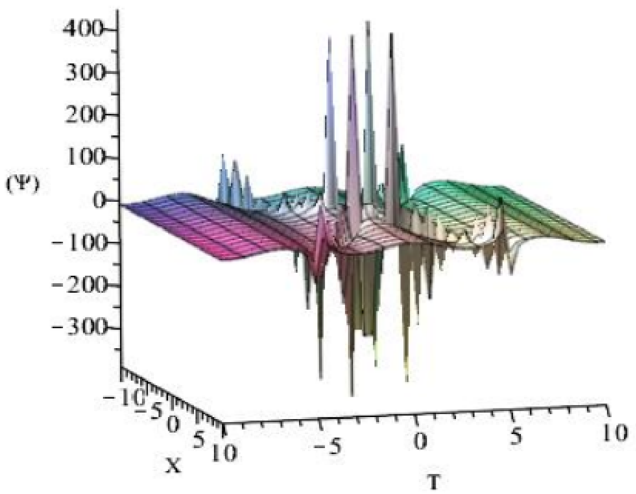


**Figure 4.** PIE for $k > 0$ , $\beta = 0.5$ and $p > 0$ here for the negative value of wavenumber $K$ and interval values of $\beta$&$p$, provide multiple Sharpe downward breather freak waves are generated.

in PI plasma for negative and positive wavenumbers, respectively. For comparison, Figures 3 and 4 show the corresponding results in PIE plasma. The inclusion of electrons in PIE plasmas leads to sharper and more elongated waveforms with enhanced amplitudes, primarily due to modified nonlinear coupling and dispersion relations.

## 3 Conclusion

This work presents a novel approach to the investigation of electrostatic freak waves by transforming the Korteweg–de Vries (KdV) equation into the nonlinear Schrödinger equation (NLSE) through the application of the wave superposition principle. The exact analytical solution of the NLSE is discussed, and the calculations for pair-ion (PI) and pair-ion–electron (PIE) plasmas are performed using the parameter and coefficient values reported in [17, 18]. The structure of PI plasma depends on $\mu$ & $K$, while PIE with free electrons depends on $\beta, p$ & $K$ under defined domain $(\mu > 0, K \neq 0,\ p \neq 0 \ \&\ \beta \in [0, \infty))$. The Plot of PI & PIE plasmas for $k < 0$; provide freak waves with a downward direction as by Figure 1, similarly for $k > 0$; provide freak waves with an upward direction as Figure 2. The Temperature ratio for PI is $\mu$ and for PIE is $\beta$, take the value of $\mu = \beta = 0.5$ & $p > 0$ at these value of coefficients PI has a freak wave structure with oscillation while PIE has a freak wave structure with sharp breathers. The amplitude of freak waves in PIE plasmas is found to be greater than that in PI plasmas for the same range of parameters. The results and analysis indicate that the variations in structure, amplitude, and behavior of freak waves in PI and PIE plasmas arise from the presence or absence of electrons. In PI plasmas, which contain only ions, the freak waves exhibit a relatively regular structure, whereas in PIE plasmas, the inclusion of electrons leads to sharper and more elongated waveforms. The combined influence of all relevant coefficients, namely dispersion, phase velocity, temperature, and density, significantly governs the characteristics of freak waves in both plasma systems. While the KdV equation describes only solitary wave solutions, the NLSE provides a more powerful framework for analyzing freak waves, making it highly suitable for plasma wave investigations.

## Data Availability Statement

Data will be made available on request.

## Funding

This work was supported without any funding.

## Conflicts of Interest

The authors declare no conflicts of interest.

## Ethical Approval and Consent to Participate

Not applicable.

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