RESEARCH ARTICLE

Reverse-Order Law for Weak Core Inverse

Amit Kumar[1,*]

[1] Department of Mathematics, National Institute of Technology Raipur, Raipur, Chhattisgarh 492010, India

Abstract

In this paper, some sufficient conditions for the reverse-order law of the weak core inverse are obtained. Several characterizations of the reverse-order law for this generalized inverse are then established. In addition, some results concerning the absorption law for the weak core inverse are proved.

Keywords: weak core inverse, core inverse, reverse-order law, absorption law.

1 Introduction

In this article, R denotes a proper ring with involution. A ring R is called as a *proper ring* if $a^*a = 0 \Rightarrow a = 0$ for all $a \in R$. An *involution* $*$ is an anti-isomorphism of order 2 that satisfies the conditions

$$\begin{aligned}&(a+b)^* = a^* + b^*,\\ &(ab)^* = b^*a^*,\\ &\text{and } (a^*)^* = a, \quad \text{for all } a, b \in R.\end{aligned}$$

An element $a \in R$ is *Moore-Penrose invertible* if there exists a unique element $x \in R$ that satisfies the equations:

$$\begin{aligned}&(1.)\ axa = a,\\ &(2.)\ xax = x,\\ &(3.)\ (ax)^* = ax,\\ &(4.)\ (xa)^* = xa.\end{aligned}$$

Then, x is called as the *Moore-Penrose inverse* [1] of a, and is denoted as $x = a^\dagger$. By $R^\dagger$, we denote the set of all Moore-Penrose invertible elements of R. The set of all elements which satisfies any of the combinations of the above four equations is denoted as $a\{i,j,k,l\}$, where $i,j,k,l \in \{1,2,3,4\}$, and is called a generalized inverse of a. The first and third generalized inverse of a is denoted as $a^{(1,3)}$. The set of first and third invertible elements of R, is denoted by $R^{(1,3)}$. An element a is called *Drazin invertible* [2] if there exists a unique element $x \in R$ such that $xa^{k+1} = a^k, ax = xa$, and $ax^2 = x$, for some positive integer k. If the Drazin inverse of a exists, then it is denoted by a^d. The smallest positive integer k is called the *Drazin index*, is denoted by $i(a)$. The set of all Drazin invertible elements of R will be denoted by R^d. If $i(a) = 1$, then the Drazin inverse of a is called as the *group inverse* of a, and is denoted by $a^\#$. The set of group invertible elements of R will be denoted by $R^\#$.

Xu *et al.* [3] proved that an element $a \in R$ is *core invertible* if and only if there exists a unique element $x \in R$ satisfying the following conditions:

$$(ax)^* = ax, \quad ax^2 = x, \quad \text{and} \quad xa^2 = a.$$

It is denoted by $a^{\textcircled{\#}}$. An element $a \in R$ is said to be *pseudo core invertible* [4] if there exists a unique element

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***Corresponding author:**
✉ Amit Kumar
amitdhull513@gmail.com

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$x \in R$ such that

$$(ax)^* = ax, \quad ax^2 = x \text{ and } xa^{m+1} = a^m,$$

for some positive integer m. The least positive integer m for which the above equations hold is called the *pseudo core index* and denoted by $I(a)$. The pseudo core inverse of a is denoted by $a^{\oplus}$. Let $a \in R$. Then, the unique element $y \in R$ is called *weak group inverse* [5] of a if it satisfies these three conditions:

$$ya^{k+1} = a^k, ay^2 = y \text{ and } (a^k)^* a^2 y = (a^k)^* a.$$

The weak group inverse of an element $a \in R$ is denoted by a^{ϖ}. The least positive integer k that satisfying the above condition is called the weak group index of a, denoted by $\text{ind}(a)$. The set of all the weak group invertible element denoted by R^{ϖ}. Let $a \in R$. Then, the unique element $y \in R$ is called *weak core inverse* [6] of a if it satisfies these three conditions:

$$ya^{k+1} = a^k, ay^2 = y \text{ and } (a^k)^* ay = (a^k)^*.$$

It is denoted by $a^{\boxplus}$. The least positive integer k that satisfies the above condition is called the *weak core index* of a, and is denoted by $\text{ind}_{wc}(a)$. The set of all the weak core invertible element denoted by $R^{\boxplus}$.

If a and b are a pair of invertible elements, then ab is also invertible and the inverse of the product ab satisfying

$$(ab)^{-1} = b^{-1}a^{-1},$$

is called as the *reverse-order law*. On the other way,

$$(ab)^{-1} = a^{-1}b^{-1}$$

is known as the *forward-order law*. While the reverse-order law does not hold for different generalized inverses, the forward-order law is not true even for invertible elements. The *absorption law* for invertible elements a and b is

$$a^{-1}(a+b)b^{-1} = a^{-1} + b^{-1}.$$

In 1966, Greville [7] first obtained sufficient conditions for which the reverse-order law holds for the Moore-Penrose inverse in matrix form, i.e., $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$. Hartwig [8] obtained several results on block generalized inverses, including a characterization of the $\{1,3\}$-inverse of an element in a ring, which will be used in the sequel. Djordjević and Dinčić [9] extended the reverse-order law involving the Moore-Penrose inverse from the matrix setting to operators on Hilbert spaces. The same problem was also considered by several authors for other generalized inverses. In 2012, Mosić and Djordjević [10] studied the reverse-order law for the group inverse in rings. Jin and Benítez [11] established equivalent conditions for the absorption law in terms of the Moore-Penrose inverse, the group inverse, the core inverse and the dual core inverse, as well as the $\{1\}$-, $\{1,2\}$-, $\{1,3\}$- and $\{1,4\}$-inverses in rings. In 2017, Chen *et al.* [12] investigated the double commutativity and the reverse-order law for the core inverse in rings with involution, and also gave characterizations and representations of the core and the dual core inverses. In 2018, Zou *et al.* [13] proved the reverse-order law for the core inverse in rings. In 2019, Xu *et al.* [14] obtained, by means of centralizers, the absorption law and the reverse-order law for the (b,c)-inverses in rings. In 2020, Zhou *et al.* [5] introduced the weak group inverse in proper $*$-rings and studied its properties. In 2021, Gao *et al.* [15] provided the absorption law and the reverse-order law for the generalized core inverse. Kumar and Mishra [16] discussed several results on the forward-order law for the core inverse, and later [17] introduced the WD and WDMP generalized inverses in rings, for which the additive property, the reverse-order law and the forward-order law were established. More recently, Li *et al.* [18] studied weak core inverses in rings with involution, where Jacobson's lemma, reverse-order laws and equivalent conditions for absorption laws of the weak core inverse were presented, while Zhou [19] investigated the reverse-order law and the absorption law for the weak group inverse in proper $*$-rings.

Although reverse-order laws and absorption laws for the weak core inverse have been considered in [18], the hypotheses adopted in the present paper are of a different nature. Here the reverse-order law is derived under the weak commutativity conditions $ab^2 = b^2a = bab$ together with $a^*b^2 = b^2a^* = ba^*b$ (Theorem 3.5), and under $aba = a^2b$ together with $aba^* = a^*ab$ (Theorem 3.12), as well as under the dual conditions $bab = b^2a$ together with $bab^* = b^*ab$ (Theorem 3.13). The first set of conditions yields the forward-order law $(ab)^{\boxplus} = a^{\boxplus}b^{\boxplus}$ as well. Moreover, Theorem 3.9 characterizes the reverse-order law by means of the Drazin inverse and the $\{1,3\}$-inverse of $(ab)^k$, thereby reducing the verification of $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}$ to the strong π-regularity of ab. Correspondingly, the equivalent conditions for the absorption law obtained in Section 4 are expressed in terms of the principal ideals $a^kR = b^kR$ and the left annihilators $^{\circ}(a^k) = {}^{\circ}(b^k)$ (Theorem 4.1), as well as in terms of $a^{\boxplus}R$ and $Ra^{\boxplus}$ (Theorem 4.2).

The paper is organized as follows. In Section 2, preliminary results are recalled. In Section 3, results useful for establishing sufficient conditions for the reverse-order law of the weak core inverse are presented, followed by the main theorem of the section. Further results concerning the equivalence of the reverse-order law for the weak core inverses of a and b are discussed. In Section 4, the absorption law for the weak core inverse is proved.

2 Preliminaries

In this section, we recall some previously established results that will be used to prove main results of this article. The following result is proved by Chen *et al.* [12] for core invertible elements.

Lemma 2.1. (*Theorem 3.2*, [12])
Let $a, b, x \in R$ with $xa = bx$ and $xa^* = b^*x$. If $a, b \in R^{\oplus}$, then $xa^{\oplus} = b^{\oplus}x$.

Recently, Sahoo *et al.* [6] obtained the following results for weak core inverse. Throughout this section, the numbering of the results quoted from [6] refers to arXiv:2005.13782v3.

Lemma 2.2. (*Proposition 3.7*, [6])
Let $a \in R^{\boxplus}$ with $ind_{wc}(a) = k$. Then $a \in R^d$ with $i(a) = k$. Moreover $a^d = (a^{\boxplus})^{k+1}a^k$.

Now, we recall the *annihilators* of an element in a ring. The left annihilator of $a \in R$ is given by ${}^{\circ}(a) = \{x \in R : xa = 0\}$ and the right annihilator of a is given by $(a)^{\circ} = \{x \in R : ax = 0\}$

Theorem 2.3. (*Theorem 3.4*, [6])
Let $a \in R$. The following assertions are equivalent:
(i) $y = a^{\boxplus}$ and $ind_{wc}(a) \leq k$;
(ii) $yay = y,\ \ yR = a^kR = a^{k+1}R$ and $a^kR \subseteq y^*R$;
(iii) $yay = y,\ \ yR = a^kR \subseteq a^{k+1}R$ and ${}^{\circ}(y^*) \subseteq {}^{\circ}(a^k)$;
(iv) $yay = y$, ${}^{\circ}(a^{k+1}) \subseteq {}^{\circ}(a^k) = {}^{\circ}(y^*)\ {}^{\circ}(y^*) \subseteq {}^{\circ}(a^k)$;

Lemma 2.4. (*Theorem 3.18*, [6])
Let $a \in R^{\boxplus}$. Then, $a^n \in R^{\boxplus}$ and $(a^{\boxplus})^n = (a^n)^{\boxplus}$ for all $n \geq 1$. Moreover, $a^{\boxplus} = a^{n-1}(a^n)^{\boxplus}$.

Theorem 2.5. (*Theorem 3.12*, [6])
Let $a \in R^d$ with $i(a) = k$. If $(a^k)^{(1,3)}$ exists, then $a \in R^{\boxplus}$. Moreover,

$$a^{\boxplus} = a^D a^k (a^k)^{(1,3)}.$$

The next result gives a characterization of $\{1,3\}$-inverse of an elements in a ring.

Lemma 2.6. ([8])
Let $a, x \in R$. Then x is a-$\{1,3\}$ inverse of a if and only if $x^*a^*a = a$.

Recall that an element $a \in R$ is called strongly $\pi-$regular in R if there exists $x, y \in R$ and a positive integer k such that

$$a^k = a^{k+1}x = ya^{k+1}.$$

Using $\pi-$regular element, we state the following characterization of drazin invertible elements in a ring obtained by Drazin [2].

Lemma 2.7. (*Theorem 4*, [2])
Let $a \in R$. Then $a \in R^d$ if and only if a is strongly π-regular in R.

3 Main Results

In this section, sufficient conditions under which the reverse-order law holds for the weak core inverse are established, and several characterizations of this law are given. An example is first provided to demonstrate that, in general, the reverse-order law does not hold for the weak core inverse in a ring with involution.

Example 3.1. *Let $R = M_{2\times 2}(\mathbb{R})$ and $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \in R$ with involution transpose. Then,*

$$(ab)^{\boxplus} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1/2 & 0 \\ -1/2 & 0 \end{bmatrix} = b^{\boxplus}a^{\boxplus}.$$

Next lemma can be proved using similar steps as Theorem 2.1. This result will be helpful while proving main results of this section.

Lemma 3.1. *Let $a, b, x \in R$ with $ax = bx$ and $a^*x = b^*x$. If $a, b \in R^{\boxplus}$, then $xa^{\boxplus} = b^{\boxplus}x$.*

For the special case $a = b$, we have the following corollary.

Corollary 3.2. *Let $a, x \in R$ with $ax = xa$ and $a^*x = xa^*$. If $a \in R^{\boxplus}$, then $xa^{\boxplus} = a^{\boxplus}x$.*

A result based on the definition of the weak core inverse is stated below.

Lemma 3.3. *Let $a \in R^{\boxplus}$, and x is the weak core inverse of a. Then,*

(i) $ax = a^m x^m$, for any positive integer m.

(ii) $a^k x^k a^k = a^k$, where k is weak core index of a.

(iii) $(ax)^* = ax$.

(iv) $xax = x$.

The following characterization of core invertible elements follows from the above result.

Theorem 3.4. *Let $a \in R$. Then $a \in R^{\oplus}$ if and only if $a \in R^{\boxplus}$ with weak core index $k = 1$.*

Proof. Let $a \in R^{\boxplus}$ with $k = 1$. Then, $ax^2 = x, xa^2 = a$, and $a^*ax = a^*$. By Lemma 3.3, we have $(ax)^* = ax$ and so $a \in R^{\oplus}$. Conversely, let $a \in R^{\oplus}$. So, $ax^2 = x$, $xa^2 = a$ and $(ax)^* = ax$. Since, $ax^2 = x$ and $xa^2 = a$, we get $ax^2a^2 = a$. Thus, $ax^2a^2 = ax(xa^2) = axa$. Taking involution both sides, we get $a^*(ax)^* = a^*$. But, $(ax)^* = ax$, therefore, $a^*ax = a^*$. Hence, $a \in R^{\oplus}$ with weak core index $k = 1$. □

Now, we are ready to prove our first main result of the paper. The following result provides some sufficient conditions for the reverse-order as well as the forward-order laws hold for the weak core inverse.

Theorem 3.5. *Let $a, b \in R^{\boxplus}$ with $ab^2 = b^2a = bab$ and $a^*b^2 = b^2a^* = ba^*b$. Then, $ab \in R^{\boxplus}$ and $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus} = a^{\boxplus}b^{\boxplus}$.*

Proof. The condition $ab^2 = b^2a = bab$ implies that $(ab)^j = a^jb^j = b^ja^j$, $a^jb^{j+1} = b^{j+1}a^j$ for any positive integer $j \geq 2$. As $a^*b^2 = b^2a^*$, taking involution both sides, we get $(b^2)^*a = a(b^2)^*$. Again, by Theorem 3.2, $ab^2 = b^2a$ and $(b^2)^*a = a(b^2)^*$ imply that $a(b^2)^{\boxplus} = (b^2)^{\boxplus}a$. Applying Lemma 2.4, we have $a(b^{\boxplus})^2 = (b^{\boxplus})^2a$. Now, by lemma 3.3 (i), we get

$$\begin{aligned} abb^{\boxplus} &= ab^2(b^{\boxplus})^2 \\ &= b^2a(b^{\boxplus})^2 \\ &= b^2a(b^2)^{\boxplus} \\ &= b^2(b^2)^{\boxplus}a \\ &= b^2(b^{\boxplus})^2a \\ &= bb^{\boxplus}a. \end{aligned}$$

Taking involution both sides, we thus have $a^*bb^{\boxplus} = bb^{\boxplus}a^*$ and therefore, $a^{\boxplus}bb^{\boxplus} = bb^{\boxplus}a^{\boxplus}$, by Theorem 3.2. Next, using definition of weak core inverse, we have

$$\begin{aligned} b^{\boxplus}a &= b(b^{\boxplus})^2a = ba(b^{\boxplus})^2 \\ &= bab(b^{\boxplus})^3 = ab^2(b^{\boxplus})^3 \\ &= a(b^2(b^{\boxplus})^2)b^{\boxplus} = a(bb^{\boxplus})b^{\boxplus} \\ &= a(b(b^{\boxplus})^2) \\ &= ab^{\boxplus}. \end{aligned}$$

Replacing a by a^*, we get $b^{\boxplus}a^* = a^*b^{\boxplus}$. By Theorem 3.2, $b^{\boxplus}a = ab^{\boxplus}$ and $a^*b^{\boxplus} = b^{\boxplus}a^*$ imply that $b^{\boxplus}a^{\boxplus} = a^{\boxplus}b^{\boxplus}$. Subsequently, we have to prove $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}$. In fact, $abb^{\boxplus}a^{\boxplus} = bb^{\boxplus}aa^{\boxplus} = aa^{\boxplus}bb^{\boxplus}$ and thus, $(abb^{\boxplus}a^{\boxplus})^* = abb^{\boxplus}a^{\boxplus}$. Now,

$$\begin{aligned} ((ab)^k)^*abb^{\boxplus}a^{\boxplus} &= (b^k)^*(a^k)^*(aa^{\boxplus}bb^{\boxplus}) \\ &= (b^k)^*((a^k)^*aa^{\boxplus})bb^{\boxplus} \\ &= (b^k)^*(a^k)^*(bb^{\boxplus})^* \\ &= (a^k)^*(b^k)^*(bb^{\boxplus}) \\ &= (a^k)^*(b^k)^* \\ &= ((ab)^k)^*, \end{aligned} \tag{1}$$

$$\begin{aligned} ab(b^{\boxplus}a^{\boxplus})^2 &= abb^{\boxplus}a^{\boxplus}b^{\boxplus}a^{\boxplus} \\ &= bb^{\boxplus}(aa^{\boxplus}a^{\boxplus})b^{\boxplus} \\ &= bb^{\boxplus}a^{\boxplus}b^{\boxplus} \\ &= bb^{\boxplus}b^{\boxplus}a^{\boxplus} \\ &= b^{\boxplus}a^{\boxplus} \end{aligned} \tag{2}$$

and

$$\begin{aligned} b^{\boxplus}a^{\boxplus}(ab)^{k+1} &= b^{\boxplus}(a^{\boxplus}a^{k+1})b^{k+1} \\ &= b^{\boxplus}a^kb^{k+1} \\ &= (b^{\boxplus}b^{k+1})a^k \\ &= b^ka^k \\ &= (ab)^k. \end{aligned} \tag{3}$$

Hence, by the definition of weak core inverse, we finally get $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}$. □

The following corollary directly follows from Theorems 2.3 and 3.3.

Corollary 3.6. *Let $a, b \in R$ with $ab^2 = b^2a = bab$ and $a^*b^2 = b^2a^* = ba^*b$. Then, following conditions are equivalent:*
(i) $ab \in R^{\boxplus}$ and $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}$, with $ind_{wc}(a) \leq k$;
(ii) $b^{\boxplus}a^{\boxplus}ab^{\boxplus}a^{\boxplus} = b^{\boxplus}a^{\boxplus}$, $b^{\boxplus}a^{\boxplus}R = a^kR = a^{k+1}R$ and $a^kR \subseteq (b^{\boxplus}a^{\boxplus})^*R$;
(iii) $b^{\boxplus}a^{\boxplus}ab^{\boxplus}a^{\boxplus} = b^{\boxplus}a^{\boxplus}$, $b^{\boxplus}a^{\boxplus}R = a^kR = a^{k+1}R$ and $^\circ((b^{\boxplus}a^{\boxplus})^*)$=$^\circ((ab)^k)$;
(iv) $b^{\boxplus}a^{\boxplus}ab^{\boxplus}a^{\boxplus} = b^{\boxplus}a^{\boxplus}$, $^\circ((b^{\boxplus}a^{\boxplus})^*)$=$^\circ((ab)^k)$ and $^\circ((ab)^{k+1})$=$^\circ((ab)^k)$=$^\circ((b^{\boxplus}a^{\boxplus}))$.

Similarly, we have another corollary which utilizes Theorem 3.5.

Corollary 3.7. *Let $a, b \in R^{\boxplus}$ with $ab = ba$ and $ab^* = b^*a$. Then, $(ab)^{\boxplus} = a^{\boxplus}b^{\boxplus} = b^{\boxplus}a^{\boxplus}$.*

Proof. Since $ab = ba$, $ab^* = b^*a$, then $ab^2 = b^2a = bab$ and $a^*b^2 = b^2a^* = ba^*b$. Thus, applying Theorem 3.5, we get $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus} = a^{\boxplus}b^{\boxplus}$. □

Next, we have some equivalent conditions for the reverse-order law for the weak core inverse.

Theorem 3.8. *Let $a, b, ab \in R^{\boxplus}$. Then, the followings are equivalent:*
(i) $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}$;
(ii) $(ab)^{\boxplus}a = b^{\boxplus}a^{\boxplus}a$ and $(ab)^{\boxplus} = (ab)^{\boxplus}aa^{\boxplus}$;
(iii) $b(ab)^{\boxplus} = bb^{\boxplus}a^{\boxplus}$ and $(ab)^{\boxplus} = b^{\boxplus}b(ab)^{\boxplus}$.

Proof. $(i) \iff (ii)$
$(ab)^{\boxplus}a = b^{\boxplus}a^{\boxplus}a$ is obvious. Since, $a^{\boxplus}$ is an outer inverse of a, we have $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}aa^{\boxplus} = (ab)^{\boxplus}aa^{\boxplus}$. Conversely, as $(ab)^{\boxplus}a = b^{\boxplus}a^{\boxplus}a$, $(ab)^{\boxplus} = (ab)^{\boxplus}aa^{\boxplus}$ and $a^{\boxplus}$ is outer inverse of a, it follows that $(ab)^{\boxplus} = (ab)^{\boxplus}aa^{\boxplus} = b^{\boxplus}a^{\boxplus}aa^{\boxplus} = b^{\boxplus}a^{\boxplus}$.
$(i) \iff (iii)$
As $b^{\boxplus}$ is an outer inverse of b, we have $(ab)^{\boxplus} = b^{\boxplus}bb^{\boxplus}a^{\boxplus} = b^{\boxplus}b(ab)^{\boxplus}$.
Conversely, if $(ab)^{\boxplus} = b^{\boxplus}b(ab)^{\boxplus}$ and $b(ab)^{\boxplus} = bb^{\boxplus}a^{\boxplus}$, then $(ab)^{\boxplus} = b^{\boxplus}bb^{\boxplus}a^{\boxplus} = b^{\boxplus}a^{\boxplus}$. □

Some more equivalent conditions are obtained next.

Theorem 3.9. *Let $a, b \in R^{\boxplus}$. Then, the followings are equivalent:*
(i) $ab \in R^{\boxplus}$ and $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}$;
(ii) $ab \in R^d$ and $[(ab)^k(b^{\boxplus}a^{\boxplus})^k]^*(ab)^k = (ab)^k$, $(ab)^d(ab)^k(b^{\boxplus}a^{\boxplus})^k = b^{\boxplus}a^{\boxplus}$ for some positive integer k;
(iii) $(ab)^k \in R^{(1,3)}$ and $(ab)^{k+1}(b^{\boxplus}a^{\boxplus})^{k+2}(ab)^{k+1} = (ab)^k$, $(ab)^k[(b^{\boxplus}a^{\boxplus})^{k+2}(ab)^{k+1}]^{k+1}(ab)^k[(ab)^k]^{(1,3)} = b^{\boxplus}a^{\boxplus}$ for some positive integer k.

Proof. $(i) \Rightarrow (ii)$
Let $ab \in R^{\boxplus}$ with $\mathrm{ind}(ab) = k$, then $ab \in R^d$ by Lemma 2.2. From $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}$, it follows that

$$\begin{aligned}\left[(ab)^k(b^{\boxplus}a^{\boxplus})^k\right]^*(ab)^k &= \left[\left[(ab)^{\boxplus}\right]^k\right]^*\left[(ab)^k\right]^*(ab)^k \\ &= \left[(ab)^k\left[(ab)^{\boxplus}\right]^k\right]^*(ab)^k \\ &= \left[(ab)(ab)^{\boxplus}\right]^*(ab)^k \\ &= ab(ab)^{\boxplus}(ab)^k = (ab)^k.\end{aligned}$$

By use of $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}$, Lemma 3.3 and definition of Drazin inverse, we get

$$\begin{aligned}(ab)^d(ab)^k(b^{\boxplus}a^{\boxplus})^k &= (ab)^d(ab)^k\left[(ab)^{\boxplus}\right]^k \\ &= (ab)^d \cdot ab(ab)^{\boxplus} \\ &= (ab)((ab)^{\boxplus})^2 \\ &= (ab)^{\boxplus} \\ &= b^{\boxplus}a^{\boxplus}.\end{aligned}$$

$(ii) \Longrightarrow (i)$ Since $[(ab)^k(b^{\boxplus}a^{\boxplus})^k]^*(ab)^k = [(b^{\boxplus}a^{\boxplus})^k]^*[(ab)^k]^*(ab)^k = (ab)^k$, then $(b^{\boxplus}a^{\boxplus})^k$ is a $\{1,3\}$-inverse of $(ab)^k$ by Lemma 2.6. Again, since $ab \in R^d$, by Theorem 2.5, $ab \in R^{\boxplus}$ and

$$\begin{aligned}(ab)^{\boxplus} &= (ab)^d(ab)^k((ab)^k)^{(1,3)} \\ &= (ab)^d(ab)^k(b^{\boxplus}a^{\boxplus})^k \\ &= b^{\boxplus}a^{\boxplus}.\end{aligned}$$

$(i) \Rightarrow (iii)$

From the assumption that $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}$ and $\mathrm{ind}(ab) = k$, it follows that

$$\begin{aligned}&(ab)^{k+1}(b^{\boxplus}a^{\boxplus})^{k+2}(ab)^{k+1} \\ &\quad= (ab)^{k+1}\left[(ab)^{\boxplus}\right]^{k+2}(ab)^{k+1} \\ &\quad= \left[(ab)^{k+1}\left[(ab)^{\boxplus}\right]^{k+1}\right](ab)^{\boxplus}(ab)^{k+1} \\ &\quad= (ab(ab)^{\boxplus})(ab)^{\boxplus}(ab)^{k+1} \\ &\quad= ab((ab)^{\boxplus})^2(ab)^{k+1} \\ &\quad= (ab)^{\boxplus}(ab)^{k+1} \\ &\quad= (ab)^k.\end{aligned}$$

Now,

$$\begin{aligned}&(ab)^k\left[(b^{\boxplus}a^{\boxplus})^{k+2}(ab)^{k+1}\right]^{k+1}(ab)^k\left[(ab)^k\right]^{(1,3)} \\ &\quad= (ab)^k\left[((ab)^{\boxplus})^{k+2}(ab)^{k+1}\right]^{k+1}(ab)^k\left[(ab)^k\right]^{(1,3)} \\ &\quad= (ab)^k\left[((ab)^{\boxplus})^{k+1}(ab)^{\boxplus}(ab)^{k+1}\right]^{k+1}(ab)^k\left[(ab)^k\right]^{(1,3)} \\ &\quad= (ab)^k\left[((ab)^{\boxplus})^{k+1}(ab)^k\right]^{k+1}(ab)^k\left[(ab)^k\right]^{(1,3)} \\ &\quad= (ab)^k\left[(ab)^d\right]^{k+1}(ab)^k\left[(ab)^k\right]^{(1,3)} \\ &\quad= (ab)^k\left[(ab)^d\right]^k(ab)^d(ab)^k\left[(ab)^k\right]^{(1,3)} \\ &\quad= (ab)(ab)^d(ab)^d(ab)^k\left[(ab)^k\right]^{(1,3)} \\ &\quad= (ab)^d(ab)^k\left[(ab)^k\right]^{(1,3)} \\ &\quad= (ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}.\end{aligned}$$

In view of Theorem 2.5, $(ab)^k \left[(b^{\boxplus}a^{\boxplus})^{k+2}(ab)^{k+1}\right]^{k+1} (ab)^k \left[(ab)^k\right]^{(1,3)} = b^{\boxplus}a^{\boxplus} = (ab)^{\boxplus}$.

$(iii) \Rightarrow (i)$

Since $(ab)^{k+1}(b^{\boxplus}a^{\boxplus})^{k+2}(ab)^{k+1} = (ab)^k$, ab is strongly $\pi-$regular, so 2.7 , we have $ab \in R^d$ and again by 2.5 we have $ab \in R^{\boxplus}$

$$(ab)^d = \left[(b^{\boxplus}a^{\boxplus})^{k+1}(ab)^k\right]$$
$$\left[(ab)^d\right]^{k+1} = \left[(b^{\boxplus}a^{\boxplus})^{k+1}(ab)^k\right]^{k+1}$$

Pre-multiply by $(ab)^k$ both sides, we get

$$(ab)^k \left[(ab)^d\right]^{k+1} = (ab)^k \left[(b^{\boxplus}a^{\boxplus})^{k+1}(ab)^k\right]^{k+1}$$
$$(ab) \left[(ab)^d\right]^{2} = (ab)^k \left[(b^{\boxplus}a^{\boxplus})^{k+1}(ab)^k\right]^{k+1}$$
$$(ab) \left[(ab)^d\right]^{2} = (ab)^k \left[(b^{\boxplus}a^{\boxplus})^{k+2}(ab)^{k+1}\right]^{k+1}$$
$$(ab)^d = (ab)^k \left[(b^{\boxplus}a^{\boxplus})^{k+2}(ab)^{k+1}\right]^{k+1}.$$

Therefore, $(ab)^{\boxplus}$ exists in light of Theorem 2.5. Moreover,

$$\begin{aligned}(ab)^{\boxplus} &= (ab)^d(ab)^k \left[(ab)^k\right]^{(1,3)} \\ &= (ab)^k \left[(b^{\boxplus}a^{\boxplus})^{k+2}(ab)^{k+1}\right]^{k+1} \\ &\quad \times (ab)^k \left[(ab)^k\right]^{(1,3)} \\ &= b^{\boxplus}a^{\boxplus}.\end{aligned}$$

This completes the proof. □

Now, we have the following lemma.

Lemma 3.10. *Let $a, b, ab \in R^{\boxplus}$ with $aba = a^2b$, and $aba^* = a^*ab$. Then,*
(i) $(a^{\boxplus}ab)^{\boxplus} = (ab)^{\boxplus}a$;
(ii) $(ab)^{\boxplus} = (a^{\boxplus}ab)^{\boxplus}a^{\boxplus}$.

Proof. Equality $aba = a^2b$ implies that $(ab)^k = a^kb^k$ for any $k \in N$. Both $aba = a^2b$ and $aba^* = a^*ab$ in conjunction with Theorem 3.2 give that

$$aba^{\boxplus} = a^{\boxplus}ab,$$
$$(ab)^{\boxplus}a = a(ab)^{\boxplus},$$
$$(ab)^{\boxplus}a^{\boxplus} = a^{\boxplus}(ab)^{\boxplus}.$$

Suppose that max{ind(a), ind(b)}=m.
(i) Note that

$$\begin{aligned}a^{\boxplus}ab(ab)^{\boxplus}a &= a^{\boxplus}aab(ab)^{\boxplus}\left[(ab)^{\boxplus}\right]^m \\ &= a^{\boxplus}aa^mb^m\left[(ab)^m\right]^m \\ &= a^mb^m\left[(ab)^{\boxplus}\right]^m \\ &= (ab)^m\left[(ab)^{\boxplus}\right]^m \\ &= ab(ab)^{\boxplus}.\end{aligned}$$

Thus, $\left[a^{\boxplus}ab(ab)^{\boxplus}a\right]^* = a^{\boxplus}ab(ab)^{\boxplus}a$. Now, we observe that

$$\begin{aligned}&((a^{\boxplus}ab)^k)^* \left[a^{\boxplus}ab(ab)^{\boxplus}a\right] \\ &= ((a^{\boxplus}ab)^k)^* \left[a^{\boxplus}ab(ab)^{\boxplus}a\right]^* \\ &= [a^{\boxplus}ab(ab)^{\boxplus}a(a^{\boxplus}ab)^k]^* \\ &= \left[a^{\boxplus}ab(ab)^{\boxplus}(ab)^ka(a^{\boxplus})^k\right]^* \\ &= \left[a^{\boxplus}(ab)^k(a^{\boxplus})^k\right]^* \\ &= \left[(ab)^ka^{\boxplus}a(a^{\boxplus})^k\right]^* \\ &= \left[(ab)^k(a^{\boxplus})^k\right]^* \\ &= \left[(a^{\boxplus}ab)^k\right]^*. \qquad (4)\end{aligned}$$

Further,

$$a^{\boxplus}ab\left[(ab)^{\boxplus}a\right]^2 = ab(ab)^{\boxplus}(ab)^{\boxplus}a = (ab)^{\boxplus}a \qquad (5)$$

and

$$\begin{aligned}(ab)^{\boxplus}a\left[a^{\boxplus}ab\right]^{m+1} &= (ab)^{\boxplus}a(a^{\boxplus})^{m+1}(ab)^{m+1} \\ &= (ab)^{\boxplus}(a^{\boxplus})^m(ab)^{m+1} \\ &= (a^{\boxplus})^m(ab)^{\boxplus}(ab)^{m+1} \qquad (6) \\ &= (a^{\boxplus})^m(ab)^m \\ &= (a^{\boxplus}ab)^m.\end{aligned}$$

In view of 4, 5 and 6, $(a^{\boxplus}ab)^{\boxplus} = (ab)^{\boxplus}a$.
(ii) It suffices to prove $(ab)^{\boxplus} = (ab)^{\boxplus}aa^{\boxplus}$, since we have proved $(a^{\boxplus}ab)^{\boxplus} = (ab)^{\boxplus}a$. That is to say, we have to check three equations for the definition of the pseudo core inverse. Firstly, observe that

$$\begin{aligned}ab(ab)^{\boxplus}aa^{\boxplus} &= aa^{\boxplus}ab(ab)^{\boxplus} \\ &= aa^{\boxplus}(ab)^m\left[(ab)^{\boxplus}\right]^m \\ &= aa^{\boxplus}a^mb^m\left[(ab)^{\boxplus}\right]^m \\ &= a^mb^m\left[(ab)^{\boxplus}\right]^m \\ &= (ab)^m\left[(ab)^{\boxplus}\right]^m \\ &= ab(ab)^{\boxplus}.\end{aligned}$$

So,

$$\left[ab(ab)^{\boxplus}aa^{\boxplus}\right]^* = ab(ab)^{\boxplus}aa^{\boxplus}. \tag{7}$$

Finally, note that

$$\begin{aligned}\left[(ab)^k\right]^* \left[ab(ab)^{\boxplus}aa^{\boxplus}\right]^* &= \left[(ab)^k\right]^* \left[ab(ab)^{\boxplus}aa^{\boxplus}\right]^* \\ &= \left[ab(ab)^{\boxplus}aa^{\boxplus}(ab)^k\right]^* \\ &= \left[ab(ab)^{\boxplus}(ab)^k aa^{\boxplus}\right]^* \\ &= \left[(ab)^k aa^{\boxplus}\right]^* \\ &= \left[aa^{\boxplus}a^k b^k\right]^* \\ &= \left[(a^k b^k)\right]^* \\ &= \left[(ab)^k\right]^*. \end{aligned} \tag{8}$$

Also,

$$ab\left[(ab)^{\boxplus}aa^{\boxplus}\right]^2 = ab(ab)^{\boxplus}(ab)^{\boxplus}aa^{\boxplus} = (ab)^{\boxplus}aa^{\boxplus} \tag{9}$$

and

$$\begin{aligned}(ab)^{\boxplus}aa^{\boxplus}(ab)^{m+1} &= aa^{\boxplus}(ab)^{\boxplus}(ab)^{m+1} \\ &= aa^{\boxplus}(ab)^m \\ &= aa^{\boxplus}a^m b^m \\ &= a^m b^m \\ &= (ab)^m. \end{aligned} \tag{10}$$

In light of 8, 9 and 10, $(ab)^{\boxplus} = (ab)^{\boxplus}aa^{\boxplus}$. This completes the proof. □

Similarly, we can prove the following result.

Lemma 3.11. *Let $a, b, ab \in R^{\boxplus}$ with $bab = b^2a$, and $bab^* = b^*ab$. Then*
(i) $(abb^{\boxplus})^{\boxplus} = b(ab)^{\boxplus}$;
(ii) $(ab)^{\boxplus} = b^{\boxplus}(abb^{\boxplus})^{\boxplus}$.

Now, we present a relation between $(ab)^{\boxplus}$ and $(a^{\boxplus}ab)^{\boxplus}$ with the help of Lemma 3.10 under some conditions.

Theorem 3.12. *Let $a, b \in R^{\boxplus}$ with $aba = a^2b$ and $aba^* = a^*ab$. Then the following are equivalent:*
(i) $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}$;
(ii) $(a^{\boxplus}ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}a$.

Proof. *(i) $\Rightarrow$ (ii)* $a^{\boxplus}ab(b^{\boxplus}a^{\boxplus}a) = a^{\boxplus}ab(ab)^{\boxplus}a = a^{\boxplus}aab(ab)^{\boxplus} = a^{\boxplus}aabb^{\boxplus}a^{\boxplus} = a^{\boxplus}aabb^{\boxplus}a^{\boxplus}aa^{\boxplus} = aa^{\boxplus}a^{\boxplus}aab(ab)^{\boxplus}aa^{\boxplus} = aa^{\boxplus}ab(ab)^{\boxplus}a^{\boxplus}aa^{\boxplus} = aa^{\boxplus}ab(ab)^{\boxplus}aa^{\boxplus}$.
Thus

$$(a^{\boxplus}abb^{\boxplus}a^{\boxplus}a)^* = a^{\boxplus}abb^{\boxplus}a^{\boxplus}a. \tag{11}$$

Note that

$$\begin{aligned}\left[(a^{\boxplus}ab)^k\right]^* a^{\boxplus}abb^{\boxplus}a^{\boxplus}a &= \left[(a^{\boxplus}ab)^k\right]^* \left[a^{\boxplus}abb^{\boxplus}a^{\boxplus}a\right]^* \\ &= \left[a^{\boxplus}abb^{\boxplus}a^{\boxplus}a(a^{\boxplus}ab)^k\right]^* \\ &= \left[a^{\boxplus}abb^{\boxplus}a^{\boxplus}(ab)^k a(a^{\boxplus})^k\right]^* \\ &= \left[a^{\boxplus}(ab)^k aa^{\boxplus}\right]^* \\ &= \left[a^{\boxplus}a(a^{\boxplus})^k(ab)^k\right]^* \\ &= \left[(a^{\boxplus})^k(ab)^k\right]^* \\ &= \left[(a^{\boxplus}ab)^k\right]^*, \end{aligned} \tag{12}$$

$$\begin{aligned}a^{\boxplus}ab\left[b^{\boxplus}a^{\boxplus}a\right]^2 &= a^{\boxplus}ab(ab)^{\boxplus}a(ab)^{\boxplus}a^2 \\ &= a^{\boxplus}(ab)^{\boxplus}a^2 \\ &= a^{\boxplus}a(ab)^{\boxplus}a \\ &= a^{\boxplus}ab^{\boxplus}a^{\boxplus}a \\ &= a^{\boxplus}a(ab)^{\boxplus}aa^{\boxplus}a \\ &= (ab)^{\boxplus}a^{\boxplus}aaa^{\boxplus}a \\ &= (ab)^{\boxplus}aa^{\boxplus}a \\ &= b^{\boxplus}a^{\boxplus}aa^{\boxplus}a \\ &= b^{\boxplus}a^{\boxplus}a. \end{aligned} \tag{13}$$

Suppose max{ind(a), ind(ab)}=k.

$$\begin{aligned}b^{\boxplus}a^{\boxplus}a(a^{\boxplus}ab)^{k+1} &= (ab)^{\boxplus}(ab)(a^{\boxplus}ab)^k \\ &= (ab)^{\boxplus}(ab)^{k+1}(a^{\boxplus})^k \\ &= (ab)^k(a^{\boxplus})^k \\ &= (a^{\boxplus}ab)^k. \end{aligned} \tag{14}$$

In view of 12, 13 and 14, $(a^{\boxplus}ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}a$.
(ii) $\Rightarrow$ (i) From the assumption that $aba = a^2b$ and $aba^* = a^*ab$, it follows that

$$(ab)^{\boxplus} = (a^{\boxplus}ab)^{\boxplus}a^{\boxplus},$$

by Lemma 3.10. Since $(a^{\boxplus}ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}a$, then $(ab)^{\boxplus} = (a^{\boxplus}ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}aa^{\boxplus} = b^{\boxplus}a^{\boxplus}$. □

The section is concluded with the following result, which is proved as Theorem 3.12.

Theorem 3.13. *Let $a, b \in R^{\boxplus}$ with $bab = b^2a$ and $bab^* = b^*ab$. Then the following are equivalent:*
(i) $(ab)^{\boxplus} = b^{\boxplus}a^{\boxplus}$;
(ii) $(abb^{\boxplus})^{\boxplus} = bb^{\boxplus}a^{\boxplus}$.

4 Absorption law for weak core inverse

In this section, we obtain some sufficient conditions under which absorption law hold in case of weak core. But, first we demonstrate that absorption law in general doesn't hold for weak core inverse by an example.

Example 4.1. *Let $R = \mathbb{Z}_{10}$ and let $a = 3, b = 6 \in R$. Then $a^{\boxplus} = 7$ and $b^{\boxplus} = 6$. However, $a^{\boxplus}(a+b)b^{\boxplus} \neq a^{\boxplus}+b^{\boxplus}$.*

Now, we prove the first result of this section which is about some equivalent conditions such that the absorption law holds.

Theorem 4.1. *Let $a, b \in R^{\boxplus}$ with $k = max\{ind(a), ind(b)\}$. Then the following are equivalent:*
(i) $a^{\boxplus}(a+b)b^{\boxplus} = a^{\boxplus} + b^{\boxplus}$;
(ii) $aa^{\boxplus} = bb^{\boxplus}$;
(iii) $a^kR = b^kR$;
(iv) $^{\circ}(a^k)$=$^{\circ}(b^k)$.

Proof. $(i) \Rightarrow (ii)$ Pre multiplying $a^{\boxplus}(a+b)b^{\boxplus} = a^{\boxplus}+b^{\boxplus}$ by $aa^{\boxplus}a$. Then,

$$aa^{\boxplus}bb^{\boxplus} = aa^{\boxplus}. \tag{15}$$

By above condition, we have $a^{\boxplus}bb^{\boxplus} = a^{\boxplus}$.
Again, post-multiplying $a^{\boxplus}(a+b)b^{\boxplus} = a^{\boxplus} + b^{\boxplus}$ by $b^{k+2}(b^{\boxplus})^{k+1}$. Then,

$$\begin{aligned} a^{\boxplus}ab^{\boxplus}b^{k+2}(b^{\boxplus})^{k+1} + a^{\boxplus}b^{k+2}(b^{\boxplus})^{k+1} &= a^{\boxplus}b^{k+2}(b^{\boxplus})^{k+1} + b^{\boxplus}b^{k+2}(b^{\boxplus})^{k+1} \\ a^{\boxplus}abb^{\boxplus} &= bb^{\boxplus}. \end{aligned}$$

Pre-miltiply by $aa^{\boxplus}$, then

$$a^{\boxplus}abb^{\boxplus} = aa^{\boxplus}bb^{\boxplus}. \tag{16}$$

So, we get

$$aa^{\boxplus}bb^{\boxplus} = bb^{\boxplus}. \tag{17}$$

Hence, $aa^{\boxplus} = bb^{\boxplus}$ in view of 15 and 17.
$(ii) \Rightarrow (i)$ Since $aa^{\boxplus} = bb^{\boxplus}$, then

$$a^{\boxplus}bb^{\boxplus} = a^{\boxplus},$$

and

$$\begin{aligned} a^{\boxplus}ab^{\boxplus} &= a^{\boxplus}bb^{\boxplus}b^{\boxplus} \\ &= a^{\boxplus}a^2a^{\boxplus}b^{\boxplus} \\ &= a^{\boxplus}a^{k+1}(a^{\boxplus})^kb^{\boxplus} \\ &= aa^{\boxplus}b^{\boxplus} \\ &= bb^{\boxplus}b^{\boxplus} \\ &= b^{\boxplus}. \end{aligned}$$

Hence, $a^{\boxplus}(a+b)b^{\boxplus} = a^{\boxplus}ab^{\boxplus} + a^{\boxplus}bb^{\boxplus} = a^{\boxplus} + b^{\boxplus}$.
$(ii) \Rightarrow (iii)$ From $aa^{\boxplus} = bb^{\boxplus}$, it follows that $b^k = b^k(b^{\boxplus})^kb^k = bb^{\boxplus}b^k = aa^{\boxplus}b^k = a^k(a^{\boxplus})^kb^k$, i.e. $b^kR \subseteq a^kR$.
Analogously, $a^kR \subseteq b^kR$. Hence, $a^kR = b^kR$.
$(iii) \Rightarrow (iv)$ It is obvious, since $a, b \in R^{\boxplus}$.
$(iv) \Rightarrow (ii)$ Since $^{\circ}(a^k)$=$^{\circ}(b^k)$, then

$$(1 - bb^{\boxplus})a^k = 0, \text{ i.e., } bb^{\boxplus}aa^{\boxplus} = aa^{\boxplus}, \tag{18}$$

and

$$(1 - aa^{\boxplus})b^k = 0, \text{ i.e., } aa^{\boxplus}bb^{\boxplus} = bb^{\boxplus}. \tag{19}$$

Applying an invlution on 19, we obtain

$$bb^{\boxplus}aa^{\boxplus} = bb^{\boxplus}. \tag{20}$$

In view of 18 and 20, $aa^{\boxplus} = bb^{\boxplus}$.

□

Now, we end this section with this result.

Theorem 4.2. *Let $a, b \in R^{\boxplus}$. Then the following are equivalent:*
(i) $a^{\boxplus}(a+b)b^{\boxplus} = a^{\boxplus} + b^{\boxplus}$;
(ii) $Ra^{\boxplus} = R(a^{\boxplus}bb^{\boxplus})$, $b^{\boxplus}R = (a^{\boxplus}ab^{\boxplus})R$;
(iii) $b^{\boxplus}R \subseteq a^{\boxplus}R$ and $Ra^{\boxplus} \subseteq Rb^{\boxplus}$.

Proof. $(i) \Rightarrow (ii)$ It is derived that

$$a^{\boxplus} = a^{\boxplus}bb^{\boxplus} \text{ and } b^{\boxplus} = a^{\boxplus}ab^{\boxplus}$$

by, respectively, pre-multiplying $a^{\boxplus}a$ and post-multiplying $bb^{\boxplus}$ on $a^{\boxplus}(a+b)b^{\boxplus} = a^{\boxplus} + b^{\boxplus}$. Hence, $Ra^{\boxplus} = R(a^{\boxplus}bb^{\boxplus})$, $b^{\boxplus}R = (a^{\boxplus}ab^{\boxplus})R$.
$(ii) \Rightarrow (iii)$ It is clear.
$(iii) \Rightarrow (i)$ $b^{\boxplus}R \subseteq a^{\boxplus}R$ implies that $b^{\boxplus} = a^{\boxplus}x$ for some $x \in R$. Since $a^{\boxplus}$ is an outer inverse of a, then $b^{\boxplus} = a^{\boxplus}x = a^{\boxplus}aa^{\boxplus}x = a^{\boxplus}ab^{\boxplus}$. Similarly, $Ra^{\boxplus} \subseteq Rb^{\boxplus}$ leads to $a^{\boxplus} = a^{\boxplus}bb^{\boxplus}$. Hence, $a^{\boxplus}(a+b)b^{\boxplus} = a^{\boxplus}ab^{\boxplus} + a^{\boxplus}bb^{\boxplus} = b^{\boxplus} + a^{\boxplus}$. □

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Amit Kumar received the Ph.D. degree in Mathematics from the National Institute of Technology Raipur, India, in 2024. (Email: amitdhull513@gmail.com)