RESEARCH ARTICLE

Time Dependent Algorithm using Modified Three-Point Superclass of Block Backward Differentiation Formula for Solving Stiff Ordinary Differential Equations

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Abstract

In this paper, we present a modified three-point superclass of Block Backward Differentiation Formula (BBDF) for the efficient numerical solution of stiff systems of ordinary differential equations (ODEs). The principal enhancement of this work is a structural modification of the classical BBDF that forms a new parameterized superclass of methods, leading to improved stability and reduced error constants compared with the standard three-point BBDFs. The proposed scheme is formulated as a fully implicit block method capable of simultaneously producing three solution approximations within each integration step. A detailed theoretical analysis is conducted to establish the order of accuracy, consistency, zero-stability, and convergence of the method. Stability analysis based on Dahlquist test equation confirms that the scheme is A-stable and suitable for a wide range of stiff ODEs. The nonlinear systems arising from the implicit formulation are efficiently solved using Newton iteration. Numerical experiments on benchmark stiff ODE problems, including systems with rapidly decaying transient components, demonstrate that the modified three-point superclass BBDF achieves higher accuracy and lower computational cost when compared with existing block implicit methods, such as the standard BBDF and diagonally implicit three-point block BDF schemes. Overall, the proposed method provides a robust and computationally efficient alternative for the numerical integration of stiff ODEs, with potential applications in chemical kinetics, control theory, and biological modeling.

Keywords: block backward differentiation formula, stiff ODEs, convergence, consistency, a-stability.

Submitted: 15 January 2026
Accepted: 03 February 2026
Published: 04 March 2026

Vol. 2, **No.** 1, 2026.
10.62762/JAM.2026.213905

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1 Introduction

Stiff ordinary differential equations (ODEs) arise in a wide range of scientific and engineering applications where dynamic processes evolve on multiple temporal scales. Such problems frequently occur in chemical kinetics, control theory, electrical circuit simulation,

Citation
Alhassan, B., Saadu, A. S., & Musa, H. (2026). Time Dependent Algorithm using Modified Three-Point Superclass of Block Backward Differentiation Formula for Solving Stiff Ordinary Differential Equations. *ICCK Journal of Applied Mathematics*, 2(1), 87–110.

atmospheric modeling, combustion processes, and biological systems characterized by rapid transient behavior followed by slow long-term evolution [1–4]. Stiffness manifests when the solution contains components that decay at substantially different rates, causing explicit numerical methods to require prohibitively small step sizes to maintain stability even when the true solution varies smoothly over a relatively large interval. Consequently, stiffness is widely regarded as a numerical property rather than solely an analytical feature of the differential equation [5]. A general initial value problem (IVP) in ordinary differential equation (ODEs) is of the following form:

$$y' = f(x, y), \quad y(a) = y_0, \quad a \leq x \leq b \tag{1}$$

Stiffness is typically associated with Jacobian matrices whose eigenvalues possess large negative real parts. This leads to a severe restriction on the allowable step size for explicit schemes. For example, the classical test equation $y' = \lambda y$ with $\Re(\lambda) < 0$ forces explicit Euler to adopt a time step $h < \frac{2}{|\lambda|}$ to remain stable. The inefficiency of explicit solvers under such conditions has motivated extensive research into the development of robust implicit or semi-implicit schemes capable of handling stiffness efficiently [6–8].

Implicit methods, especially backward differentiation formulas (BDFs), have traditionally been the preferred class of numerical integrators for stiff IVPs due to their favorable stability properties, including A-stability and L-stability [7]. However, classical BDFs face limitations such as order reduction, high computational cost from nonlinear iterations, and restrictions on stability for orders higher than six. These limitations have led to the development of more advanced families of methods such as diagonally implicit Runge-Kutta schemes, singly diagonally implicit Runge-Kutta (SDIRK) methods, Rosenbrock methods, and various block and hybrid multistep integrators [8–15].

Block methods, in particular, have gained substantial attention due to their ability to compute approximations at multiple points within a single integration step, thereby enhancing accuracy, efficiency, and stability. Recent advancements include the construction of generalized and superclass extensions of classical BDFs, variable-step block backward differentiation formulas, and hybrid block approaches that incorporate off-step points to improve the order and stability of the overall numerical scheme [16–28]. These developments seek to simultaneously address key challenges in stiff ODE integration, such as reducing function evaluations, improving stability regions, and minimizing the computational burden associated with implicitness.

Despite these advances, the search for highly stable, accurate, and computationally efficient stiff solvers remains active. Modern applications involving stiff systems such as reactive fluid dynamics, epidemiological modeling, and large-scale multi-physics simulations require numerical methods that balance strong stability properties with low computational overhead. This need has motivated the exploration of new block and hybrid implicit schemes, including multi-point superclass variable step of BDF-type methods, which offer flexibility in achieving high-order accuracy and wide stability regions without excessive cost [21, 23, 24, 29–31].

Implicit linear multistep methods (LMMs) remain one of the most effective classes of numerical integrators for handling stiff ODEs. Consider the general linear k-step formula of the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2}$$

where α_j and β_j are the real constant coefficients associated with the first and second characteristics polynomial, respectively. Here, h denotes the constant step size, $x_n = a + nh$, y_n represents the numerical approximation to the exact solution $y(x_n)$ and $f_n = f(x_n, y_n)$, $n \geq 0$. A special subclass of (2) arises when

$$\beta_0 = \beta_1 = \beta_2 = \cdots = \beta_{k-1} = 0 \quad \text{and} \quad \beta_k \neq 0,$$

Under this condition, (2) reduces to an implicit backward differentiation formula (BDF), expressed as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k f_{n+k}, \tag{3}$$

BDF methods are well known for their favorable stability properties, especially A-stability and stiff decay, making them suitable for the integration of stiff systems. Since practical fixed-step methods for stiff ODEs must be implicit, attention has been directed toward schemes where only the last two coefficients of the second characteristic polynomial are nonzero. Consider therefore the case:

$$\beta_0 = \beta_1 = \beta_2 = \cdots = \beta_{k-2} = 0,$$
$$\beta_{k-1} \neq 0 \quad \text{and} \quad \beta_k \neq 0$$

Under this condition, equation (2) becomes

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k (f_{n+k} - \rho f_{n+k-1}), \tag{4}$$

where the relation $\beta_{k-1} = -\rho\beta_k$ holds. The parameter ρ plays a central role in the stability behavior of the resulting method. For absolute stability, ρ is restricted to the interval $(-1, 1)$, and the absolute stability properties for this interval have been rigorously established in [32]. While the formulas (3) and (4) compute only one solution value per integration step, recent research efforts have emphasized the development of implicit block methods capable of generating multiple solution values simultaneously. Such block methods are particularly valuable for stiff ODEs, as they enhance stability, minimize computational effort, and improve accuracy by using multi-point approximations within each step. Although several algorithms have been proposed for the block integration of stiff systems, there remains significant interest in constructing more efficient and highly stable implicit block methods.

This research contributes to the ongoing development of block methods by constructing a modified three-point block scheme of the form (4), designed to approximate the solution sequence $y(x_n)$ and generate three numerical solution values at each integration step. The proposed method incorporates the conventional three-point block backward differentiation formula (BBDF) developed in [16] as a special case. Greater flexibility is achieved through the introduction of a free parameter ρ, which generates a family of methods with adjustable stability and accuracy properties.

The free parameter ρ plays a central role in the BBDF formulation. It acts as a stability-control parameter that regulates the contribution of selected function evaluations within the block structure. By appropriately choosing ρ, the stability polynomial can be reshaped to enlarge the region of absolute stability and enhance A-stability characteristics without sacrificing the order of accuracy. Moreover, ρ allows the method to recover the classical three-point BBDF as a special case for a particular choice of its value, while other selections of ρ yield improved damping of stiff components and reduced error constants. The principal structural modification introduced in this study is the introduction the term $+\rho\beta_{k,i} f_{n+k-2}$ into the existing classical BBDF formulation [16], which significantly alters the stability behavior and leads to improved performance for highly stiff systems.

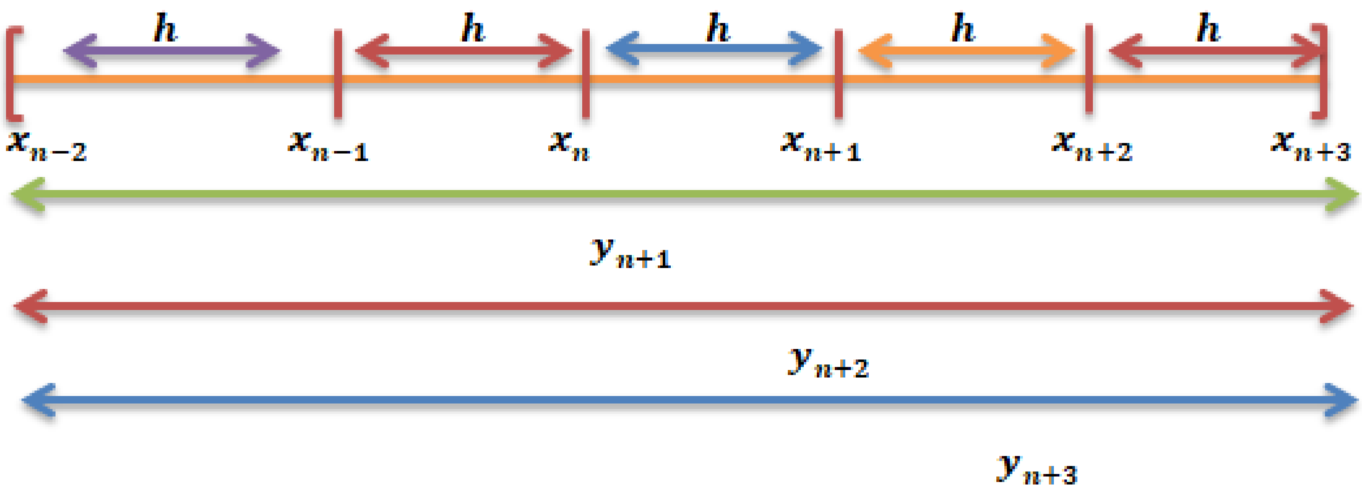

Figure 1. Interpolation points involved in the M3SBBDF method.

2 Mathematical Formulation of the Method

Consider the conventional 3-point block backward differentiation formula (BBDF) developed in [16], which is expressed as:

$$\sum_{j=0}^{5} \alpha_j y_{n+j-2} = h\beta_{k,i} f_{n+k}, \quad k = i = 1, 2, 3 \tag{5}$$

where $\beta_{0,i} = \beta_{1,i} = \beta_{2,i} = \cdots = \beta_{k-1,i} = 0$ and $\beta_{k,i} \neq 0$; $k = i = 1$, $k = i = 2$, and $k = i = 3$ correspond respectively to the first, second and third points of the 3-point BBDF method. Figure 1 shows the interpolation points involved in the proposed M3SBBDF method, illustrating how the three solution values are computed simultaneously within each integration step.

In this paper, we extend this framework by relaxing the restriction on the β-coefficients and introducing additional flexibility through two non-zero trailing coefficients. Specifically, we consider the case

$$\beta_{0,i} = \beta_{1,i} = \beta_{2,i} = \cdots = \beta_{k-1,i} = 0, \quad \beta_{k-2,i} \neq 0 \quad \text{and} \quad \beta_{k,i} \neq 0$$

and on this basis, we define a superclass of the method (5). This superclass formulation generalizes the conventional 3-point BBDF by incorporating an additional degree of freedom, enabling the development of more stable, highly accurate, and computationally efficient block implicit methods for stiff ODEs. The conventional BBDF in (5) appears as a special case when $\beta_{k-2,i} = 0$. This generalization allows the generation of families of methods parameterized by the ratio

$$\rho = -\frac{\beta_{k-2,i}}{\beta_{k,i}}$$

providing enhanced stability properties, including wider regions of absolute stability when the free parameter ρ is defined in the interval $(-1, 1)$ as in [32].

Definition 1. *The fully implicit modified three-point super class of block backward differentiation formula (M3SBBDF) is defined by*

$$\sum_{j=0}^{5} \alpha_j y_{n+j-2} = h\beta_{k,i}(f_{n+k} + \rho f_{n+k-2}), \quad k = i = 1, 2, 3 \tag{6}$$

With three back values at x_{n-2}, x_{n-1} and x_n, we aim to derive a formula to simultaneously compute the approximate values y_{n+1}, y_{n+2} and y_{n+3} at x_{n+1}, x_{n+2} and x_{n+3} respectively.

Definition 2. *The Linear operator L_i associated with the first, second and third-points of the M3SBBDF method is defined as:*

$$L_i[y(x_n), h] : \alpha_{0,i} y_{n-2} + \alpha_{1,i} y_{n-1} + \alpha_{2,i} y_n + \alpha_{3,i} y_{n+1} + \alpha_{4,i} y_{n+2} + \alpha_{5,i} y_{n+3} - h\beta_{k,i} f_{n+k} - h\rho\beta_{k,i} f_{n+k-2} = 0, \quad k = i = 1, 2, 3 \tag{7}$$

To obtain the coefficient of the first point y_{n+1}. We substitute $k = i = 1$ in (7) and define the linear operator L_1 as:

$$L_1[y(x_n), h] : \alpha_{0,1} y_{n-2} + \alpha_{1,1} y_{n-1} + \alpha_{2,1} y_n + \alpha_{3,1} y_{n+1} + \alpha_{4,1} y_{n+2} + \alpha_{5,1} y_{n+3} - h\beta_{1,1} f_{n+1} - h\rho\beta_{1,1} f_{n-1} = 0 \tag{8}$$

Expanding equation (8) as a Taylor's series about any point x_n and grouping the like terms yields:

$$C_{0,1} y(x_n) + C_{1,1} h y'(x_n) + C_{2,1} h^2 y''(x_n) + C_{3,1} h^3 y'''(x_n) + \cdots = 0. \tag{9}$$

where

$$
\begin{aligned}
C_{0,1} &= \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} + \alpha_{4,1} + \alpha_{5,1} = 0 \\
C_{1,1} &= -2\alpha_{0,1} - \alpha_{1,1} + \alpha_{3,1} + 2\alpha_{4,1} + 3\alpha_{5,1} - \beta_{1,1}(1+\rho) = 0 \\
C_{2,1} &= 2\alpha_{0,1} + \frac{1}{2}\alpha_{1,1} + \frac{1}{2}\alpha_{3,1} + 2\alpha_{4,1} + \frac{9}{2}\alpha_{5,1} - \beta_{1,1}(1-\rho) = 0 \\
C_{3,1} &= -\frac{4}{3}\alpha_{0,1} - \frac{1}{6}\alpha_{1,1} + \frac{1}{6}\alpha_{3,1} + \frac{4}{3}\alpha_{4,1} + \frac{9}{2}\alpha_{5,1} - \beta_{1,1}\left(\frac{1}{2} + \frac{1}{2}\rho\right) = 0 \\
C_{4,1} &= \frac{2}{3}\alpha_{0,1} + \frac{1}{24}\alpha_{1,1} + \frac{1}{24}\alpha_{3,1} + \frac{2}{3}\alpha_{4,1} + \frac{27}{8}\alpha_{5,1} - \beta_{1,1}\left(\frac{1}{6} - \frac{1}{6}\rho\right) = 0 \\
C_{5,1} &= -\frac{4}{15}\alpha_{0,1} - \frac{1}{120}\alpha_{1,1} + \frac{1}{120}\alpha_{3,1} + \frac{4}{15}\alpha_{4,1} + \frac{81}{40}\alpha_{5,1} - \beta_{1,1}\left(\frac{1}{24} + \frac{1}{24}\rho\right) = 0
\end{aligned}
\tag{10}
$$

When deriving the first point y_{n+1}, the coefficient $\alpha_{3,1}$ is normalized to 1. Solving the system of linear equation in (10) for the values of $\alpha_{j,i}$ and $\beta_{j,i}$ gives:

Table 1. The coefficient of the first point for M3SBBDF.

$\alpha_{0,1}$	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{4,1}$	$\alpha_{5,1}$	$\beta_{1,1}$
$\frac{1}{10}\cdot\frac{6\rho+1}{3\rho-1}$	$\frac{1}{4}\cdot\frac{-3+13\rho}{3\rho-1}$	$-\frac{3(-1+2\rho)}{3\rho-1}$	1	$-\frac{1}{2}\cdot\frac{2\rho+3}{3\rho-1}$	$\frac{3}{20}\cdot\frac{1+\rho}{3\rho-1}$	$-\frac{3}{3\rho-1}$

The coefficients obtained in Table 1 are then substituted into equation (8) to derive the formula for the first point. Substituting these values in equation (8), we obtain:

$$
\begin{aligned}
y_{n+1} = &-\frac{1}{10}\frac{6\rho+1}{3\rho-1}y_n - \frac{1}{4}\frac{-3+13\rho}{3\rho-1}y_{n-1} + \frac{3(-1+2\rho)}{3\rho-1}y_n + \frac{12\rho+3}{23\rho-1}y_{n+2} - \frac{3}{20}\frac{1+\rho}{3\rho-1}y_n + \frac{3}{3\rho-1}hf_{n+1} \\
&- \frac{3\rho}{3\rho-1}hf_{n-1}
\end{aligned}
\tag{11}
$$

The derivation of the second point y_{n+2} follows a similar procedure to that of the first point formula. The linear operator L_2 corresponding to the second point is defined as:

$$
L_2[y(x_n), h] : \alpha_{0,2}y_{n-2} + \alpha_{1,2}y_{n-1} + \alpha_{2,2}y_n + \alpha_{3,2}y_{n+1} + \alpha_{4,2}y_{n+2} + \alpha_{5,2}y_{n+3} - h\beta_{2,2}f_{n+2} - h\rho\beta_{2,2}f_n = 0 \tag{12}
$$

The corresponding approximate relationship for equation (12) can be expressed as:

$$
\begin{aligned}
&\alpha_{0,2}y(x_n - 2h) + \alpha_{1,2}y(x_n - h) + \alpha_{2,2}y(x_n) + \alpha_{3,2}y(x_n + h) + \alpha_{4,2}y(x_n + 2h) + \alpha_{5,2}y(x_n + 3h) \\
&- h\beta_{2,2}f(x_n + 2h) - h\rho\beta_{2,2}f(x_n) = 0
\end{aligned}
\tag{13}
$$

Expanding equation (13) as a Taylor's series around x_n and grouping the like terms yields:

$$
C_{0,2}y(x_n) + C_{1,2}hy'(x_n) + C_{2,2}h^2y''(x_n) + C_{3,2}h^3y''''(x_n) + \cdots = 0. \tag{14}
$$

where,

$$
\begin{aligned}
C_{0,2} &= \alpha_{0,2} + \alpha_{1,2} + \alpha_{2,2} + \alpha_{3,2} + \alpha_{4,2} + \alpha_{5,2} = 0 \\
C_{1,2} &= -2\alpha_{0,2} - \alpha_{1,2} + \alpha_{3,2} + 2\alpha_{4,2} + 3\alpha_{5,2} - \beta_{2,2}(1+\rho) = 0 \\
C_{2,2} &= 2\alpha_{0,2} + \frac{1}{2}\alpha_{1,2} + \frac{1}{2}\alpha_{3,2} + 2\alpha_{4,2} + \frac{9}{2}\alpha_{5,2} - 2\beta_{2,2} = 0 \\
C_{3,2} &= -\frac{4}{3}\alpha_{0,2} - \frac{1}{6}\alpha_{1,2} + \frac{1}{6}\alpha_{3,2} + \frac{4}{3}\alpha_{4,2} + \frac{9}{2}\alpha_{5,2} - 2\beta_{2,2} = 0 \\
C_{4,2} &= \frac{2}{3}\alpha_{0,2} + \frac{1}{24}\alpha_{1,2} + \frac{1}{24}\alpha_{3,2} + \frac{27}{8}\alpha_{5,2} - \frac{4}{3}\beta_{2,2} = 0 \\
C_{5,2} &= -\frac{4}{15}\alpha_{0,2} - \frac{1}{120}\alpha_{1,2} + \frac{1}{120}\alpha_{3,2} + \frac{4}{15}\alpha_{4,2} + \frac{81}{40}\alpha_{5,2} - \frac{2}{3}\beta_{2,2} = 0
\end{aligned}
\tag{15}
$$

In deriving the second point y_{n+2} the coefficient $\alpha_{4,2}$ is normalized to 1. Solving the linear systems of simultaneous equations in (15) for the values of $\alpha_{j,i}$ and $\beta_{j,i}$ gives: The coefficients presented in Table 2

Table 2. The coefficient of the second point for M3SBBDF.

$\alpha_{0,2}$	$\alpha_{1,2}$	$\alpha_{2,2}$	$\alpha_{3,2}$	$\alpha_{4,2}$	$\alpha_{5,2}$	$\beta_{2,2}$
$-\frac{3}{5}\frac{1+\rho}{3\rho-13}$	$\frac{2(3\rho+2)}{3\rho-13}$	$\frac{4(\rho-3)}{3\rho-13}$	$\frac{12(\rho-2)}{3\rho-13}$	1	$-\frac{2}{5}\frac{\rho+6}{3\rho-13}$	$\frac{12}{3\rho-13}$

are then substituted into equation (12) to obtain the formula for the second point. Substituting these values in equation (12), we obtain:

$$
\begin{aligned}
y_{n+2} = \frac{3}{5}\frac{1+\rho}{3\rho-13}y_{n-2} - \frac{2(3\rho+2)}{3\rho-13}y_{n-1} - \frac{4(\rho-3)}{3\rho-13}y_n + \frac{12(\rho-2)}{3\rho-13}y_{n+1} + \frac{2}{5}\frac{\rho+6}{3\rho-13}y_{n+3} \\
+ \frac{12h}{3\rho-13}f_{n+2} - \frac{12h\rho}{3\rho-13}f_n
\end{aligned}
\tag{16}
$$

Similarly, the third point y_{n+3} is obtained by defining the operator L_3 as:

$$
L_3[y(x_n),h] : \alpha_{0,3}y_{n-2} + \alpha_{1,3}y_{n-1} + \alpha_{2,3}y_n + \alpha_{3,3}y_{n+1} + \alpha_{4,3}y_{n+2} + \alpha_{5,3}y_{n+3} - h\beta_{3,3}f_{n+3} - h\rho\beta_{3,3}f_{n+1} = 0 \tag{17}
$$

To obtain:

$$
\begin{aligned}
y_{n+3} = -\frac{2(\rho+6)}{3\rho-137}y_{n-2} + \frac{15(\rho+5)}{\rho-137}y_{n-1} - \frac{20(3\rho+10)}{3\rho-137}y_n + \frac{20(\rho+15)}{3\rho-137}y_{n+1} + \frac{30(\rho-10)}{3\rho-137}y_{n+2} \\
- \frac{60h}{3\rho-137}f_{n+3} - \frac{60h\rho}{3\rho-137}f_{n+1}
\end{aligned}
\tag{18}
$$

By combining equations (11), (16), and (18), we obtain a fully implicit modified 3-point super class of block backward differentiation formula (M3SBBDF) as:

$$
\begin{aligned}
y_{n+1} &= -\frac{1}{10}\frac{6\rho+1}{3\rho-1}y_{n-2} - \frac{1}{4}\frac{-3+13\rho}{3\rho-1}y_{n-1} + \frac{3(-1+2\rho)}{3\rho-1}y_n + \frac{1}{2}\frac{2\rho+3}{3\rho-1}y_{n+2} - \frac{3}{20}\frac{1+\rho}{3\rho-1}y_{n+3} \\
&\quad - \frac{3}{3\rho-2}hf_{n+1} - \frac{3\rho}{3\rho-2}f_{n-1} \\
y_{n+2} &= \frac{3}{5}\frac{1+\rho}{3\rho-13}y_{n-2} - \frac{2(3\rho+2)}{3\rho-13}y_{n-1} - \frac{4(\rho-3)}{3\rho-13}y_n + \frac{12(\rho-2)}{3\rho-13}y_{n+1} + \frac{2}{5}\frac{\rho+6}{3\rho-13}y_{n+3} \\
&\quad - \frac{12h}{3\rho-13}f_{n+2} - \frac{12h\rho}{3\rho-13}f_n \\
y_{n+3} &= -\frac{2(\rho+6)}{3\rho-137}y_{n-2} + \frac{15(\rho+5)}{\rho-137}y_{n-1} - \frac{20(3\rho+10)}{3\rho-137}y_n + \frac{20(\rho+15)}{3\rho-137}y_{n+1} + \frac{30(\rho-10)}{3\rho-137}y_{n+2} - \frac{60h}{3\rho-137}f_{n+3} \\
&\quad - \frac{60h\rho}{3\rho-137}f_{n+1}
\end{aligned}
\tag{19}
$$

The matrix formulation of (19) is compatible represented as:

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix}
&=
\begin{bmatrix}
\frac{1}{10}\frac{6\rho+1}{3\rho-137} & -\frac{1}{4}\frac{-3+13\rho}{3\rho-1} & \frac{3(-1+2\rho)}{3\rho-1} \\
\frac{3}{5}\frac{1+\rho}{3\rho-13} & -\frac{2(3\rho+2)}{3\rho-13} & -\frac{4(\rho-3)}{3\rho-13} \\
-\frac{2(\rho+6)}{3\rho-137} & \frac{15(\rho+5)}{3\rho-137} & -\frac{20(3\rho+10)}{3\rho-137}
\end{bmatrix}
\begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}
+
\begin{bmatrix}
0 & \frac{12\rho+3}{3\rho-1} & -\frac{3}{20}\frac{1+\rho}{3\rho-1} \\
\frac{12(\rho-2)}{3\rho-13} & 0 & \frac{2}{5}\frac{\rho}{3\rho-13} \\
\frac{20(\rho+15)}{3\rho-137} & \frac{30(\rho-10)}{3\rho-137} & 0
\end{bmatrix}
\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} \\
&+ h
\begin{bmatrix}
\frac{3}{3\rho-2} & 0 & 0 \\
0 & \frac{12}{3\rho-13} & 0 \\
-\frac{60\rho}{3\rho-137} & 0 & -\frac{60\rho}{3\rho-137}
\end{bmatrix}
\begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}
+ h
\begin{bmatrix}
0 & \frac{3\rho}{3\rho-1} & 0 \\
0 & 0 & -\frac{12\rho}{3\rho-13} \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}
\end{aligned}
\tag{20}
$$

For the purpose of maintaining absolute stability of the method (19), we restrict the value of the free parameter ρ to be within the interval $-1 < \rho < 1$ as in [1, 2, 32, 33]. It's worth noting that the fully implicit modified three-point superclass of block backward differentiation formula (19) reduces to the fully implicit almost A-stable fifth-order 3-point block backward differentiation formula (3BBDF) for solving Stiff ODEs, as developed by [16], when the value of ρ is set to zero.

3 Order, Error Constant and Consistency of the Method

To determine the order and error constant of the M3SBBDF method corresponding to the formulae in (19), the formulae can also be rewritten as:

$$
\begin{aligned}
&\frac{1}{10}\frac{6\rho+1}{3\rho-1}y_{n-2}+\frac{1}{4}\frac{-3+13\rho}{3\rho-1}y_{n-1}-\frac{3(-1+2\rho)}{3\rho-1}y_n+y_{n+1}-\frac{1}{2}\frac{2\rho+3}{3\rho-1}y_{n+2}+\frac{3}{20}\frac{1+\rho}{3\rho-1}y_{n+3}\\
&=-\frac{3\rho}{3\rho-1}hf_{n-1}-\frac{3}{3\rho-1}hf_{n+1}\\
&-\frac{3}{5}\frac{1+\rho}{3\rho-13}y_{n-2}+\frac{2(3\rho+2)}{3\rho-13}y_{n-1}+\frac{4(3\rho-3)}{3\rho-13}y_n-\frac{12(2\rho-2)}{3\rho-13}y_{n+1}+y_{n+2}-\frac{2}{5}\frac{\rho+6}{3\rho-13}y_{n+3}\\
&=-\frac{12h\rho}{3\rho-13}f_n-\frac{12h}{3\rho-13}f_{n+2}\\
&\frac{2(\rho+6)}{3\rho-137}y_{n-2}-\frac{15(\rho+5)}{\rho-137}y_{n-1}+\frac{20(3\rho+10)}{3\rho-137}y_n-\frac{20(\rho+15)}{3\rho-137}y_{n+1}-\frac{30(\rho-10)}{3\rho-137}y_{n+2}+y_{n+3}\\
&=-\frac{60h\rho}{3\rho-137}f_{n+1}-\frac{60h}{3\rho-137}f_{n+3}
\end{aligned}
\tag{21}
$$

The formulae in (21) can also be expressed as:

$$
\begin{aligned}
&\begin{bmatrix}\frac{1}{10}\frac{6\rho+1}{3\rho-1} & \frac{1}{4}\frac{-3+13\rho}{3\rho-1} & -\frac{3(-1+2\rho)}{3\rho-1}\\ -\frac{3}{5}\frac{1+\rho}{5\rho-13} & \frac{2(3\rho+2)}{3\rho-13} & \frac{4(3\rho-3)}{3\rho-13}\\ \frac{2(\rho+6)}{3\rho-137} & -\frac{15(\rho+5)}{\rho-137} & \frac{20(3\rho+10)}{3\rho-137}\end{bmatrix}\begin{bmatrix}y_{n-2}\\ y_{n-1}\\ y_n\end{bmatrix}+\begin{bmatrix}1 & -\frac{1}{2}\frac{12\rho+3}{3\rho-1} & \frac{3}{20}\frac{1+\rho}{3\rho-1}\\ -\frac{12(2\rho-2)}{3\rho-13} & 1 & -\frac{2}{5}\frac{\rho+6}{3\rho-13}\\ -\frac{20(\rho+15)}{3\rho-137} & -\frac{30(\rho-10)}{3\rho-137} & 1\end{bmatrix}\begin{bmatrix}y_{n+1}\\ y_{n+2}\\ y_{n+3}\end{bmatrix}\\
&=h\begin{bmatrix}0 & -\frac{3\rho}{3\rho-2} & 0\\ 0 & 0 & -\frac{12\rho}{3\rho-13}\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}f_{n-2}\\ f_{n-1}\\ f_n\end{bmatrix}+h\begin{bmatrix}-\frac{3}{3\rho-2} & 0 & 0\\ 0 & -\frac{12}{3\rho-13} & 0\\ -\frac{60}{3\rho-137} & 0 & -\frac{60}{3\rho-137}\end{bmatrix}\begin{bmatrix}f_{n+1}\\ f_{n+2}\\ f_{n+3}\end{bmatrix}
\end{aligned}
\tag{22}
$$

The matrix equation (22) can also be defined and expressed as:

$$
\sum_{j=0}^{1} C_j^* Y_{m+j-1} = h\sum_{j=0}^{1} B_j^* F_{m+j-1}
\tag{23}
$$

where C_0^*, C_1^*, D_0^* and D_1^* are block square matrices defined by

$$
C_0^*=\begin{bmatrix}\frac{1}{10}\frac{6\rho+1}{3\rho-1} & \frac{1}{4}\frac{-3+13\rho}{3\rho-1} & -\frac{3(-1+2\rho)}{3\rho-1}\\ -\frac{3}{5}\frac{1+\rho}{3\rho-13} & \frac{2(3\rho+2)}{3\rho-13} & \frac{4(\rho-3)}{3\rho-13}\\ \frac{2(\rho+6)}{3\rho-137} & -\frac{15(\rho+5)}{\rho-137} & \frac{20(3\rho+10)}{3\rho-137}\end{bmatrix},\quad C_1^*=\begin{bmatrix}1 & -\frac{1}{2}\frac{2\rho+3}{3\rho-1} & \frac{3}{20}\frac{1+\rho}{3\rho-1}\\ -\frac{12(\rho-2)}{3\rho-13} & 1 & -\frac{2}{5}\frac{\rho+6}{3\rho-13}\\ -\frac{20(\rho+15)}{3\rho-137} & -\frac{30(\rho-10)}{3\rho-137} & 1\end{bmatrix}
$$

$$
B_0^*=\begin{bmatrix}0 & -\frac{3\rho}{3\rho-1} & 0\\ 0 & 0 & -\frac{12\rho}{3\rho-13}\\ 0 & 0 & 0\end{bmatrix},\quad B_1^*=\begin{bmatrix}-\frac{3}{3\rho-1} & 0 & 0\\ 0 & -\frac{12}{3\rho-13} & 0\\ -\frac{60}{3\rho-137} & 0 & -\frac{60}{3\rho-137}\end{bmatrix}
$$

and Y_{m-1}, Y_m, F_{m-1} and F_m are column vectors defined by

$$Y_{m-1}=\begin{bmatrix}y_{n-2}\\y_{n-1}\\y_n\end{bmatrix},\quad Y_m=\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix},\quad F_{m-1}=\begin{bmatrix}f_{n-2}\\f_{n-1}\\f_n\end{bmatrix},\quad F_m=\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix}.$$

It should be noted that the block matrices

$$C_0^*=(C_0\ C_1\ C_2),\quad C_1^*=(C_3\ C_4\ C_5),\quad B_0^*=(B_0\ B_1\ B_2)\quad \text{and}\quad B_1^*=(B_3\ B_4\ B_5).$$

where,

$$C_0=\begin{bmatrix}\frac{1}{10}\frac{6\rho+1}{3\rho-13}\\-\frac{3}{5}\frac{1+\rho}{3\rho-13}\\\frac{2(\rho+6)}{3\rho-137}\end{bmatrix},\quad C_1=\begin{bmatrix}\frac{1}{4}\frac{3\rho-1}{3\rho-13}\\\frac{2(3\rho+2)}{3\rho-13}\\-\frac{15(\rho+5)}{\rho-137}\end{bmatrix},\quad C_2=\begin{bmatrix}-\frac{3(-1+2\rho)}{3\rho-1}\\\frac{4(\rho-3)}{3\rho-13}\\\frac{20(3\rho+10)}{3\rho-137}\end{bmatrix},\quad C_3=\begin{bmatrix}1\\-\frac{12(\rho-2)}{3\rho-13}\\-\frac{20(\rho+15)}{3\rho-137}\end{bmatrix}$$

$$C_4=\begin{bmatrix}-\frac{1}{2}\frac{2\rho+3}{3\rho-1}\\1\\-\frac{30(\rho-10)}{3\rho-137}\end{bmatrix},\quad C_5=\begin{bmatrix}\frac{3}{20}\frac{1+\rho}{3\rho-1}\\-\frac{2}{5}\frac{\rho+6}{3\rho-13}\\1\end{bmatrix},\quad B_0=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad B_1=\begin{bmatrix}-\frac{3\rho}{3\rho-2}\\0\\0\end{bmatrix}$$

$$B_2=\begin{bmatrix}0\\-\frac{12\rho}{3\rho-13}\\0\end{bmatrix},\quad B_3=\begin{bmatrix}-\frac{3}{3\rho-1}\\0\\-\frac{60\rho}{3\rho-137}\end{bmatrix},\quad B_4=\begin{bmatrix}0\\-\frac{12}{3\rho-13}\\0\end{bmatrix},\quad B_5=\begin{bmatrix}0\\0\\-\frac{60}{3\rho-137}\end{bmatrix}$$

Definition 3. *The order of the block method (19) and its associated linear difference operator given by:*

$$L[y(x);h]=\sum_{j=0}^{k-5}[C_jy(x+jh)]-h\sum_{j=0}^{5}[B_jy'(x+jh)] \tag{24}$$

is a unique P such that $E_0=E_1=E_2=\cdots=E_p=0$ and $E_{p+1}\neq 0$, where E_{p+1} is a column vector which is called the error constant. The constant column vectors $E_0, E_1, \ldots, E_q$ are defined by:

$$\begin{aligned}E_0&=C_0+C_1+\cdots+C_k\\E_1&=C_1+2C_2+\cdots+kC_k-(B_0+B+\cdots+B_k)\\&\vdots\\E_q&=\frac{1}{q!}(C_1+2^qC_2+\cdots+k^qC_k)-\frac{1}{(q-1)!}(B_1+2^{q-1}B_2+\cdots+(k)^{q-1}B_k),\quad q=2,3,\ldots\end{aligned} \tag{25}$$

where

$$E_0=\sum_{j=0}^{5}C_j=C_0+C_1+C_2+C_3+C_4+C_5=\begin{bmatrix}\frac{1}{10}\frac{6\rho+1}{3\rho-1}\\-\frac{3}{5}\frac{1+\rho}{3\rho-13}\\\frac{2(\rho+6)}{3\rho-137}\end{bmatrix}+\begin{bmatrix}\frac{1}{4}\frac{-3+13\rho}{3\rho-1}\\\frac{2(3\rho+2)}{3\rho-13}\\\frac{15(\rho+5)}{\rho-137}\end{bmatrix}+\begin{bmatrix}-\frac{3(-1+2\rho)}{3\rho-1}\\\frac{4(\rho-3)}{3\rho-13}\\\frac{20(3\rho+10)}{3\rho-137}\end{bmatrix}$$

$$+\begin{bmatrix}1\\-\frac{12(\rho-2)}{3\rho-13}\\-\frac{20(\rho+15)}{3\rho-137}\end{bmatrix}+\begin{bmatrix}-\frac{1}{2}\frac{2\rho+3}{3\rho-1}\\1\\-\frac{30(\rho-10)}{3\rho-137}\end{bmatrix}+\begin{bmatrix}\frac{3}{20}\frac{1+\rho}{3\rho-1}\\-\frac{2(\rho+6)}{5\rho-13}\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$E_1 = \sum_{j=0}^{5} jC_j - \sum_{j=0}^{5} B_j = ((0)C_0 + (1)C_1 + (2)C_2 + (3)C_3 + (4)C_4 + (5)C_5) - (B_0 + B_1 + B_2 + B_3 + B_4 + B_5)$$

$$= \left[(0) \begin{bmatrix} \frac{1}{10}\frac{6\rho+1}{3\rho-1} \\ -\frac{3}{5}\frac{1+\rho}{3\rho-13} \\ \frac{2(\rho+6)}{3\rho-137} \end{bmatrix} + (1) \begin{bmatrix} \frac{1}{4}\frac{-3+13\rho}{3\rho-1} \\ \frac{2(3\rho+2)}{3\rho-13} \\ -\frac{15(\rho+5)}{\rho-137} \end{bmatrix} + (2) \begin{bmatrix} -\frac{3(-1+2\rho)}{3\rho-1} \\ \frac{4(\rho-3)}{3\rho-13} \\ \frac{20(3\rho+10)}{3\rho-137} \end{bmatrix} + (3) \begin{bmatrix} 1 \\ -\frac{12(\rho-2)}{3\rho-13} \\ -\frac{20(\rho+15)}{3\rho-137} \end{bmatrix} + (4) \begin{bmatrix} -\frac{1}{2}\frac{2\rho+3}{3\rho-1} \\ 1 \\ -\frac{30(\rho-10)}{3\rho-137} \end{bmatrix} \right.$$

$$\left. +(5) \begin{bmatrix} \frac{3}{20}\frac{1+\rho}{3\rho-1} \\ -\frac{2}{5}\frac{\rho+6}{3\rho-13} \\ 1 \end{bmatrix} \right] - \left[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{3\rho}{3\rho-1} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{12\rho}{3\rho-13} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{3}{3\rho-2} \\ 0 \\ -\frac{60\rho}{3\rho-137} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{12}{3\rho-13} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{60}{3\rho-137} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E_2 = \frac{1}{2!}\sum_{j=0}^{5} j^2 C_j - \frac{1}{1!}\sum_{j=0}^{5} jB_j = \frac{1}{2!}\left((0)^2C_0 + (1)^2C_1 + (2)^2C_2 + (3)^2C_3 + (4)^2C_4 + (5)^2C_5\right)$$

$$-\frac{1}{1!}\left((0)^1B_0 + (1)^1B_1 + (2)^1B_2 + (3)^1B_3 + (4)^1B_4 + (5)^1B_5\right)$$

$$= \frac{1}{2!}\left[(0)^2 \begin{bmatrix} \frac{1}{10}\frac{6\rho+1}{3\rho-1} \\ -\frac{3}{5}\frac{1+\rho}{3\rho-13} \\ \frac{2(\rho+6)}{3\rho-137} \end{bmatrix} + (1)^2 \begin{bmatrix} \frac{1}{4}\frac{-3+13\rho}{3\rho-1} \\ \frac{2(3\rho+2)}{3\rho-13} \\ -\frac{15(\rho+5)}{\rho-137} \end{bmatrix} + (2)^2 \begin{bmatrix} -\frac{3(-1+2\rho)}{3\rho-1} \\ \frac{4(\rho-3)}{3\rho-13} \\ \frac{20(3\rho+10)}{3\rho-137} \end{bmatrix} + (3)^2 \begin{bmatrix} 1 \\ -\frac{12(\rho-2)}{3\rho-13} \\ -\frac{20(\rho+15)}{3\rho-137} \end{bmatrix} + (4)^2 \begin{bmatrix} -\frac{12\rho+3}{2\rho-1} \\ 1 \\ -\frac{30(\rho-10)}{3\rho-137} \end{bmatrix} \right.$$

$$\left. +(5)^2 \begin{bmatrix} \frac{3}{20}\frac{1+\rho}{3\rho-1} \\ -\frac{2(\rho+6)}{5\rho-13} \\ 1 \end{bmatrix} \right] - \frac{1}{1!}\left[(0)^1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (1)^1 \begin{bmatrix} -\frac{3\rho}{3\rho-1} \\ 0 \\ 0 \end{bmatrix} + (2)^1 \begin{bmatrix} 0 \\ -\frac{12\rho}{3\rho-13} \\ 0 \end{bmatrix} + (3)^1 \begin{bmatrix} -\frac{3}{3\rho-2} \\ 0 \\ -\frac{60\rho}{3\rho-137} \end{bmatrix} \right.$$

$$\left. +(4)^1 \begin{bmatrix} 0 \\ -\frac{3}{3\rho-13} \\ 0 \end{bmatrix} + (5)^1 \begin{bmatrix} 0 \\ 0 \\ -\frac{60}{3\rho-137} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E_3 = \frac{1}{3!}\sum_{j=0}^{5} j^3 C_j - \frac{1}{2!}\sum_{j=0}^{5} j^2 B_j = \frac{1}{6}\left((0)^3C_0 + (1)^3C_1 + (2)^3C_2 + (3)^3C_3 + (4)^3C_4 + (5)^3C_5\right)$$

$$-\frac{1}{2}\left((0)^2B_0 + (1)^2B_1 + (2)^2B_2 + (3)^2B_3 + (4)^2B_4 + (5)^2B_5\right)$$

$$= \frac{1}{3!}\left[(0)^3 \begin{bmatrix} \frac{1}{10}\frac{6\rho+1}{3\rho-1} \\ \frac{3}{5}\frac{1+\rho}{3\rho-13} \\ \frac{2(\rho+6)}{3\rho-137} \end{bmatrix} + (1)^3 \begin{bmatrix} \frac{1}{4}\frac{-3+13\rho}{3\rho-1} \\ \frac{2(3\rho+2)}{3\rho-13} \\ -\frac{15(\rho+5)}{\rho-137} \end{bmatrix} + (2)^3 \begin{bmatrix} -\frac{3(-1+2\rho)}{3\rho-1} \\ \frac{4(\rho-3)}{3\rho-13} \\ \frac{20(3\rho+10)}{3\rho-137} \end{bmatrix} + (3)^3 \begin{bmatrix} 1 \\ -\frac{12(\rho-2)}{3\rho-13} \\ -\frac{20(\rho+15)}{3\rho-137} \end{bmatrix} + (4)^3 \begin{bmatrix} -\frac{1}{2}\frac{2\rho+3}{3\rho-1} \\ 1 \\ -\frac{30(\rho-10)}{3\rho-137} \end{bmatrix} \right.$$

$$\left. +(5)^3 \begin{bmatrix} \frac{3}{20}\frac{1+\rho}{3\rho-1} \\ -\frac{2}{5}\frac{\rho+6}{3\rho-13} \\ 1 \end{bmatrix} \right] - \frac{1}{2!}\left[(0)^2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (1)^2 \begin{bmatrix} -\frac{3\rho}{3\rho-1} \\ 0 \\ 0 \end{bmatrix} + (2)^2 \begin{bmatrix} 0 \\ -\frac{12\rho}{3\rho-13} \\ 0 \end{bmatrix} + (3)^2 \begin{bmatrix} -\frac{3}{3\rho-2} \\ 0 \\ -\frac{60\rho}{3\rho-137} \end{bmatrix} \right.$$

$$\left. +(4)^2 \begin{bmatrix} 0 \\ -\frac{12}{3\rho-13} \\ 0 \end{bmatrix} + (5)^2 \begin{bmatrix} 0 \\ 0 \\ -\frac{60}{3\rho-137} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\begin{aligned}
E_4 &= \frac{1}{4!}\sum_{j=0}^{5} j^4 C_j - \frac{1}{3!}\sum_{j=0}^{5} j^3 B_j = \frac{1}{24}\left((0)^4C_0 + (1)^4C_1 + (2)^4C_2 + (3)^4C_3 + (4)^4C_4 + (5)^4C_5\right) \\
&\quad - \frac{1}{6}\left((0)^3B_0 + (1)^3B_1 + (2)^3B_2 + (3)^3B_3 + (4)^3B_4 + (5)^3B_5\right) \\
&= \frac{1}{4!}\left[(0)^4\begin{bmatrix} \frac{1}{10}\frac{6\rho+1}{3\rho-1} \\ -\frac{3}{5}\frac{1+\rho}{3\rho-13} \\ \frac{2(\rho+6)}{3\rho-137} \end{bmatrix} + (1)^4\begin{bmatrix} \frac{1}{4}\frac{-3+13\rho}{3\rho-1} \\ \frac{2(3\rho+2)}{3\rho-13} \\ \frac{15(\rho+5)}{\rho-137} \end{bmatrix} + (2)^4\begin{bmatrix} -\frac{3(-1+2\rho)}{3\rho-1} \\ \frac{4(\rho-3)}{3\rho-13} \\ \frac{20(3\rho+10)}{3\rho-137} \end{bmatrix} + (3)^4\begin{bmatrix} 1 \\ -\frac{12(\rho-2)}{3\rho-13} \\ -\frac{20(\rho+15)}{3\rho-137} \end{bmatrix} + (4)^4\begin{bmatrix} -\frac{12\rho+3}{2\rho-1} \\ 1 \\ -\frac{30(\rho-10)}{3\rho-137} \end{bmatrix}\right. \\
&\quad \left. +(5)^4\begin{bmatrix} \frac{3}{20}\frac{1+\rho}{3\rho-1} \\ -\frac{2(\rho+6)}{5\rho-13} \\ 1 \end{bmatrix}\right] - \frac{1}{6}\left[(0)^3\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (1)^3\begin{bmatrix} -\frac{3\rho}{3\rho-1} \\ 0 \\ 0 \end{bmatrix} + (2)^3\begin{bmatrix} 0 \\ -\frac{12\rho}{3\rho-13} \\ 0 \end{bmatrix} + (3)^3\begin{bmatrix} 0 \\ 0 \\ -\frac{60\rho}{3\rho-137} \end{bmatrix}\right. \\
&\quad \left. +(4)^3\begin{bmatrix} -\frac{3}{3\rho-1} \\ 0 \\ 0 \end{bmatrix} + (5)^3\begin{bmatrix} 0 \\ -\frac{12}{3\rho-13} \\ 0 \end{bmatrix} + (6)^3\begin{bmatrix} 0 \\ 0 \\ -\frac{60}{3\rho-137} \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
E_5 &= \frac{1}{5!}\sum_{j=0}^{5} j^5 C_j - \frac{1}{4!}\sum_{j=0}^{5} j^4 B_j = \frac{1}{5!}((0)^5C_0 + (1)^5C_1 + (2)^5C_2 + (3)^5C_3 + (4)^5C_4 + (5)^5C_5) \\
&\quad - \frac{1}{4!}((0)^4B_0 + (1)^4B_1 + (2)^4B_2 + (3)^4B_3 + (4)^4B_4 + (5)^4B_5) \\
&= \frac{1}{5!}\left[(0)^5\begin{bmatrix} \frac{1}{10}\frac{6\rho+1}{3\rho-1} \\ -\frac{3}{5}\frac{\rho+1}{3\rho-13} \\ \frac{2(\rho+6)}{3\rho-137} \end{bmatrix} + (1)^5\begin{bmatrix} \frac{1}{4}\frac{-3+13\rho}{3\rho-1} \\ \frac{2(3\rho+2)}{3\rho-13} \\ -\frac{15(\rho+5)}{\rho-137} \end{bmatrix} + (2)^5\begin{bmatrix} -\frac{3(-1+2\rho)}{3\rho-1} \\ \frac{4(\rho-3)}{3\rho-13} \\ \frac{20(3\rho+10)}{3\rho-137} \end{bmatrix} + (3)^5\begin{bmatrix} 1 \\ \frac{12(\rho-3)}{3\rho-13} \\ -\frac{20(\rho+15)}{3\rho-137} \end{bmatrix} + (4)^5\begin{bmatrix} -\frac{1}{2}\frac{2\rho+3}{3\rho-1} \\ 1 \\ -\frac{30(\rho-10)}{3\rho-137} \end{bmatrix}\right. \\
&\quad \left. +(5)^5\begin{bmatrix} \frac{3}{20}\frac{1+\rho}{3\rho-1} \\ \frac{2}{5}\frac{\rho+6}{3\rho-13} \\ 1 \end{bmatrix}\right] - \frac{1}{4!}\left[(0)^4\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (1)^4\begin{bmatrix} -\frac{3\rho}{3\rho-2} \\ 0 \\ 0 \end{bmatrix} + (2)^4\begin{bmatrix} 0 \\ -\frac{12\rho}{3\rho-13} \\ 0 \end{bmatrix} + (3)^4\begin{bmatrix} -\frac{3}{3\rho-2} \\ 0 \\ -\frac{60\rho}{3\rho-137} \end{bmatrix} + (4)^4\begin{bmatrix} 0 \\ -\frac{12}{3\rho-13} \\ 0 \end{bmatrix}\right. \\
&\quad \left. +(5)^4\begin{bmatrix} 0 \\ 0 \\ -\frac{60}{3\rho-137} \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
E_6 &= \frac{1}{6!}\sum_{j=0}^{5} j^6 C_j - \frac{1}{5!}\sum_{j=0}^{5} j^5 B_j = \frac{1}{6!}((0)^6 C_0 + (1)^6 C_1 + (2)^6 C_2 + (3)^6 C_3 + (4)^6 C_4 + (5)^6 C_5) \\
&- \frac{1}{5!}((0)^5 B_0 + (1)^5 B_1 + (2)^5 B_2 + (3)^5 B_3 + (4)^5 B_4 + (5)^5 B_5) \\
&= \frac{1}{6!}\left[(0)^6 \begin{bmatrix} \frac{1}{10}\frac{6\rho+1}{3\rho-1} \\ -\frac{3}{5}\frac{1-\rho}{3\rho-13} \\ \frac{2(\rho+6)}{3\rho-137} \end{bmatrix} + (1)^6 \begin{bmatrix} \frac{1}{4}\frac{-3+13\rho}{3\rho-1} \\ \frac{2(3\rho+2)}{3\rho-13} \\ -\frac{15(\rho+5)}{\rho-137} \end{bmatrix} + (2)^6 \begin{bmatrix} -\frac{3(-1+2\rho)}{3\rho-1} \\ \frac{4(\rho-3)}{3\rho-13} \\ \frac{20(3\rho+10)}{3\rho-137} \end{bmatrix} + (3)^6 \begin{bmatrix} 1 \\ -\frac{12(\rho-2)}{3\rho-13} \\ -\frac{20(\rho+15)}{3\rho-137} \end{bmatrix} + (4)^6 \begin{bmatrix} -\frac{1}{2}\frac{2\rho+3}{3\rho-1} \\ 1 \\ -\frac{30(\rho-10)}{3\rho-137} \end{bmatrix}\right. \\
&\left.+(5)^6 \begin{bmatrix} \frac{3}{20}\frac{1+\rho}{3\rho-1} \\ -\frac{2}{5}\frac{\rho+6}{3\rho-13} \\ 1 \end{bmatrix}\right] - \frac{1}{5!}\left[(0)^5 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (1)^5 \begin{bmatrix} -\frac{3\rho}{3\rho-2} \\ 0 \\ 0 \end{bmatrix} + (2)^5 \begin{bmatrix} 0 \\ \frac{12\rho}{3\rho-13} \\ 0 \end{bmatrix} + (3)^5 \begin{bmatrix} -\frac{3}{3\rho-2} \\ 0 \\ -\frac{60\rho}{3\rho-137} \end{bmatrix} + (4)^5 \begin{bmatrix} 0 \\ -\frac{12}{3\rho-13} \\ 0 \end{bmatrix}\right. \\
&\left.+(5)^5 \begin{bmatrix} 0 \\ 0 \\ -\frac{60}{3\rho-137} \end{bmatrix}\right] = \begin{bmatrix} \frac{1}{20}\frac{1-2\rho}{3\rho-1} \\ -\frac{1}{5}\frac{2+\rho}{3\rho-13} \\ \frac{10+\rho}{3\rho-137} \end{bmatrix}
\end{aligned}
$$

In accordance with the aforementioned definition 3.1, then the M3SBBDF method is of order five with error constant given by:

$$
E_6 = \begin{bmatrix} \frac{1}{20}\cdot\frac{1+2\rho}{3\rho-1} \\ -\frac{1}{5}\cdot\frac{2+\rho}{3\rho-13} \\ \frac{10+\rho}{3\rho-137} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Definition 4 (Consistency)**.** *A LMM is consistent if it has order $p \geq 1$.*

Since our M3SBBDF method is order five which is greater than one, then we conclude that the method is consistent.

4 Stability and Convergence Analysis of the Method

Stability analysis plays a central role in assessing the reliability and practical applicability of any LMM, particularly when such methods are intended for stiff IVPs. For stiff systems, the numerical solution can easily become unstable if the method does not effectively control the propagation of errors [5]. Therefore, establishing the stability properties of the M3SBBDF is essential.

In general, a numerical method is considered stable if the numerical errors introduced at any step remain bounded as the computation proceeds. This ensures that the numerical solution does not diverge and remains a meaningful approximation to the analytical solution. Stability becomes especially important for stiff ODEs, where conventional explicit schemes fail due to severe restrictions on the step size [7, 8].

To analyze the stability of the proposed method, we investigate the standard stability concepts associated with LMMs namely zero-stability and A-stability.

Definition 5 (Zero-Stability)**.** *LMM is said to be zero-stable if all the roots of its first characteristic polynomial have moduli less than or equal to unity, and any root lying on the unit circle is simple* [34]*.*

Definition 6 (A-Stability)**.** *A LMM is said to be A-stable if its absolute stability region encompasses the entire left half of the complex plane* [35]*.*

The stability properties of the method are determined by applying the linear test differential equation of the form:

$$y' = \lambda y \tag{26}$$

where $\lambda < 0$ is a complex constant with negative real part, by applying (26) in (19), we have following formulae:

$$\begin{aligned}
y_{n+1} &= -\frac{1}{10}\frac{6\rho+1}{3\rho-1}y_{n-2} - \frac{1}{4}\frac{-3+13\rho}{3\rho-1}y_{n-1} + \frac{3(-1+2\rho)}{3\rho-1}y_n + \frac{1}{2}\frac{2\rho+3}{3\rho-1}y_{n+2} - \frac{3}{20}\frac{1+\rho}{3\rho-1}y_{n+3} + \frac{3}{3\rho-2}h\lambda y_{n+1} - \frac{3\rho}{3\rho-2}h\lambda y_{n-1}\\
y_{n+2} &= \frac{3}{5}\frac{1+\rho}{3\rho-13}y_{n-2} - \frac{2(3\rho+2)}{3\rho-13}y_{n-1} - \frac{4(\rho-3)}{3\rho-13}y_n + \frac{12(\rho-2)}{3\rho-13}y_{n+1} + \frac{2}{5}\frac{\rho+6}{3\rho-13}y_{n+3} - \frac{12}{3\rho-13}h\lambda y_{n+2} - \frac{12\rho}{3\rho-13}h\lambda f_n\\
y_{n+3} &= -\frac{2(\rho+6)}{3\rho-137}y_{n-2} + \frac{15(\rho+5)}{\rho-137}y_{n-1} - \frac{20(3\rho+10)}{3\rho-137}y_n + \frac{20(\rho+15)}{3\rho-137}y_{n+1} + \frac{30(\rho-10)}{3\rho-137}y_{n+2} - \frac{60h}{3\rho-137}h\lambda y_{n+3} - \frac{60\rho}{3\rho-137}h\lambda y_{n+1}
\end{aligned} \tag{27}$$

Rearranging and collecting the like terms of equation (27) leads to:

$$\begin{aligned}
&\left(1+\frac{3}{3\rho-1}\lambda h\right)y_{n+1} - \frac{1}{2}\frac{2\rho+3}{3\rho-1}y_{n+2} + \frac{3}{20}\frac{1+\rho}{3\rho-1}y_{n+3} = -\frac{1}{10}\frac{6\rho+1}{3\rho-1}y_{n-2} + \left(-\frac{1}{4}\frac{-3+13\rho}{3\rho-1} - \frac{3\rho}{3\rho-1}\lambda h\right)y_{n-1} + \frac{3(-1+2\rho)}{3\rho-1}y_n\\
&-\frac{12(\rho-2)}{3\rho-13}y_{n+1} + \left(1+\frac{12}{3\rho-13}\lambda h\right)y_{n+2} - \frac{2}{5}\frac{\rho+6}{3\rho-13}y_{n+3} = \frac{3}{5}\frac{1+\rho}{3\rho-13}y_{n-2} - \frac{2(3\rho+2)}{3\rho-13}y_{n-1} + \left(-\frac{4(\rho-3)}{3\rho-13} - \frac{12\rho}{3\rho-13}\lambda h\right)y_n\\
&\left(-\frac{20(\rho+15)}{3\rho-137} + \frac{60\rho}{3\rho-137}\lambda h\right)y_{n+1} - \frac{30(\rho-10)}{3\rho-137}y_{n+2} + \left(1+\frac{60}{3\rho-137}\lambda h\right)y_{n+3} = -\frac{2(\rho+6)}{3\rho-137}y_{n-2} + \frac{15(\rho+5)}{\rho-137}y_{n-1} - \frac{20(3\rho+10)}{3\rho-137}y_n
\end{aligned} \tag{28}$$

The matrix representation of these equations is given by:

$$\begin{aligned}
&\begin{bmatrix} \left(1+\frac{3\lambda\bar{h}}{3\rho-2}\right) & -\frac{1}{2}\frac{2\rho+3)}{3\rho-1} & \frac{3}{20}\frac{1+\rho}{3\rho-1} \\ -\frac{12(\rho-2)}{3\rho-13} & \left(1+\frac{12\lambda\bar{h}}{3\rho-13}\right) & 0 \\ \left(-\frac{20(\rho+15)}{3\rho-137}+\frac{60\rho\lambda\bar{h}}{3\rho-137}\right) & -\frac{30(\rho-10)}{3\rho-137} & \left(1+\frac{60\lambda\bar{h}}{3\rho-137}\right) \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix}\\
&= \begin{bmatrix} -\frac{1}{10}\frac{6\rho+1}{3\rho-1} & -\left(\frac{1}{4}\frac{-3+13\rho}{3\rho-1}+\frac{3\rho\lambda h}{3\rho-1}\right) & \frac{3(-1+2\rho)}{3\rho-1} \\ \frac{3}{5}\frac{1+\rho}{3\rho-13} & -\frac{2(3\rho+2)}{3\rho-13} & \left(-\frac{4(\rho-3)}{3\rho-13}-\frac{12\rho\lambda h}{3\rho 13}\right) \\ -\frac{2(\rho+6)}{3\rho-137} & \frac{15(\rho+5)}{\rho-137} & -\frac{20(3\rho+10)}{3\rho-137} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}
\end{aligned} \tag{29}$$

Putting $\bar{h} = \lambda h$ in matrix equation (29), we have:

$$\begin{aligned}
&\begin{bmatrix} \left(1+\frac{3\bar{h}}{3\rho-2}\right) & -\frac{12(\rho-2)}{3\rho-13} & 0 \\ -\frac{12(\rho-2)}{3\rho-13} & \left(1+\frac{12\bar{h}}{3\rho-13}\right) & 0 \\ \left(-\frac{20(\rho+15)}{3\rho-137}+\frac{60\rho\bar{h}}{3\rho-137}\right) & -\frac{30(\rho-10)}{3\rho-137} & \left(1+\frac{60\rho\bar{h}}{3\rho-137}\right) \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} =\\
&\begin{bmatrix} -\frac{1}{10}\frac{6\rho+1}{3\rho-1} & -\left(\frac{1}{4}\frac{-3+13\rho+3\rho}{3\rho-1}+\frac{3\rho\bar{h}}{3\rho-1}\right) & \frac{3(-1+2\rho)}{3\rho-1} \\ \frac{3}{5}\frac{1+\rho}{3\rho-13} & -\frac{2(3\rho+2)}{3\rho-13} & -\left(\frac{4(\rho-3)}{3\rho-13}-\frac{12\rho\bar{h}}{3\rho-13}\right) \\ -\frac{2(\rho+6)}{3\rho-137} & \frac{15(\rho+5)}{\rho-137} & -\frac{20(3\rho+10)}{3\rho-137} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}
\end{aligned} \tag{30}$$

If m is the number of block and r is the number of points in the block, then $n = mr$, where $r = 3$ and $n = 3m$. By [17], we let

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \end{bmatrix}, \quad Y_{m-1} = \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_{3m-2} \\ y_{3m-1} \\ y_{3m} \end{bmatrix} = \begin{bmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \end{bmatrix}$$

Equation (30) can also be written as:

$$AY_m = BY_{m-1} \tag{31}$$

where,

$$A = \begin{bmatrix} \left(1 + \frac{3\bar{h}}{3\rho-2}\right) & -\frac{1}{2}\frac{2\rho+3}{3\rho-1} & \frac{3}{20}\frac{1+\rho}{3\rho-1} \\ -\frac{12(\rho-2)}{3\rho-13} & \left(1 + \frac{12\bar{h}}{3\rho-13}\right) & 0 \\ \left(-\frac{20(\rho+15)}{3\rho-137} + \frac{60\rho\bar{h}}{3\rho-137}\right) & -\frac{30(\rho-10)}{3\rho-137} & \left(1 + \frac{60\bar{h}}{3\rho-137}\right) \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{1}{10}\frac{6\rho+1}{3\rho-1} & -\left(\frac{1}{4}\frac{-3+13\rho}{3\rho-1} + \frac{3\rho\bar{h}}{3\rho-1}\right) & \frac{3(-1+2\rho)}{3\rho-1} \\ \frac{3}{5}\frac{1+\rho)}{3\rho-13} & -\frac{2(3\rho+2)}{3\rho-13} & \left(-\frac{4(\rho-3)}{3\rho-13} - \frac{12\rho\bar{h}}{3\rho-13}\right) \\ -\frac{2(\rho+6)}{3\rho-137} & \frac{15(\rho+5)}{\rho-137} & -\frac{20(3\rho+10)}{3\rho-137} \end{bmatrix}$$

The characteristic polynomial of the proposed method is determined by evaluating

$$\pi(t, \bar{h}) = \det(At - B) \tag{32}$$

This implies that

$$\pi(t, \bar{h}) = \det\left(\begin{bmatrix} \left(1 + \frac{3\bar{h}}{3\rho-2}\right) & -\frac{1}{2}\cdot\frac{2\rho+3}{3\rho-1} & \frac{3}{20}\cdot\frac{1+\rho}{3\rho-1} \\ -\frac{12(\rho-2)}{3\rho-13} & \left(1 + \frac{12\bar{h}}{3\rho-13}\right) & 0 \\ \left(-\frac{20(\rho+15)}{3\rho-137} + \frac{60\rho\bar{h}}{3\rho-137}\right) & -\frac{30(\rho-10)}{3\rho-137} & \left(1 + \frac{60\bar{h}}{3\rho-137}\right) \end{bmatrix} t - \begin{bmatrix} -\frac{1}{10}\frac{6\rho+1}{3\rho-1} & -\left(\frac{1}{4}\frac{(-3+13\rho)}{3\rho-1} + \frac{3\rho\bar{h}}{3\rho-1}\right) & \frac{3(-1+2\rho)}{3\rho-1} \\ \frac{3}{5}\frac{1+\rho}{3\rho-13} & -\frac{2(3\rho+2)}{3\rho-13} & \left(-\frac{4(\rho-3)}{3\rho-13} - \frac{12\rho\bar{h}}{3\rho-13}\right) \\ -\frac{2(\rho+6)}{3\rho-137} & \frac{15(\rho+5)}{\rho-137} & -\frac{20(3\rho+10)}{3\rho-137} \end{bmatrix} \right) = 0 \tag{33}$$

The determinant in (33) was evaluated using the Maple 18 software environment, which yields:

$$\begin{aligned} \pi(t, \bar{h}) = \frac{1}{(3\rho-1)(3\rho-13)(3\rho-137)} \Big(& 2160\bar{h}^3\rho^3 t - 792\bar{h}^2\rho^2 t^3 - 108\bar{h}^2\rho^2 t^3 - 3\bar{h}\rho^3 t^3 + 2160\bar{h}^3 t^3 + 2340\bar{h}^2\rho^2 t^3 \\ & + 6048\bar{h}^2\rho^2 t^2 + 2700\bar{h}^2\rho t^3 + 2160\bar{h}\rho^3 t^2 + 225\bar{h}\rho^2 t^3 + 10\rho^3 t^3 + 72\bar{h}^2\rho^3 - 12852\bar{h}^2\rho^2 t - 13608\bar{h}^2\rho t^2 \\ & - 7992\bar{h}^2 t^3 + 4815\bar{h}\rho^3 t - 12078\bar{h}^2\rho t^2 - 7425\bar{h}\rho^3 t^2 - 2037\rho^3 t^2 - 4140\bar{h}\rho^2 t^3 + 1627\bar{h} t^3 + 1614\rho^3 t \\ & + 9387\rho^2 t^2 + 6162\rho t^3 + 1548\bar{h}\rho^2 + 4365\bar{h}\rho t + 9738\bar{h} t^2 + 413\rho^3 - 10854\rho^2 t - 10071\rho t^2 - 8018 t^3 \\ & - 90\bar{h}\rho - 495\bar{h} t + 1677\rho^2 + 4086\rho t + 8745 t^2 - 177\rho - 726 t - 1 \Big) = 0 \end{aligned} \tag{34}$$

To obtain the first characteristic polynomial of the method, we substitute $\bar{h} = 0$ in (34), we have

$$\frac{1}{(3\rho-1)(3\rho-13)(3\rho-137)}\Big(10\rho^3t^3 - 2037\rho^3t^2 + 1614\rho^3t + 9387\rho^2t^2 + 6162\rho t^3 + 413\rho^3 - 10854\rho^2t - 10071\rho t^2 - 8018t^3 + 1677\rho^2 + 4086\rho t + 8745t^2 - 177\rho - 726t - 1\Big) = 0 \tag{35}$$

Solving (35) for t, we have the following roots:

$$t = 1,$$

$$t = \frac{2027\rho^3 - 9177\rho^2 + 3909\rho - 727 + 9\sqrt{50929\rho^6 - 4627758\rho^5 + 1343559\rho^4 - 573524\rho^3 - 364481\rho^2 - 390\rho + 6921}}{4(5\rho^3 - 105\rho^2 + 3081\rho - 4009)}$$

$$t = -\frac{-2027\rho^3 + 9177\rho^2 - 3909\rho + 727 + 9\sqrt{50929\rho^6 - 4627758\rho^5 + 1343559\rho^4 - 573524\rho^3 - 364481\rho^2 - 390\rho + 6921}}{4(5\rho^3 - 105\rho^2 + 3081\rho - 4009)}$$

By substituting two different values of the free parameter ρ into the above values of t, the roots of the first characteristic polynomial were determined as follows:

(i) When $\rho = \frac{4}{5}$

$$t = 1, \quad t = 0.5957821465, \quad t = 0.5957821465$$

(ii) When $\rho = -\frac{1}{5}$

$$t = 1, \quad t = 0.1029730174, \quad t = 0.1029730174$$

These roots confirm that the M3SBBDF method is zero-stable, as no root of the first characteristic polynomial lies outside the unit circle, and the root with modulus one is simple (non-repeated).

The boundaries of the absolute stability regions for the M3SBBDF method, for different values of ρ, were determined by substituting $t = e^{i\theta}$, where $0 \leq \theta \leq 2\pi$ into the stability polynomial defined in equation (33). The corresponding stability region plots for M3SBBDF method were generated using Maple 18 software and are presented in Figures 2 and 3.

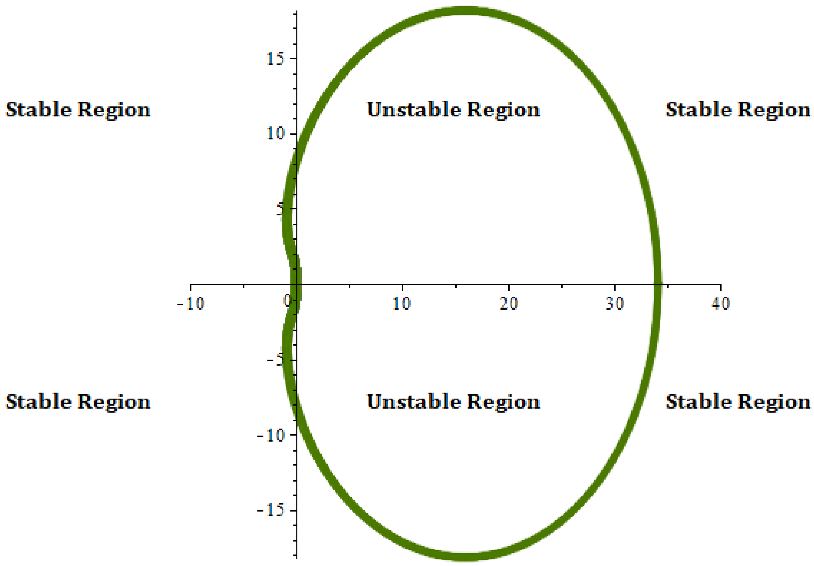

Figure 2. Stability region of M3SBBDF when ρ=4/5.

Therefore, the stability region of the method extends beyond the circular boundary, indicating that the M3SBBDF method is A-stable when $\rho = -\frac{1}{5}$ and almost A-stable when $\rho = \frac{4}{5}$, as the region covers nearly the entire left half of the complex plane. Having satisfied both the zero-stability and A-stability conditions, the M3SBBDF method is deemed suitable for the numerical solution of first-order stiff IVPs.

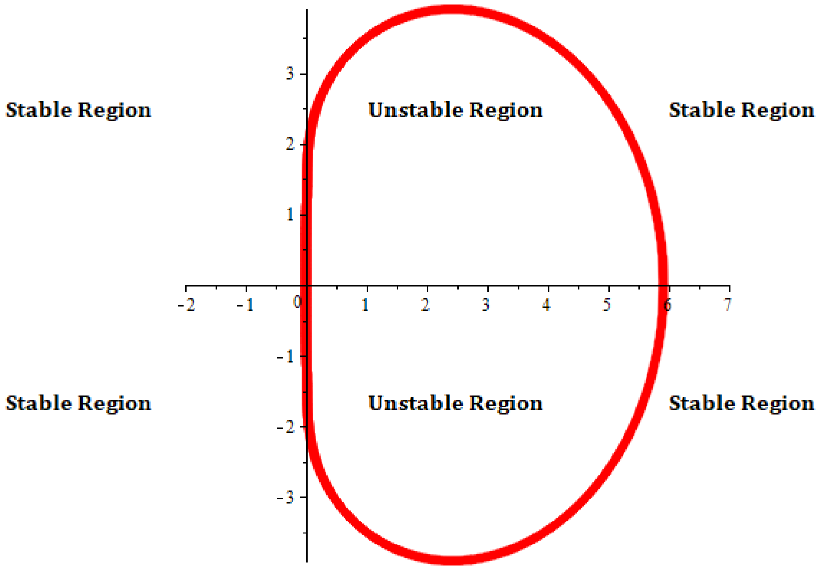

Figure 3. Stability region of M3SBBDF when ρ=-1/5.

Theorem 4.1 (Dahlquist Equivalence Theorem)**.** *A LMM is convergent if it is both consistent and zero-stable.*

Theorem 4.2. *The M3SBBDF method is convergent.*

Proof. The convergence of a LMM follows directly from the fundamental Dahlquist equivalence theorem, which states that a consistent LMM is convergent if and only if it is zero-stable. This result provides the theoretical basis for establishing the convergence of the proposed M3SBBDF method.

The M3SBBDF method has been shown to be consistent, as it satisfies the linear multistep consistency conditions and achieves order five accuracy. In particular, the local truncation error tends to zero as the step size $h \to 0$, confirming that the method accurately approximates the underlying differential equation.

Furthermore, the zero-stability of the M3SBBDF method has been established through the analysis of its first characteristic polynomial. All roots of the characteristic equation lie within or on the unit circle, and the roots on the unit circle are simple. This satisfies the root condition required for zero-stability, ensuring that perturbations in the initial data or round-off errors do not grow uncontrollably as the integration proceeds.

Since the M3SBBDF method is both consistent and zero-stable, it follows immediately from the Dahlquist equivalence theorem that the method is convergent. Consequently, the global error of the method tends to zero as the step size decreases, and the numerical solution converges to the exact solution of the stiff IVPs.

Therefore, the M3SBBDF method is guaranteed to produce reliable and accurate approximations for sufficiently small step sizes, confirming its theoretical soundness for the numerical solution of stiff IVPs. □

5 Implementation of the Method

Newton's iteration is applied for the implementation of the method. The description of the iteration is given below. We first start by defining the error.

Definition 7 (Absolute Error)**.** *Let y_i and $y(x_i)$ be the approximate and exact solutions of* (1) *respectively. Then the absolute error is given by*

$$(error_i)_t = |(y_i)_t - (y(x_i))_t| \tag{36}$$

The maximum error is defined by:

$$MAXE = \max_{1 \le i \le T} \frac{\max(error_i)_t}{1 \le i \le N} \tag{37}$$

where T is the total number of steps and N is the number of equations.

Define:

$$\begin{aligned}
F_1 &= y_{n+1} - \frac{1}{2}\frac{2\rho+3}{3\rho-1}y_{n+2} + \frac{3}{20}\frac{1+\rho}{3\rho-1}y_{n+3} + \frac{3}{3\rho-2}hf_{n+1} + \frac{3\rho}{3\rho-2}hf_{n-1} - \varepsilon_1 \\
F_2 &= y_{n+2} - \frac{12(\rho-2)}{3\rho-13}y_{n+1} - \frac{2}{5}\frac{\rho+6}{3\rho-13}y_{n+3} + \frac{12h}{3\rho-13}f_{n+2} + \frac{12h\rho}{3\rho-13}f_n - \varepsilon_2 \\
F_3 &= y_{n+3} - \frac{20(\rho+15)}{3\rho-137}y_{n+1} - \frac{30(\rho-10)}{3\rho-137}y_{n+2} + \frac{60h}{3\rho-137}f_{n+3} + \frac{60h\rho}{3\rho-137}f_{n+1} - \varepsilon_3
\end{aligned} \tag{38}$$

where,

$$\begin{aligned}
\varepsilon_1 &= -\frac{1}{10}\frac{6\rho+1}{3\rho-1}y_{n-2} - \frac{1}{4}\frac{-3+13\rho}{3\rho-1}y_{n-1} + \frac{3(-1+2\rho)}{3\rho-1}y_n \\
\varepsilon_2 &= \frac{3}{5}\frac{1+\rho}{3\rho-13}y_{n-2} - \frac{2(3\rho+2)}{3\rho-13}y_{n-1} - \frac{4(\rho-3)}{3\rho-13}y_n \\
\varepsilon_3 &= -\frac{2(\rho+6)}{3\rho-137}y_{n-2} + \frac{15(\rho+5)}{\rho-137}y_{n-1} - \frac{20(3\rho+10)}{3\rho-137}y_n
\end{aligned}$$

are the back values.

Let $y_{n+j}^{(i+1)}$, $j = 1, 2, 3$ denote the $(i+1)^{th}$ iterative values of y_{n+j} and define

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, \quad j = 1, 2, 3 \tag{39}$$

Newton's iteration for the M3SBBDF takes the form:

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \left(F_j'\left(y_{n+j}^{(i)}\right)\right)^{-1}\left(F_j\left(y_{n+j}^{(i)}\right)\right), \quad j = 1, 2, 3 \tag{40}$$

This can be written as

$$\left(F_j'\left(y_{n+j}^{(i)}\right)\right)e_{n+j}^{(i+1)} = -\left(F_j\left(y_{n+j}^{(i)}\right)\right), \quad j = 1, 2, 3 \tag{41}$$

Equation (41) is equivalent to the following matrix form:

$$\begin{aligned}
&\begin{bmatrix}
\left(1 + \frac{3h}{3\rho-2}\frac{\partial f_{n+1}}{\partial y_{n+1}}\right) & -\frac{12\rho+3}{3\rho-13} & \frac{3}{20}\frac{1+\rho}{3\rho-1} \\
-\frac{12(\rho-2)}{3\rho-13} & \left(1 + \frac{12h}{3\rho-13}\frac{\partial f_{n+2}}{\partial y_{n+2}}\right) & -\frac{2}{5}\frac{\rho+6}{3\rho-13} \\
\left(-\frac{20(\rho+15)}{3\rho-137} + \frac{60h}{3\rho-137}\frac{\partial f_{n+1}}{\partial y_{n+1}}\right) & -\frac{30(\rho-10)}{3\rho-137} & \left(1 + \frac{60h}{3\rho-137}\frac{\partial f_{n+3}}{\partial y_{n+3}}\right)
\end{bmatrix}
\begin{bmatrix} e_{n+1}^{(i+1)} \\ e_{n+2}^{(i+1)} \\ e_{n+3}^{(i+1)} \end{bmatrix} \\
&= \begin{bmatrix}
-1 & \frac{1}{2}\frac{2\rho+3}{3\rho-1} & -\frac{3}{20}\frac{1+\rho}{3\rho-1} \\
\frac{12(\rho-2)}{3\rho-13} & -1 & \frac{2}{5}\frac{\rho+6}{3\rho-13} \\
\frac{20(\rho+15)}{3\rho-137} & \frac{30(\rho-10)}{3\rho-137} & -1
\end{bmatrix}
\begin{bmatrix} y_{n+1}^{(i)} \\ y_{n+2}^{(i)} \\ y_{n+3}^{(i)} \end{bmatrix}
+ h\begin{bmatrix}
-\frac{3}{3\rho-2} & 0 & 0 \\
0 & -\frac{12}{3\rho-13} & 0 \\
-\frac{60\rho}{3\rho-137} & 0 & -\frac{60}{3\rho-137}
\end{bmatrix}
\begin{bmatrix} f_{n+1}^{(i)} \\ f_{n+2}^{(i)} \\ f_{n+3}^{(i)} \end{bmatrix} \\
&+ h\begin{bmatrix}
0 & -\frac{3\rho}{3\rho-2} & 0 \\
0 & 0 & -\frac{12\rho}{3\rho-13} \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} f_{n-2}^{(i)} \\ f_{n-1}^{(i)} \\ f_n^{(i)} \end{bmatrix}
+ \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}
\end{aligned} \tag{42}$$

5.1 Newton's Iteration Matrix of the Method

The Newton's iteration matrix of the fully implicit modified 3-point super class of block backward differentiation formula (M3SBBDF) as given in (42) is derived for various values of ρ as follows:

i. For $\rho = 4/5$

$$\begin{bmatrix} \left(1+\frac{15h}{7}\frac{\partial f_{n+1}}{\partial y_{n+1}}\right) & -\frac{23}{14} & \frac{27}{140} \\ -\frac{72}{53} & \left(1-\frac{60h}{53}\frac{\partial f_{n+2}}{\partial y_{n+2}}\right) & \frac{68}{53} \\ \left(\frac{1580}{673}-\frac{240h}{673}\frac{\partial f_{n+1}}{\partial y_{n+1}}\right) & -\frac{1380}{673} & \left(1-\frac{300h}{673}\frac{\partial f_{n+3}}{\partial y_{n+3}}\right) \end{bmatrix} \begin{bmatrix} e^{(i+1)}_{n+1} \\ e^{(i+1)}_{n+2} \\ e^{(i+1)}_{n+3} \end{bmatrix} = \begin{bmatrix} -1 & \frac{23}{14} & -\frac{27}{140} \\ \frac{72}{53} & -1 & -\frac{68}{53} \\ \frac{1580}{673} & \frac{1380}{673} & -1 \end{bmatrix} \begin{bmatrix} y^{(i)}_{n+1} \\ y^{(i)}_{n+2} \\ y^{(i)}_{n+3} \end{bmatrix}$$
$$+h\begin{bmatrix} -\frac{15}{7} & 0 & 0 \\ 0 & \frac{60}{53} & 0 \\ \frac{240}{673} & 0 & \frac{300}{673} \end{bmatrix} \begin{bmatrix} f^{(i)}_{n+1} \\ f^{(i)}_{n+2} \\ f^{(i)}_{n+3} \end{bmatrix} + h\begin{bmatrix} 0 & -\frac{12}{7} & 0 \\ 0 & 0 & \frac{48}{53} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f^{(i)}_{n-2} \\ f^{(i)}_{n-1} \\ f^{(i)}_{n} \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \tag{43}$$

ii. For $\rho = -\frac{1}{5}$

$$\begin{bmatrix} 1-\frac{15h}{8}\frac{\partial f_{n+1}}{\partial y_{n+1}} & \frac{13}{16} & -\frac{3}{40} \\ -\frac{33}{17} & \left(1-\frac{15h}{17}\frac{\partial f_{n+2}}{\partial y_{n+2}}\right) & \frac{29}{170} \\ \left(\frac{185}{86}+\frac{15h}{172}\frac{\partial f_{n+1}}{\partial y_{n+1}}\right) & -\frac{765}{344} & \left(1-\frac{75h}{172}\frac{\partial f_{n+3}}{\partial y_{n+3}}\right) \end{bmatrix} \begin{bmatrix} e^{(i+1)}_{n+1} \\ e^{(i+1)}_{n+2} \\ e^{(i+1)}_{n+3} \end{bmatrix} = \begin{bmatrix} -1 & -\frac{13}{16} & \frac{3}{40} \\ \frac{33}{17} & -1 & -\frac{29}{170} \\ -\frac{185}{86} & \frac{765}{344} & -1 \end{bmatrix} \begin{bmatrix} y^{(i)}_{n+1} \\ y^{(i)}_{n+2} \\ y^{(i)}_{n+3} \end{bmatrix}$$
$$+h\begin{bmatrix} \frac{15}{8} & 0 & 0 \\ 0 & \frac{15}{17} & 0 \\ -\frac{15}{172} & 0 & \frac{75}{172} \end{bmatrix} \begin{bmatrix} f^{(i)}_{n+1} \\ f^{(i)}_{n+2} \\ f^{(i)}_{n+3} \end{bmatrix} + h\begin{bmatrix} 0 & -\frac{3}{8} & 0 \\ 0 & 0 & -\frac{3}{17} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f^{(i)}_{n-2} \\ f^{(i)}_{n-1} \\ f^{(i)}_{n} \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \tag{44}$$

Thus, a computer code in C programming language is written for the implementation of the method only when $\rho = 4/5$ and $\rho = -1/5$.

6 Test Problems and Numerical Results

The following stiff IVPs were carefully selected to evaluate and compare the performance of the proposed methods. These problems consist of both linear and non-linear first-order stiff systems, drawn from existing literature, and are commonly used as benchmark tests for assessing the accuracy, stability, and efficiency of numerical methods for stiff equations.

Table 3. Notations used in the numerical results tables.

Notation	Description
h	Step Size
METHOD	The Methods Used
TS	Total Steps
MAXE	Maximum Absolute Error
CPU TIME	Computation Time In Seconds
3BBDF	3-Point Block Backward Differentiation Formula
3DBBDF	Diagonally Implicit 3-Point Block Backward Differentiation Formula
3DSBEBDF	Modified Fully Implicit 3-Point Super Class of Block Backward Differentiation Formula

Problem 1:

This problem is a linear stiff system with oscillatory behavior [36]:

$$y_1' = -3y_1 + 2y_2 + 3\cos(x) - 3\sin(x), \quad y_1(0) = 1, \quad 0 \le x \le 20$$
$$y_2' = 2y_1 - 3y_2 - \cos(x) + 3\sin(x), \quad y_2(0) = 0.$$

The exact solutions are:

$$y_1(x) = \cos(x), \quad y_2(x) = \sin(x).$$

The solutions are purely oscillatory, while the system matrix possesses negative real eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -5$, which introduce rapidly decaying transient components. The coexistence of oscillatory forcing terms and significantly different eigenvalue magnitudes makes this problem moderately stiff with oscillatory characteristics.

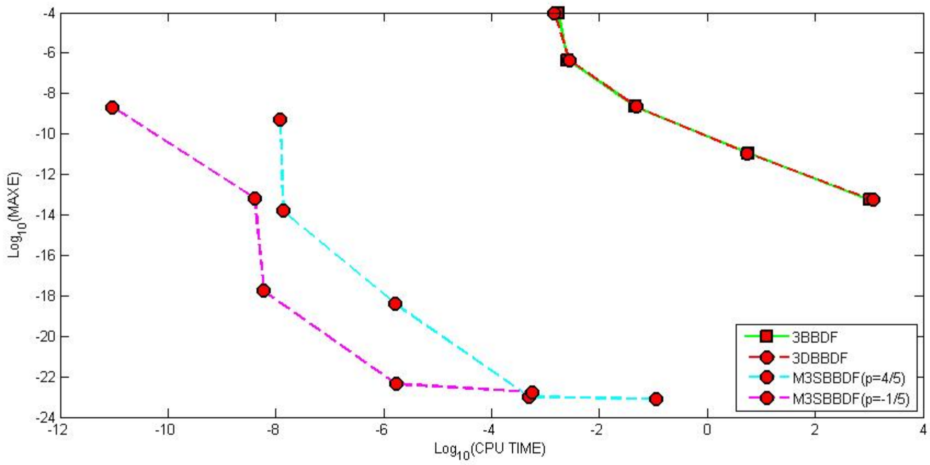

Figure 4. Efficiency Graph for Problem 1.

Table 4 presents the numerical results for Problem 1, and Figure 4 illustrates the efficiency of the methods through the log-log plot of maximum error versus CPU time.

Table 4. Numerical Result for Problem 1.

H	METHOD	TS	MAXE	CPU TIME
10^{-2}	3BBDF	666	1.79395E-02	6.26800E-02
	3DBBDF	666	1.79396E-02	5.86900E-02
	M3SBBDF($\rho = 4/5$)	666	9.06872E-05	3.64100E-04
	M3SBBDF($\rho = -1/5$)	666	1.69647E-04	1.62400E-05
10^{-3}	3BBDF	6666	1.76790E-03	7.48400E-02
	3DBBDF	6666	1.76790E-03	7.72200E-02
	M3SBBDF($\rho = 4/5$)	6666	1.01330E-06	3.82100E-04
	M3SBBDF($\rho = -1/5$)	6666	1.89025E-06	2.24800E-04
10^{-4}	3BBDF	66666	1.76533E-04	2.64900E-01
	3DBBDF	66666	1.76533E-04	2.65600E-01
	M3SBBDF($\rho = 4/5$)	66666	1.02508E-08	3.04100E-03
	M3SBBDF($\rho = -1/5$)	66666	1.92712E-08	2.69400E-04
10^{-5}	3BBDF	666666	1.76511E-05	2.14000E+00
	3DBBDF	666666	1.76511E-05	2.09500E+00
	M3SBBDF($\rho = 4/5$)	666666	1.02627E-10	3.62100E-02
	M3SBBDF($\rho = -1/5$)	666666	1.93255E-10	3.12400E-03
10^{-6}	3BBDF	6666666	1.76511E-06	2.04000E+01
	3DBBDF	6666666	1.76512E-06	2.15700E+01
	M3SBBDF($\rho = 4/5$)	6666666	9.24720E-11	3.86900E-01
	M3SBBDF($\rho = -1/5$)	6666666	1.29422E-10	3.92400E-02

Problem 2:
This is a highly stiff problem, particularly for small values of ϵ [36].

$$y_1' = -2y_1 + y_2 + 2\sin(x), \quad y_1(0) = 2, \quad 0 \leq x \leq 10$$
$$y_2' = -(\epsilon^{-1} + 2)y_1 + (\epsilon^{-1} + 1)(y_2 - \cos(x) + \sin(x)), y_2(0) = 3.$$

The analytical exact solutions are:

$$y_1(x) = 2e^{-x} + \sin(x), \quad y_2(x) = 2e^{-x} + \cos(x).$$

The eigenvalues are:

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = \epsilon^{-1},$$

which exhibit a large separation in magnitude when ϵ is small, a defining feature of stiffness. Although trigonometric functions appear in the forcing terms, the dominant behavior of the solution is governed by exponential decay rather than sustained oscillations.

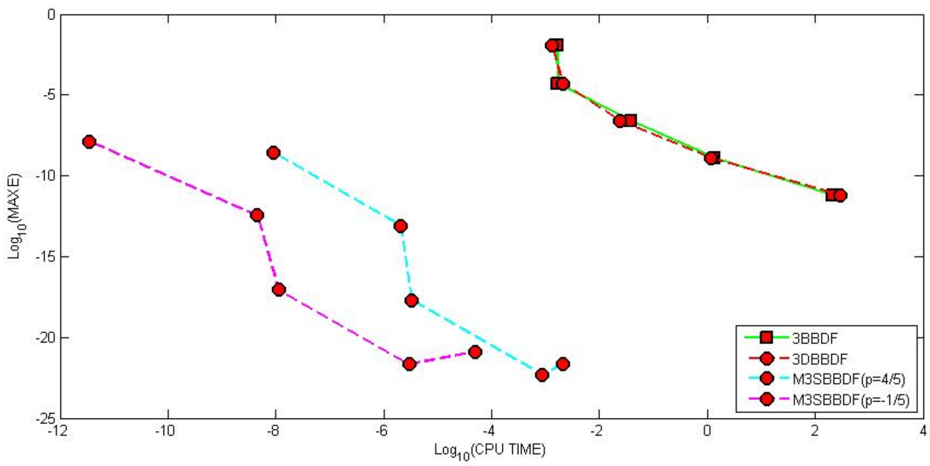

Figure 5. Efficiency Graph for Problem 2.

Table 5 presents the numerical results for Problem 2, and Figure 5 illustrates the efficiency comparison between the methods through the log-log plot of maximum error versus CPU time.

Problem 3:
This is a severe stiff problem taken from [37]:

$$y_1' = -0.1y_1 - 49.9y_2, \quad y_1(0) = 2, \quad 0 \leq x \leq 10$$
$$y_2' = -50y_2, \quad y_2(0) = 1,$$
$$y_3' = 70y_2 - 120y_3, \quad y_3(0) = 2.$$

The exact solutions are:

$$y_1(x) = e^{-0.1x} + e^{-50x},$$
$$y_2(x) = e^{-50x},$$
$$y_3(x) = e^{-50x} + e^{-120x}.$$

The eigenvalues are:

$$\lambda_1 = -0.1, \quad \lambda_2 = -120, \quad \lambda_3 = -50.$$

Table 5. Numerical Result for Problem 2.

H	METHOD	TS	MAXE	CPU TIME
10^{-2}	3BBDF	333	1.42161E-01	6.17200E-02
	3DBBDF	333	1.42198E-01	5.62000E-02
	M3SBBDF($\rho = 4/5$)	333	1.89703E-04	3.18400E-04
	M3SBBDF($\rho = -1/5$)	333	3.80501E-04	1.06200E-05
10^{-3}	3BBDF	3333	1.39302E-02	6.23100E-02
	3DBBDF	3333	1.39306E-02	6.84900E-02
	M3SBBDF($\rho = 4/5$)	3333	2.03681E-06	3.40600E-03
	M3SBBDF($\rho = -1/5$)	3333	3.86032E-06	2.38400E-04
10^{-4}	3BBDF	33333	1.39012E-03	2.42200E-01
	3DBBDF	33333	1.39012E-03	1.96100E-01
	M3SBBDF($\rho = 4/5$)	33333	2.05120E-08	4.17600E-03
	M3SBBDF($\rho = -1/5$)	33333	3.86521E-08	3.52100E-04
10^{-5}	3BBDF	333333	1.38983E-04	1.14000E+00
	3DBBDF	333333	1.38983E-04	1.05800E+00
	M3SBBDF($\rho = 4/5$)	333333	2.05266E-10	4.68400E-02
	M3SBBDF($\rho = -1/5$)	333333	3.86567E-10	3.96400E-03
10^{-6}	3BBDF	3333333	1.38982E-05	1.03000E+01
	3DBBDF	3333333	1.38982E-05	1.18700E+01
	M3SBBDF($\rho = 4/5$)	3333333	3.84314E-10	6.81200E-02
	M3SBBDF($\rho = -1/5$)	3333333	8.58105E-10	1.34100E-02

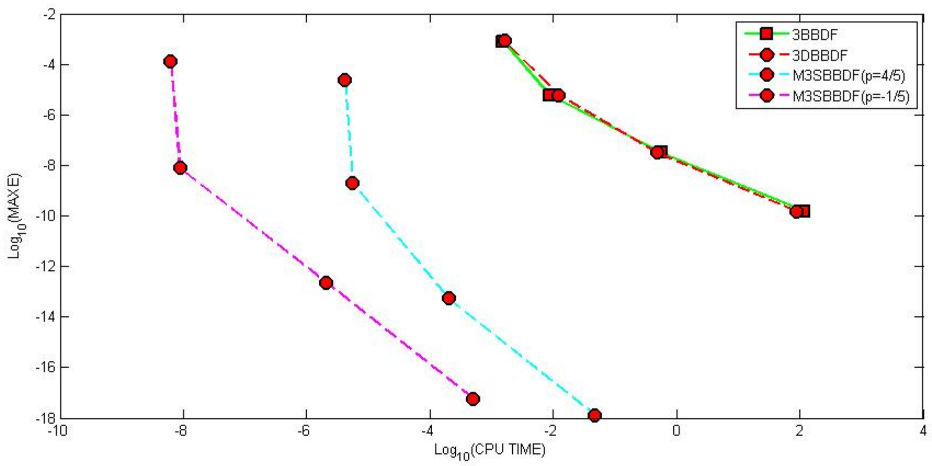

Figure 6. Efficiency Graph for Problem 3.

All eigenvalues are real and negative, leading to purely exponential decay with multiple time scales. The presence of very fast decaying modes alongside slow components results in strong stiffness, but no oscillatory behavior is observed in the exact solutions.

Table 6 presents the numerical results for Problem 3, and Figure 6 provides a visual comparison of the methods' efficiency for this severely stiff system.

The problems given below are solved using 3-point Block Backward Differentiation Formula, 3-point diagonally implicit Block Backward Differentiation Formula and the modified fully implicit 3-point Super class Block Backward Differentiation Formula methods. In order to compare the accuracy and efficiency of the methods, the maximum absolute errors obtained for different step size h are given for each problem. The number of steps

Table 6. Numerical Result for Problem 3.

H	METHOD	TS	MAXE	CPU TIME
10^{-2}	3BBDF	333	1.29757E+114	7.72800E-02
	3DBBDF	333	3.68219E+104	7.67800E-02
	M3SBBDF($\rho = 4/5$)	333	4.73808E-01	4.03100E-04
	M3SBBDF($\rho = -1/5$)	333	1.76147E-01	2.53100E-05
10^{-3}	3BBDF	3333	4.52009E-02	5.97000E-02
	3DBBDF	3333	4.64814E-02	6.11800E-02
	M3SBBDF($\rho = 4/5$)	3333	9.98962E-03	4.62400E-03
	M3SBBDF($\rho = -1/5$)	3333	2.04717E-02	2.72100E-04
10^{-4}	3BBDF	33333	5.44166E-03	1.27400E-01
	3DBBDF	33333	5.44039E-03	1.47400E-01
	M3SBBDF($\rho = 4/5$)	33333	1.63860E-04	5.24100E-03
	M3SBBDF($\rho = -1/5$)	33333	3.01867E-04	3.16100E-04
10^{-5}	3BBDF	333333	5.51066E-04	7.85300E-01
	3DBBDF	333333	5.51049E-04	7.35500E-01
	M3SBBDF($\rho = 4/5$)	333333	1.72474E-06	2.48100E-02
	M3SBBDF($\rho = -1/5$)	333333	3.23016E-06	3.41600E-03
10^{-6}	3BBDF	3333333	5.51749E-05	7.97200E+00
	3DBBDF	3333333	5.51748E-05	7.00400E+00
	M3SBBDF($\rho = 4/5$)	3333333	1.73363E-08	2.67200E-01
	M3SBBDF($\rho = -1/5$)	3333333	3.26243E-08	3.68100E-02

taken to solve each problem and the computation time are also given. The notations used in the numerical results tables are summarized in Table 3. To make referencing easy, below is the 3-point BBDF method [16].

$$\begin{aligned}
y_{n+1} &= \frac{1}{10}y_{n-2} - \frac{3}{4}y_{n-1} + 3y_n - \frac{3}{2}y_{n+2} + \frac{3}{20}y_{n+3} + 3hf_{n+1}, \\
y_{n+2} &= -\frac{3}{65}y_{n-2} + \frac{4}{13}y_{n-1} - \frac{12}{13}y_n + \frac{24}{13}y_{n+1} - \frac{12}{65}y_{n+3} + \frac{12}{13}hf_{n+2}, \\
y_{n+3} &= \frac{12}{137}y_{n-2} - \frac{75}{137}y_{n-1} + \frac{200}{137}y_n - \frac{300}{137}y_{n+1} + \frac{300}{137}y_{n+2} + \frac{60}{137}hf_{n+3}.
\end{aligned} \tag{45}$$

and diagonally implicit 3-point BBDF method proposed in [34] is expressed as:

$$\begin{aligned}
y_{n+1} &= \frac{2}{11}y_{n-2} - \frac{9}{11}y_{n-1} + \frac{18}{11}y_n + \frac{6}{11}hf_{n+1}, \\
y_{n+2} &= -\frac{3}{25}y_{n-2} + \frac{16}{25}y_{n-1} - \frac{36}{25}y_n + \frac{48}{25}y_{n+1} + \frac{12}{25}hf_{n+2}, \\
y_{n+3} &= \frac{12}{137}y_{n-2} - \frac{75}{137}y_{n-1} + \frac{200}{137}y_n - \frac{300}{137}y_{n+1} + \frac{300}{137}y_{n+2} + \frac{60}{137}hf_{n+3}.
\end{aligned} \tag{46}$$

For a clearer visual comparison of the methods' performance, the relationship between accuracy and computational cost is illustrated by plotting $log_{10}(MAXE)$ versus $log_{10}(CPU\,TIME)$.

7 Discussion

The numerical performance of the classical 3BBDF, 3DBBDF, and the proposed M3SBBDF methods with $\rho = 4/5$ and $\rho = -1/5$ was evaluated across a range of step sizes. The comparison was based on maximum error (MAXE) and CPU time, with efficiency curves plotted as $\log_{10}(\text{MAXE})$ versus $\log_{10}(\text{CPU TIME})$.

For Problems 1 and 2, the classical methods exhibit similar behavior, achieving the expected order of convergence but with relatively large errors. Their CPU times increase significantly for smaller step sizes. In contrast, M3SBBDF consistently produces significantly smaller errors while maintaining considerably lower computational cost. Efficiency curves confirm that the modified scheme achieves higher accuracy at reduced CPU time, demonstrating improved error constants and enhanced stability.

For Problem 3, a highly stiff test problem, the classical methods fail at larger step sizes $h = 10^{-2}$, yielding extremely large errors and illustrating their instability for highly stiff systems. The proposed M3SBBDF, however, remains stable and produces bounded errors even at coarse steps. As h decreases, the classical methods gradually stabilize, but their accuracy remains inferior to that of M3SBBDF. The modified method achieves smaller errors and reduced CPU times across all step sizes, confirming superior stiff-mode damping, enhanced stability, and computational efficiency.

Overall, the results demonstrate that the introduction of the free parameter ρ and the structural modification in the BBDF framework significantly improve stability, reduce the error constant, and preserve the theoretical order of accuracy. Across all test problems, M3SBBDF consistently outperforms the classical 3BBDF and 3DBBDF methods, providing a robust, accurate, and computationally efficient tool for the numerical solution of stiff ODEs.

8 Conclusion

This paper presented the development, analysis, and numerical validation of a modified three-point superclass of block backward differentiation formulae (M3SBBDF) for the efficient numerical solution of stiff IVPs. The proposed method was formulated by incorporating a free parameter into the classical block BDF framework, allowing improved control over stability and error behavior while maintaining fifth-order accuracy.

A detailed theoretical investigation of the methods was carried out, including consistency, zero-stability, convergence, and stiff-stability analyses. The results of these analyses confirm that the proposed schemes are convergent and possess strong stability properties that make them well suited for stiff systems, including problems with widely separated eigenvalues and oscillatory components.

Extensive numerical experiments were conducted on a set of standard benchmark problems involving both oscillatory and non-oscillatory stiff systems. The performance of the proposed M3SBBDF methods was compared with existing fifth order block BDF schemes, namely the 3BBDF and 3DBBDF methods. The numerical results clearly demonstrate that the proposed methods achieve significantly higher accuracy, often reducing the maximum error by several orders of magnitude, while simultaneously requiring substantially less computational time. In particular, the proposed schemes remain stable and reliable for highly stiff problems where the classical methods exhibit numerical instability or excessive error growth.

The influence of the free parameter was also examined, and the results indicate that appropriate parameter selection can further enhance computational efficiency without compromising stability or accuracy. Among the parameter choices considered, the scheme with ρ=-1/5 consistently provided the best overall performance in terms of accuracy–efficiency balance.

Data Availability Statement

Data will be made available on request.

Funding

This work was supported without any funding.

Conflicts of Interest

The authors declare no conflicts of interest.

AI Use Statement

The authors declare that generative AI (ChatGPT, version 5.2) was used for language assistance in the preparation of this manuscript. All content was reviewed and approved by the authors.

Ethical Approval and Consent to Participate

Not applicable.

References

[1] Bavarian, M., Soroush, M., Kevrekidis, I. G., & Benziger, J. B. (2010). Mathematical modeling, steady-state and dynamic behavior, and control of fuel cells: A review. *Industrial & engineering chemistry research, 49*(17), 7922-7950. [CrossRef]

[2] Dameris, M., Grewe, V., Ponater, M., Deckert, R., Eyring, V., Mager, F., ... & Giorgetta, M. A. (2005). Long-term changes and variability in a transient simulation with a chemistry-climate model employing realistic forcing. *Atmospheric Chemistry and Physics, 5*(8), 2121-2145. [CrossRef]

[3] Zhao, Z., Tan, D., & Li, K. (2019). Transient behaviors of multiscale megawatt power electronics systems—Part I: Characteristics and analysis. *IEEE Journal of Emerging and Selected Topics in Power Electronics, 7*(1), 7-17. [CrossRef]

[4] Subedi, S., Rauniyar, M., Ishaq, S., Hansen, T. M., Tonkoski, R., Shirazi, M., ... & Cicilio, P. (2021). Review of methods to accelerate electromagnetic transient simulation of power systems. *IEEE access, 9*, 89714-89731. [CrossRef]

[5] Lambert, J. D. (1973). *Computational Methods in Ordinary Differential Equations*. Chichester, New York: Wiley.

[6] Ibrahim, Z. B., Ijam, H. M., Aksah, S. J., & Abd Rasid, N. (2024). Enhancing accuracy and efficiency in stiff ODE integration using variable step diagonal BBDF approaches. *Malaysian Journal of Fundamental and Applied Sciences, 20*(5), 1083-1100. [CrossRef]

[7] Gear, C. W. (1971). *Numerical Initial Value Problems in Ordinary Differential Equations*. London: Prentice-Hall.

[8] Wanner, G., & Hairer, E. (1996). *Solving ordinary differential equations II* (Vol. 375, p. 98). New York: Springer Berlin Heidelberg.

[9] Alexander, R. (1977). Diagonally implicit Runge–Kutta methods for stiff ODE's. *SIAM Journal on Numerical analysis, 14*(6), 1006-1021. [CrossRef]

[10] Mahnken, R., & Westermann, H. (2024). Construction of A-stable explicit last-stage diagonal implicit Runge–Kutta (ELDIRK) methods. *Computational Mechanics, 74*(3), 485-507. [CrossRef]

[11] Liu, X., Xia, Y., Luo, H., & Xuan, L. (2016). A comparative study of Rosenbrock-type and implicit Runge-Kutta time integration for discontinuous Galerkin method for unsteady 3D compressible Navier-Stokes equations. *Communications in Computational Physics, 20*(4), 1016-1044. [CrossRef]

[12] Rothe, S., Hamkar, A. W., Quint, K. J., & Hartmann, S. (2012). Comparison of diagonal-implicit, linear-implicit and half-explicit Runge–Kutta methods in non-linear finite element analyses. *Archive of Applied Mechanics, 82*(8), 1057-1074. [CrossRef]

[13] Ferracina, L., & Spijker, M. N. (2008). Strong stability of singly-diagonally-implicit Runge–Kutta methods. *Applied Numerical Mathematics, 58*(11), 1675-1686. [CrossRef]

[14] Hazizah, M. I., Saufianin, J. A., Ahmad, F. N. R., Norshakila, A. R., Athraa, A., Nurul, H. M. A., & Fatimah, H. (2024). Numerical Solutions of Stiff Chemical Reaction Problems using Hybrid Block Backward Differentiation Formula. *Journal of Advanced Research, 25*(1), 100-115.

[15] Azhari, N. A. I., Abd Rasid, N., Ijam, H. M., Arsad, R., Kamal, M. H. A., Salih, N. M., & Aksah, S. J. (2025). Designing an implicit Block Backward Differentiation Formula (BBDF) for stiff ordinary differential equations. *Data Analytics and Applied Mathematics (DAAM)*, 16-30. [CrossRef]

[16] Ibrahim, Z. B., Othman, K. I., & Suleiman, M. (2007). Implicit r-point block backward differentiation formula for solving first-order stiff ODEs. *Applied Mathematics and Computation, 186*(1), 558-565. [CrossRef]

[17] Yakubu, S. D., & Sibanda, P. (2025). A novel one-step optimized hybrid block method for solving general second-order ordinary differential equations. *Recent Advances in Natural Sciences*, 135-135. [CrossRef]

[18] Gupta, G. K., Sacks-Davis, R., & Tescher, P. E. (1985). A review of recent developments in solving ODEs. *ACM Computing Surveys (CSUR), 17*(1), 5-47. [CrossRef]

[19] Byrne, G. D., & Hindmarsh, A. C. (1987). Stiff ODE solvers: A review of current and coming attractions. *Journal of Computational physics, 70*(1), 1-62. [CrossRef]

[20] Singla, R., Singh, G., Ramos, H., & Kanwar, V. (2023). Development of a Higher-Order $\mathcal{A}$-Stable Block Approach with Symmetric Hybrid Points and an Adaptive Step-Size Strategy for Integrating Differential Systems Efficiently. *Symmetry, 15*(9), 1635. [CrossRef]

[21] Söderlind, G. (2002). Automatic control and adaptive time-stepping. *Numerical Algorithms, 31*(1), 281-310. [CrossRef]

[22] Tao, M., Owhadi, H., & Marsden, J. E. (2010). Nonintrusive and structure preserving multiscale integration of stiff ODEs, SDEs, and Hamiltonian systems with hidden slow dynamics via flow averaging. *Multiscale Modeling & Simulation, 8*(4), 1269-1324.

[23] Tao, M., Owhadi, H., & Marsden, J. E. (2010). Nonintrusive and structure preserving multiscale integration of stiff ODEs, SDEs, and Hamiltonian systems with hidden slow dynamics via flow averaging. *Multiscale Modeling & Simulation, 8*(4), 1269-1324. [CrossRef]

[24] Ascher, U. M., & Petzold, L. R. (1993). Stability of computational methods for constrained dynamics systems. *SIAM Journal on Scientific Computing, 14*(1), 95-120. [CrossRef]

[25] Arnold, M., Burgermeister, B., & Eichberger, A. (2007). Linearly implicit time integration methods in real-time applications: DAEs and stiff ODEs. *Multibody System*

Dynamics, *17*(2), 99-117. [CrossRef]

[26] Musa, H., Suleiman, M. B., Ismail, F., Senu, N., & Ibrahim, Z. B. (2013). An accurate block solver for stiff initial value problems. *International Scholarly Research Notices*, *2013*(1), 567451. [CrossRef]

[27] Chu, Y., & You, F. (2015). Model-based integration of control and operations: Overview, challenges, advances, and opportunities. *Computers & Chemical Engineering*, *83*, 2-20. [CrossRef]

[28] Geart, C. W., & Watanabe, D. S. (1974). Stability and convergence of variable order multistep methods. *SIAM Journal on Numerical Analysis*, *11*(5), 1044-1058. [CrossRef]

[29] Suleiman, M. B., Musa, H., Ismail, F., & Senu, N. (2013). A new variable step size block backward differentiation formula for solving stiff initial value problems. *International Journal of Computer Mathematics*, *90*(11), 2391-2408. [CrossRef]

[30] Fronk, C., & Petzold, L. (2024). Training stiff neural ordinary differential equations with implicit single-step methods. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *34*(12). [CrossRef]

[31] Martín-Vaquero, J., & Vigo-Aguiar, J. (2007). Adapted BDF algorithms: higher-order methods and their stability. *Journal of Scientific Computing*, *32*(2), 287-313. [CrossRef]

[32] Vijitha, H. Y. K. (1985). *Variable stepsize variable order multistep methods for stiff ordinary differential equations* (Doctoral dissertation). Iowa State University.

[33] Enright, W. H., & Pryce, J. D. (1987). Two FORTRAN packages for assessing initial value methods. *ACM Transactions on Mathematical Software (TOMS)*, *13*(1), 1-27. [CrossRef]

[34] Alhassan, B., Yusuf, H., & Adamu, A. (2024). Analysis of convergence and stability properties of diagonally implicit 3-point block backward differentiation formula for first order stiff initial value problems. *UMYU scientifica*, *3*(2), 202-210. [CrossRef]

[35] Ibrahim, Z. B., & Zawawi, I. S. M. (2019). A Stable Fourth Order Block Backward Differentiation Formulas α for Solving Stiff Initial Value Problems. *ASM Science Journal*, *Special Issue 6*, 60-66.

[36] Aminikhah, H., & Hemmatnezhad, M. (2011). An effective modification of the homotopy perturbation method for stiff systems of ordinary differential equations. *Applied Mathematics Letters*, *24*(9), 1502-1508. [CrossRef]

[37] Wu, X. Y. (1998). A sixth-order A-stable explicit one-step method for stiff systems. *Computers & Mathematics with Applications*, *35*(9), 59-64. [CrossRef]

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