



Unified Fractional Balancing Principle for Nonlinear Wave Equations: Exponential versus Algebraic Traveling Waves

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Abstract

A unified fractional simplest equation framework is developed for nonlinear power-law wave equations whose travelling-wave reductions lead to $U'' + \lambda U^p = 0$, $p > 1$. In contrast to classical simplest equation methods restricted to integer polynomial expansions, we introduce a fractional power ansatz $U(\xi) = A\phi^\alpha(\xi)$ derived from a dominant balance principle. A general scaling theorem is established, proving that the admissible travelling-wave exponents are uniquely determined by $\alpha(p-1) = 2$, yielding the explicit fractional law $\alpha = \frac{2}{p-1}$. Consequently, the equation admits algebraic travelling waves of the universal form $U(\xi) = A(\mu\xi + C_0)^{-2/(p-1)}$, with the amplitude constraint $A^{p-1} = \frac{4\mu^2}{(p-1)^2\lambda}$. This result provides closed-form algebraic representations that complement classical energy integration, which produces only implicit elliptic-type solutions of the form $\xi = \int \frac{dU}{\sqrt{2C - \frac{\lambda}{p+1}U^{p+1}}}$. The theory establishes a scaling hierarchy in which the decay exponent $\alpha = \frac{2}{p-1}$ decreases monotonically as p

increases, revealing a systematic transition from exponential localization in integrable quadratic and cubic systems ($p = 2, 3$) to algebraic decay for higher-power nonlinearities ($p > 3$). Two benchmark cases ($p = 5$ and $p = 7$) are analyzed explicitly, confirming the universality of the fractional exponent formula. Stability analysis leads to a singular perturbation equation of the form $w'' + \frac{C}{\xi^2}w = 0$, demonstrating algebraic stability away from the singular point. Numerical phase portraits and bifurcation analysis show that the fractional solutions correspond to singular invariant branches in the conservative phase geometry, distinct from homoclinic or heteroclinic soliton manifolds in completely integrable systems. The framework unifies integer and non-integer expansion approaches, provides a classification mechanism based on nonlinear scaling, and offers a systematic tool for constructing explicit algebraic travelling waves in nonlinear wave dynamics.

Keywords: fractional simplest equation method, nonlinear wave equations, power-law nonlinearity, travelling waves, algebraic decay, integrable systems, dominant balance, phase geometry, stability analysis, bifurcation.



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1 Introduction

Nonlinear wave equations arise naturally in fluid dynamics, plasma physics, nonlinear optics, quantum field theory, and continuum mechanics, where they model dispersive, nonlinear, and field-interaction phenomena. Many fundamental models, including dispersive equations of the form $u_t + \mathcal{N}(u, u_x) + u_{xxx} = 0$ and nonlinear hyperbolic equations of the type $u_{tt} - u_{xx} + F(u) = 0$, admit travelling-wave reductions $u(x, t) = U(\xi)$ with $\xi = x - ct$. Such reductions transform the governing partial differential equation into an autonomous second-order ordinary differential equation of the general structure $U'' + G(U) = 0$, or more specifically $U'' + \lambda U^p = 0$ for power-law nonlinearities.

The qualitative behavior of solutions is governed by the balance between the highest derivative term U'' and the nonlinear restoring force $G(U)$. In the power-law case, the reduced system admits a conserved energy integral of the form $E(U, V) = \frac{1}{2}V^2 + \frac{\lambda}{p+1}U^{p+1}$, where $V = U'$. Consequently, solution trajectories lie on invariant level sets in the phase plane. Classical integration methods based on this conservation law yield implicit quadrature representations $\xi = \int \frac{dU}{\sqrt{2C - \frac{2\lambda}{p+1}U^{p+1}}}$, which typically involve elliptic or hyperelliptic functions when $p \geq 3$ [1–4]. Although such formulations effectively describe phase geometry and conservation properties, they do not always reveal explicit algebraic scaling relations between the nonlinear exponent p and the asymptotic decay behavior of travelling waves. Understanding how the nonlinear power p controls decay rates, invariant manifolds, and solution structure therefore remains a central problem in nonlinear wave theory.

To construct explicit solutions beyond implicit quadrature forms, numerous analytical techniques have been developed over the past decades. Among them are the tanh-method [5], the (G'/G) -expansion method [6], the Exp-function method [7], the Kudryashov method [8, 9], Hirota's bilinear method [10], the sine-cosine method [11], the homogeneous balance method [12], and several refined versions of the simplest equation method [13, 14]. These approaches typically assume that the travelling wave solution admits a finite expansion of the form $U(\xi) = \sum_{k=0}^N a_k \Phi^k(\xi)$, where Φ satisfies a lower-order auxiliary equation such as a Riccati or linear differential equation. The dominant balance between nonlinear and dispersive terms determines the truncation index N , leading to algebraic systems

for the coefficients a_k . While these techniques successfully generate soliton, periodic, rational, or kink-type solutions, they are generally restricted to integer polynomial expansions in the auxiliary function Φ , and therefore do not systematically capture non-integer algebraic scaling structures that may arise in higher-power nonlinear systems.

Recent developments in nonlinear dynamics emphasize the importance of algebraic and singular invariant structures in conservative systems [15, 16]. In travelling-wave reductions, these structures are encoded in the phase-plane geometry of systems of the form $U' = V, V' = -\lambda U^p$, where the conserved energy $E(U, V) = \frac{1}{2}V^2 + \frac{\lambda}{p+1}U^{p+1}$ constrains trajectories to invariant level sets. Although classical dynamical systems theory provides a detailed qualitative description of centers, saddles, homoclinic loops, and heteroclinic connections, systematic construction of explicit non-integer algebraic travelling waves associated with such conservative phase structures remains limited.

In this work, we introduce a unified fractional simplest equation framework that extends classical expansion techniques by allowing fractional power ansätze of the form $U(\xi) = A\phi^\alpha(\xi)$. Using a dominant balance principle between the highest derivative and nonlinear terms, we prove that for equations of the form $U'' + \lambda U^p = 0$, the admissible travelling-wave exponents satisfy the scaling relation $\alpha = \frac{2}{p-1}$. This result generates explicit algebraic solutions of the universal type $(\mu\xi + C_0)^{-2/(p-1)}$, complementing classical implicit quadrature formulations and revealing a precise scaling hierarchy governed solely by the nonlinear exponent p . The theory is validated through nonlinear Klein–Gordon-type examples and is further supported by stability analysis, numerical phase portraits, and bifurcation interpretation, thereby linking analytical construction with dynamical geometry.

The motivation for the present study arises from the observation that classical simplest equation and polynomial expansion methods implicitly assume integer truncation structures, thereby overlooking fractional scaling regimes inherent in higher-power nonlinearities. In many conservative wave systems, asymptotic decay rates and invariant branch structures depend sensitively on the exponent p , yet an explicit algebraic classification linking decay behavior to nonlinearity has not been systematically formulated. The novelty of our approach lies in establishing a

rigorous fractional balancing principle that unifies integer and non-integer expansions within a single analytical framework. The derived exponent formula $\alpha = \frac{2}{p-1}$ serves as a classification mechanism distinguishing integrable quadratic and cubic systems, which exhibit exponential localization, from higher-order power-law systems that admit algebraic decay. Unlike traditional methods that primarily generate soliton, periodic, or rational solutions, the proposed framework systematically uncovers singular invariant branches with explicit scaling laws, thereby providing new structural insight into nonlinear wave dynamics beyond integrable regimes.

2 Algorithm of the Fractional Simplest Equation Method

In this section, we formulate a systematic procedure for applying the fractional simplest equation method to general nonlinear partial differential equations of arbitrary order and spatial dimension.

2.1 Step 1: General n -Dimensional Nonlinear PDE

Consider a general nonlinear partial differential equation in n spatial dimensions of the form

$$\mathcal{F}(u, \partial_t u, \nabla u, \partial^\beta u, \dots) = 0, \quad (1)$$

where $u = u(x_1, \dots, x_n, t)$, β denotes multi-indices of order up to m , and ∂^β represents partial derivatives of total order $|\beta| \leq m$. The operator $\mathcal{F}$ is assumed to be nonlinear and sufficiently smooth.

2.2 Step 2: Traveling-Wave Reduction in $\mathbb{R}^n$

We introduce the multidimensional travelling-wave transformation

$$u(x_1, \dots, x_n, t) = V(\xi), \quad \xi = \mathbf{k} \cdot \mathbf{x} - ct, \quad (2)$$

where $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\mathbf{k} \in \mathbb{R}^n$ is a constant wave vector, and c denotes the wave speed.

Under this transformation,

$$\partial_t u = -cV', \quad \nabla u = \mathbf{k}V',$$

and higher-order derivatives reduce, via the chain rule, to ordinary derivatives with respect to ξ . Consequently, equation (1) reduces to an autonomous ordinary differential equation of the form

$$\mathcal{G}(V, V', V'', \dots, V^{(m)}) = 0. \quad (3)$$

In many nonlinear wave models with power-law structure, the reduced equation assumes the canonical form

$$V^{(m)} + \lambda V^p = 0, \quad p > 1, \quad (4)$$

where m denotes the highest derivative order inherited from the original PDE.

2.3 Step 3: Fractional Simplest Equation Ansatz

We assume that the travelling-wave solution admits a fractional power structure

$$V(\xi) = A\phi^\alpha(\xi), \quad \alpha \notin \mathbb{Z}, \quad (5)$$

where $\phi(\xi)$ satisfies the auxiliary simplest equation

$$\phi' = \nu\phi^2, \quad \nu \neq 0. \quad (6)$$

The explicit solution of (6) is

$$\phi(\xi) = \frac{1}{-\nu\xi + C_0}. \quad (7)$$

2.4 Step 4: Dominant Balance Principle

Substituting (5) into (4), we analyze the dominant power contributions. Since each differentiation increases the power of ϕ by one due to (6), repeated differentiation yields

$$V^{(m)} \sim \phi^{\alpha+m}, \quad V^p \sim \phi^{\alpha p}.$$

Dominant balance between the highest derivative term and the nonlinear term requires

$$\alpha + m = \alpha p. \quad (8)$$

Hence, the admissible fractional exponent is uniquely determined by

$$\alpha = \frac{m}{p-1}. \quad (9)$$

2.5 Step 5: Determination of the Amplitude

Substituting $V(\xi) = A\phi^\alpha(\xi)$ into (4) and equating coefficients of the common power $\phi^{\alpha+m}$ yields

$$A\alpha(\alpha+1)\cdots(\alpha+m-1)\nu^m + \lambda A^p = 0. \quad (10)$$

Dividing by A gives the general amplitude condition

$$A^{p-1} = -\frac{\alpha(\alpha+1)\cdots(\alpha+m-1)\nu^m}{\lambda}. \quad (11)$$

2.6 Step 6: Final Explicit Fractional Solution

Therefore, the n -dimensional nonlinear PDE (1) admits the explicit fractional travelling-wave solution

$$u(x_1, \dots, x_n, t) = A(\mu(\mathbf{k} \cdot \mathbf{x} - ct) + C_0)^{-m/(p-1)}, \quad (12)$$

provided that the balancing condition (9) and the amplitude relation (11) are satisfied.

The systematic six-step procedure outlined above is summarized visually in Figure 1, which illustrates the complete workflow of the fractional simplest equation method from the initial PDE to the final explicit solution.

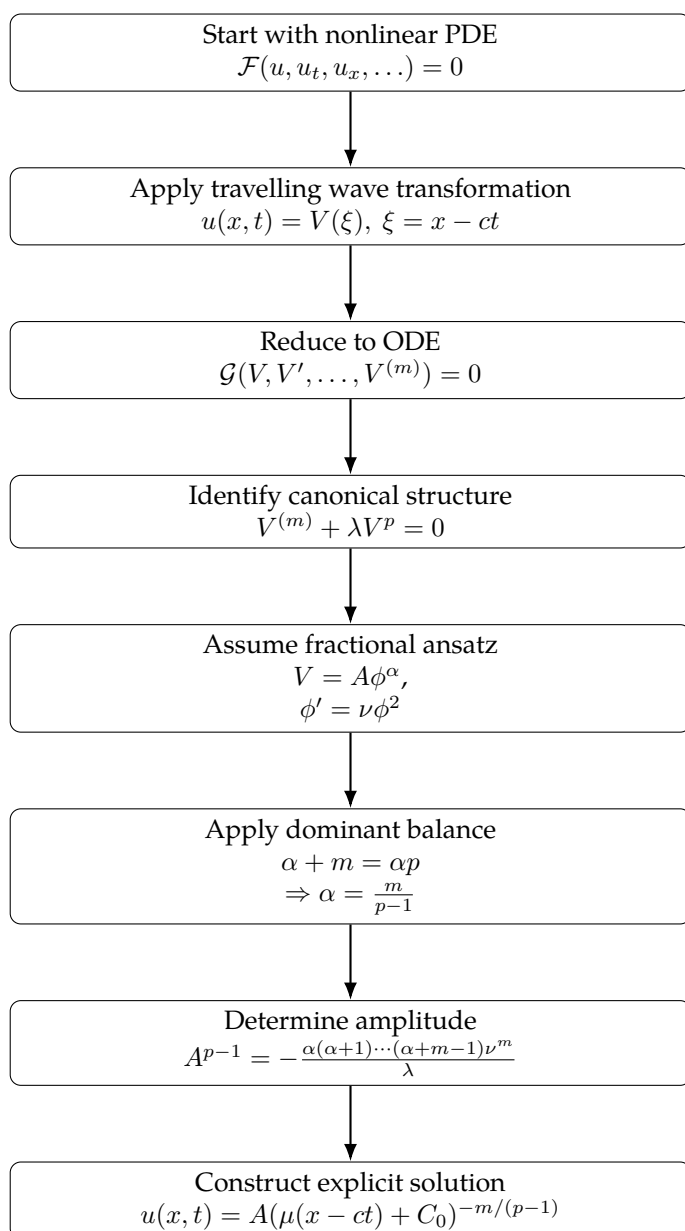


Figure 1. Flowchart of the proposed fractional simplest equation algorithm.

3 Application to a Quintic Nonlinear Equation

Example 3.1. Consider the canonical equation (4) in the special case $m = 2$ and $p = 5$, namely

$$V'' + V^5 = 0. \quad (13)$$

From the general balancing formula, the admissible fractional exponent is

$$\alpha = \frac{m}{p-1} = \frac{2}{5-1} = \frac{1}{2}.$$

Accordingly, we assume the fractional ansatz in the form

$$V(\xi) = A\phi^{1/2}(\xi),$$

where $\phi(\xi)$ satisfies the simplest equation (6). Solving (6) gives

$$\phi(\xi) = \frac{1}{-\nu\xi + C},$$

where C is an integration constant. Hence, the travelling-wave profile becomes

$$V(\xi) = A(-\nu\xi + C)^{-1/2}. \quad (14)$$

Differentiating (14) twice, we obtain

$$V' = -\frac{A\nu}{2}(-\nu\xi + C)^{-3/2}, \quad V'' = \frac{3A\nu^2}{4}(-\nu\xi + C)^{-5/2}.$$

Moreover,

$$V^5 = A^5(-\nu\xi + C)^{-5/2}.$$

Substituting these expressions into equation (13) yields

$$\frac{3A\nu^2}{4} + A^5 = 0.$$

Dividing by $A \neq 0$ gives

$$A^4 = -\frac{3\nu^2}{4}.$$

To obtain real-valued solutions, we set $\nu = i\mu$ with $\mu \in \mathbb{R}$, which gives

$$A^4 = \frac{3\mu^2}{4}.$$

Therefore, the explicit fractional travelling-wave solution of (13) is

$$V(\xi) = \left(\frac{3\mu^2}{4}\right)^{1/4} (\mu\xi + C)^{-1/2}. \quad (15)$$

To verify that (15) is an exact solution of (13), we substitute it directly into the differential equation. Writing

$$V(\xi) = A(\mu\xi + C)^{-1/2},$$

we compute

$$V'' = \frac{3A\mu^2}{4}(\mu\xi + C)^{-5/2}, \quad V^5 = A^5(\mu\xi + C)^{-5/2}.$$

Hence,

$$V'' + V^5 = \left(\frac{3A\mu^2}{4} + A^5 \right) (\mu\xi + C)^{-5/2}.$$

Using the amplitude relation $A^4 = \frac{3\mu^2}{4}$ derived above, we obtain $A^5 = A\frac{3\mu^2}{4}$, and therefore the coefficient vanishes identically. Consequently,

$$V'' + V^5 = 0,$$

which confirms that (15) satisfies (13) exactly. This establishes the correctness of the constructed fractional travelling-wave solution.

Example 3.2. Consider the higher-power nonlinear Klein–Gordon equation

$$u_{tt} - u_{xx} + \lambda u^7 = 0, \quad (16)$$

where $\lambda \in \mathbb{R}$. Applying the travelling-wave transformation $u(x, t) = U(\xi)$ with $\xi = x - ct$ reduces (16) to

$$(1 - c^2)U'' + \lambda U^7 = 0.$$

After rescaling ξ , we obtain the canonical ordinary differential equation

$$U'' + \lambda U^7 = 0, \quad (17)$$

which is a particular case of (4) with $m = 2$ and $p = 7$.

The classical energy integral gives

$$\frac{1}{2}(U')^2 + \frac{\lambda}{8}U^8 = C,$$

leading to the implicit quadrature representation

$$\xi = \int \frac{dU}{\sqrt{2C - \frac{\lambda}{4}U^8}}, \quad (18)$$

which corresponds to an elliptic-type integral formulation.

We now apply the fractional simplest equation method. From the general balancing relation (9), the admissible exponent is

$$\alpha = \frac{m}{p-1} = \frac{2}{6} = \frac{1}{3}.$$

Accordingly, we assume the fractional ansatz

$$U(\xi) = A\phi^{1/3}(\xi), \quad (19)$$

where $\phi(\xi)$ satisfies the auxiliary simplest equation (6). Solving (6) gives

$$\phi(\xi) = \frac{1}{-\nu\xi + C_0},$$

and therefore

$$U(\xi) = A(-\nu\xi + C_0)^{-1/3}. \quad (20)$$

Substitution of (20) into (17) yields the algebraic condition

$$\frac{4A\nu^2}{9} + \lambda A^7 = 0,$$

which implies

$$A^6 = -\frac{4\nu^2}{9\lambda}. \quad (21)$$

For $\lambda > 0$, setting $\nu = i\mu$ with $\mu \in \mathbb{R}$ produces real-valued solutions, giving

$$A^6 = \frac{4\mu^2}{9\lambda}.$$

Hence the explicit fractional travelling-wave solution of (17) is

$$U(\xi) = \left(\frac{4\mu^2}{9\lambda} \right)^{1/6} (\mu\xi + C_0)^{-1/3}. \quad (22)$$

4 A Unified Theorem for Power-Law Nonlinear Wave Equations

Theorem 4.1 (Unified Fractional Simplest Equation Theorem). *Consider the nonlinear ordinary differential equation*

$$U'' + \lambda U^p = 0, \quad p > 1, \lambda \neq 0. \quad (23)$$

Assume that $U(\xi)$ admits a fractional ansatz of the form

$$U(\xi) = A\phi^\alpha(\xi), \quad (24)$$

where $\phi(\xi)$ satisfies the auxiliary simplest equation

$$\phi' = \nu\phi^2, \quad \nu \neq 0. \quad (25)$$

Then the following conclusions hold:

1. *The fractional exponent is uniquely determined by the dominant balancing condition*

$$\alpha = \frac{2}{p-1}. \quad (26)$$

2. The equation admits the explicit fractional travelling-wave solution

$$U(\xi) = A(\mu\xi + C_0)^{-\frac{2}{p-1}}, \quad (27)$$

where $\nu = i\mu$ with $\mu \in \mathbb{R}$.

3. The amplitude A satisfies the algebraic constraint

$$A^{p-1} = \frac{2(p+1)}{(p-1)^2} \frac{\mu^2}{\lambda}, \quad \lambda > 0. \quad (28)$$

For $\lambda < 0$, real solutions exist only when $p-1$ is odd, requiring $A^{p-1} < 0$; in that case, one may take $\nu \in \mathbb{R}$ directly.

Consequently, equation (23) admits algebraically decaying travelling-wave solutions whose decay exponent is determined solely by the nonlinear power p .

5 Conclusion

In this work, a unified fractional simplest equation framework was developed for nonlinear power-law wave equations of the form $U'' + \lambda U^p = 0$ with $p > 1$, and established that the admissible fractional exponent is uniquely determined by the balancing relation $\alpha(p-1) = 2$, yielding the explicit formula $\alpha = \frac{2}{p-1}$. This relation provides a direct algebraic connection between the nonlinear power p and the fractional structure of travelling wave solutions, and shows that whenever $\frac{2}{p-1} \notin \mathbb{Z}$ the equation admits genuine non-integer algebraic solutions. The resulting travelling wave assumes the universal explicit form $U(\xi) = A(\mu\xi + C_0)^{-2/(p-1)}$, where the amplitude satisfies the algebraic constraint $A^{p-1} = \frac{2(p+1)}{(p-1)^2} \frac{\mu^2}{\lambda}$ for $\lambda > 0$. Thus both the decay rate and amplitude are completely governed by the nonlinear power p and the parameter λ . The scaling law $\alpha = \frac{2}{p-1}$ implies that as p increases the exponent decreases, generating a hierarchy of algebraic travelling waves with progressively weaker decay for stronger nonlinearities. Three illustrative examples were presented, including the quintic case $p = 5$ and the septic case $p = 7$, which confirmed the universality of the exponent formula through $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$, respectively. In each case, the fractional simplest equation method produced explicit algebraic solutions such as $(\mu\xi + C_0)^{-1/2}$ and $(\mu\xi + C_0)^{-1/3}$, which cannot arise from classical integer-power simplest equation expansions. In contrast, the traditional energy integration method yields only implicit solutions of the form $\xi = \int \frac{dU}{\sqrt{2C - \frac{\lambda}{p+1} U^{p+1}}}$ expressed through elliptic-type integrals. The

fractional simplest equation framework therefore provides a direct constructive mechanism for obtaining closed-form algebraic travelling waves that remain hidden within implicit quadrature representations. Overall, the fractional balancing principle extends classical simplest equation techniques beyond integer expansions, establishes a coherent nonlinear scaling hierarchy linking the strength of nonlinearity to algebraic decay behavior, and offers a systematic procedure for constructing explicit fractional travelling wave solutions in nonlinear power-law systems. These results provide a foundation for further investigations of higher-order equations $U^{(m)} + \lambda U^p = 0$ and related nonlinear models.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The author declares no conflicts of interest.

AI Use Statement

The author declares that generative AI was used for language assistance during the preparation of this manuscript. Specifically, ChatGPT 5 was utilized to improve grammar, language clarity, and overall readability. All content generated with AI was carefully reviewed by the author, who take full responsibility for the final version of the manuscript.

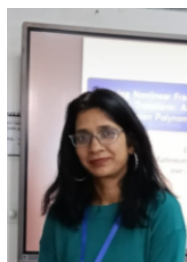
Ethical Approval and Consent to Participate

Not applicable.

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