RESEARCH ARTICLE

A New Variant of Benes Network: Its Topological Characterisation and Comparative Analysis

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Abstract

The modern era is characterized by rapid advancements in technology, wherein the design and topology of interconnection networks play a pivotal role in enabling efficient communication systems. The analysis of topological and structural characteristics of interconnection networks, however, remains a challenging task. Graph theory facilitates this process by enabling analytical and efficient solutions through the use of numerical parameters known as topological descriptors. These descriptors have proven to be highly significant, with applications spanning computer science, chemistry, biology, and related domains. This paper deals with the evaluation of topological descriptors for an n-dimensional multistage interconnection network, namely the benes network, denoted as $BB(n)$. Additionally, a new variant of the interconnection network is derived from the benes network, called the augmented benes network and denoted as $BB^*(n)$. The topological descriptors for the derived network are also determined in this work. Furthermore, the benes network and the augmented benes network undergo a comparative analysis based on selected network parameters, which helps in understanding the efficiency of the newly derived network. A broadcasting algorithm for the augmented benes network is also presented.

Keywords: complex networks, benes network, augmented benes network, distance based descriptors, Network parameters, broadcasting algorithm, network structures.

1 Introduction

Networking was first made use in 1950s, in telecommunications to connect phone calls through switchboards that switched between electric connections or switches to make the connections. In the computing world, networking provides a method for fast communication between multiple processors of a computer and between multiple computers connected to a network. The network is composed of links and switches. A single processor, a cluster of processors, or a set of memory modules can make up each node in the network, which is made up

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of interconnections and switches that help transfer data from the source to the destination. A network is defined by its layout, transit scheme, switching mechanism, and diffusion strategy. Topology is indeed the arrangement used to link individual switches to other components such as processors, memories and other switches. The interconnection network, a crucial component of parallel systems, facilitates data interchange among processors through its architectural configuration, determined by switch layout and wiring patterns. Structurally, this network enables task division and simultaneous execution, reducing processing time. Key factors influencing its topology and routing algorithms include switch degree, routing technique, and internal buffer capacity. Effective interconnectivity is essential for achieving higher transmission speeds between parallel system components, underscoring the importance of structural analysis in optimizing network performance.

The design of an interconnection network and analysing its efficiency is an important research problem in parallel computing. In the field of computer science, average distance is a principal parameter that is used to measure the communication cost of networks. Researchers have also developed various other parameters to analyse and differentiate interconnection network models. These parameters include degree, diameter, message traffic density, network throughput, graph density, total connectivity, packing density and network cost. A network is said to be efficient if it has low and constant degree, minimum diameter, minimum average distance, low message traffic density, high network throughput, high graph density, high total connectivity, low network cost and high packing density [1]. The interconnection networks can be tailored into a graph with vertices representing the processors and edges representing the communication links. Even, graphs can be used to design new interconnection networks. From the graph models of the networks, efficiently one can characterise its topological properties. Therefore, it is purposeful to study the topological descriptors of networks using the graph models. The idea of topological descriptors was first introduced by H. Wiener in 1947 [2]. Eventually many other topological descriptors were introduced by researchers. The parameter average distance is nothing, but the Wiener index itself. These descriptors are related with the efficiency of networks, and hence, can be used to study and develop new networks.

The butterfly network is one of the notable multistage interconnection networks, which is mainly used to perform fast Fourier transform [3] and other appealing applications are given in [6]. The n-dimensional butterfly is denoted as $BF(n)$ and it consists of $2^n(n+1)$ nodes arranged in $n+1$ levels, such that each level has 2^n nodes. Each node of $BF(n)$ can be assigned with a label (j,k), where j is a n-bit binary number that denotes the column of the node and k is the level of the node ranging from 0 to n. Two nodes (j,k) and (j',k') of $BF(n)$ are linked, if $k' = k+1$ and either j and j' are identical or j and j' differ precisely in k^{th} bit. The benes network is back to back butterflies, whose n^{th} dimensional network is denoted as $BB(n)$ and it consists of $2^n(2n+1)$ nodes arranged in $2n+1$ levels. In $BB(n)$, each node can be labelled as (j,k) where j is a n-bit binary number and $0 \leq k \leq 2n$. Two nodes of $BB(n)$, (j,k) and (j',k') are linked, if $k' = k+1$ and either j and j' are identical or j and j' differ precisely in k^{th} bit for $0 \leq k \leq n-1$ and they differ in $(2n-k-1)^{\text{th}}$ bit for $n \leq k \leq 2n$. The Benes network is a multi-stage interconnection network, which has considerable applications and is known for permutation routing [3, 6]. In topological aspect, each node is considered as a vertex and the links between them as edges. Manuel et al. [3] has described the diamond representation for butterfly network as well as benes network together with their vertex labelling. The 3-dimensional benes network, $BB(3)$ is presented in Figures 1 and 2 respectively.

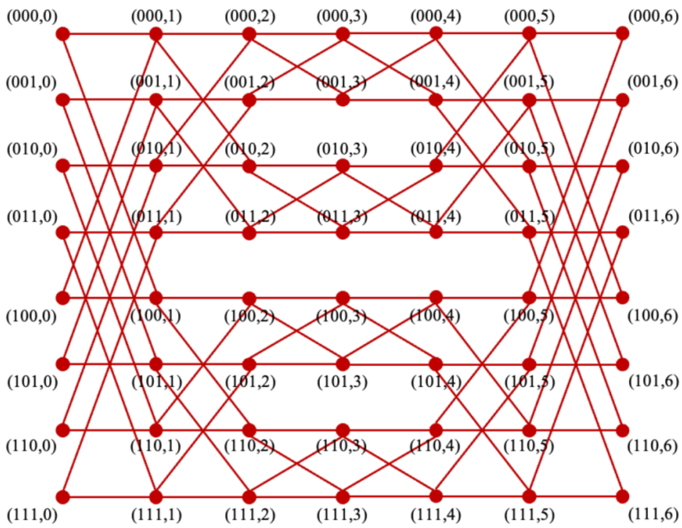


Figure 1. The normal representation of 3-dimensional benes network, $\mathbf{BB(3)}$.

In this paper, a new variant of the Benes network, named the augmented Benes network, is derived. The n-dimensional augmented benes network is denoted as $BB^*(n), n \geq 1$ consists of $2^n(2n+1)$ nodes arranged

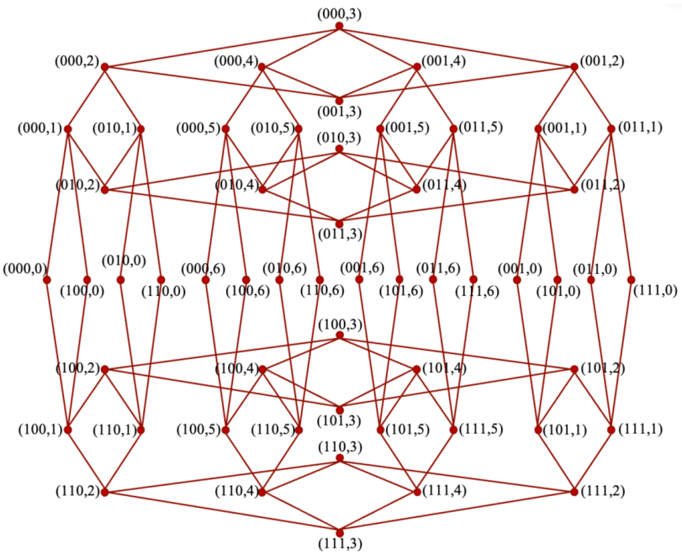


Figure 2. The diamond representation for 3-dimensional benes network, $\mathbf{BB(3)}$.

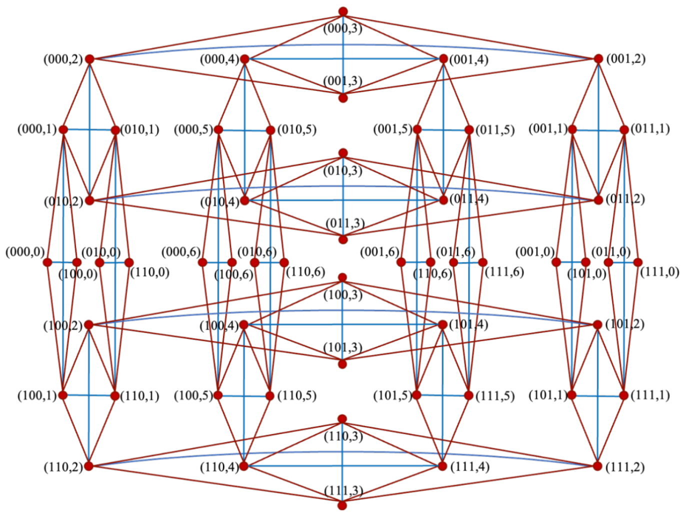


Figure 3. The 3-dimensional augmented benes network, $\mathbf{BB}*(\mathbf{3})$.

in $2n+1$ levels and two nodes (j,k) and (j',k') are linked, if the following conditions are satisfied.

1. $k' = k+1$ and j, j' are identical or j, j' differ in k^{th} bit for $0 \leq k \leq n-1$ and they differ in $(2n-k-1)^{\text{th}}$ bit for $n \leq k \leq 2n$.

2. $k' = k$ and j, j' differ in k^{th} bit for $k = 0$, or j, j' differ in k^{th} bit or precisely in $(k-1)^{\text{th}}$ bit for $1 \leq k \leq n$, and for $n+1 \leq k \leq 2n$ they differ in $(2n-k)^{\text{th}}$ or $(2n-k-1)^{\text{th}}$ bit.

Hence, the number of vertices of the n-dimensional augmented benes network remains the same as that of the n-dimensional benes network, whereas the number of edges increases to $2^{n-1}(12n-1)$. The 3-dimensional augmented benes network, $BB^*(3)$ is illustrated in Figure 3, which provides a better constructional clarity.

The Benes network has been studied by researchers from different perspectives, highlighting its functional capabilities and reliability [7, 8, 10–12, 20]. Topological indices and algorithmic aspects of these networks have also been investigated [10, 11]. Several researchers have investigated distance-based and degree-based topological descriptors for butterfly and Benes networks [13–16]. The literature indicates that studies on degree-based descriptors for the Benes network exist [13, 15, 16]; however, studies on distance-based descriptors are limited. In this paper, this research gap is addressed by determining the distance-based topological descriptors for the Benes network, denoted by $BB(n)$. Additionally, a new network derived from the Benes network, denoted as $BB^*(n)$, is introduced, and its topological descriptors are analyzed. Furthermore, both the Benes network and the augmented Benes network are examined using selected topological parameters, and a comparative study is conducted to understand the efficiency of the newly derived network. A broadcasting algorithm for the augmented Benes network is also proposed in this study.

2 Mathematical Terminologies

Consider a simple connected graph ζ with vertex set, $V(\zeta)$ and edge set, $E(\zeta)$. The number of edges incident to a vertex μ is the degree of that vertex and is denoted as $\deg(\mu)$. Let $d_\zeta(\mu,\eta)$ denote the shortest-path distance between the vertices μ and η in ζ. The neighborhood of a vertex μ is denoted as $N_\zeta(\mu)$, which is the set of vertices adjacent to μ, or equivalently the subgraph of ζ induced by all the vertices adjacent to μ. If the vertex μ is included in the set or induced subgraph, it is said to be closed neighborhood and is denoted as $N_\zeta[\mu]$. For more basic definitions refer to [17, 19].

The concepts of isometric subgraph, partial cubes, convex subgraph and Djokovic-Winkler (Θ) relation are to be recollected while working on topological indices [14]. The Djokovic-Winkler relation is a major concept applied in computing topological indices. For two edges $\varepsilon = \mu\eta$ and $\rho = \kappa\nu$ of ζ, if

$$d_\zeta(\eta,\nu) + d_\zeta(\mu,\kappa) \neq d_\zeta(\mu,\nu) + d_\zeta(\eta,\kappa),$$

then we say ε is related to ρ. The relation Θ is reflexive, symmetric and transitive in case of partial cubes. Its transitive closure Θ^* forms an equivalence relation in general and partitions the edge set into convex components. Let

$$\mathbf{F} = \{F_1, F_2, \ldots, F_\chi\}$$

be the Θ^* equivalence class. A partition

$$\mathcal{E} = \{\xi_1, \xi_2, \ldots, \xi_\delta\}$$

of $E(\zeta)$ is said to be coarser than partition $\mathbf{F}$ if each set ξ_i is the union of one or more Θ^*-classes of $\mathbf{F}$. The concept of strength weighted graph is presented in [14]. A strength weighted graph is denoted as

$$\zeta_{sw} = (\zeta, [w_v, s_v], s_e),$$

whose vertex weight, vertex strength and edge strength are denoted as w_v, s_v and s_e, respectively. For convenience, let us consider strength-weighted graph ζ_{sw} as ζ. For an edge $\varepsilon = \mu\eta$, the sets

$$N_\mu(\varepsilon|\zeta) = \{\lambda \in V(\zeta) : d_\zeta(\mu, \lambda) < d_\zeta(\eta, \lambda)\}$$

and

$$M_\mu(\varepsilon|\zeta) = \{\chi \in E(\zeta) : d_\zeta(\mu, \chi) < d_\zeta(\eta, \chi)\}$$

are defined.

The cardinality of sets $N_\mu(\varepsilon|\zeta)$ and $M_\mu(\varepsilon|\zeta)$ are described as

$$n_\mu(\varepsilon|\zeta) = \sum_{\lambda \in N_\mu(\varepsilon|\zeta)} w_v(\lambda),$$

and

$$m_\mu(\varepsilon|\zeta) = \sum_{\lambda \in N_\mu(\varepsilon|\zeta)} s_v(\lambda) + \sum_{\chi \in M_\mu(\varepsilon|\zeta)} s_e(\chi),$$

respectively, and the values of

$$n_\eta(\varepsilon|\zeta) \quad \text{and} \quad m_\eta(\varepsilon|\zeta)$$

are analogous.

For more terminologies and definitions refer to [14, 18]. The notion of Wiener index for vertex-weighted graphs and its chemical interpretations were studied in [4]. A comprehensive theory and applications of the Wiener index for trees are presented in [5]. The distance based topological descriptors [18] and their corresponding mathematical expression for strength-weighted graph ζ are given in Table 1.

Theorem 1. *Let ζ be a simple connected weighted graph. Let $X_i, 1 \leq i \leq a$ and $Y_j, 1 \leq j \leq b$ be subgraphs of ζ such that $X_i, Y_j \cong K_m, m \geq 2\ \forall i, j$, with $w_v(\mu) = \alpha\ \forall \mu \in V(X_i)$ and $w_v(\eta) = \beta,\ \forall \eta \in V(Y_j)$. If*

1. *Closed neighborhood of a vertex $\mu \in V(X_i)$, $N_\zeta[\mu]$ is a subgraph induced by the vertex set $\bigcup_{j=1}^{b} V(Y_j) \cup V(X_i)$ and $N_\zeta[\mu] \cong N_\zeta[\eta]$, for $\mu \in V(X_i), \eta \in V(X_h), 1 \leq i, h \leq a$.*
2. *Closed neighborhood of a vertex $\mathfrak{p} \in V(Y_j)$, $N_\zeta[\mathfrak{p}]$ is a subgraph induced by the vertex set $\bigcup_{i=1}^{a} V(X_i) \cup V(Y_j)$ and $N_\zeta[\mathfrak{p}] \cong N_\zeta[\mathfrak{q}]$, for $\mathfrak{p} \in V(Y_j), \mathfrak{q} \in V(Y_h), 1 \leq j, h \leq b$.*

Then,

$$\begin{aligned} W(\zeta) = & m^2\big(a(a-1)\alpha^2 + b(b-1)\beta^2 + ab\alpha\beta\big) \\ & + \binom{m}{2}(a\alpha^2 + b\beta^2). \end{aligned} \tag{1}$$

Proof. The graph ζ under the given conditions can be depicted as given in Figure 4. As an example, the graph ζ with $m = 3$, $a = 3$ and $b = 2$ is presented in Figure 5.

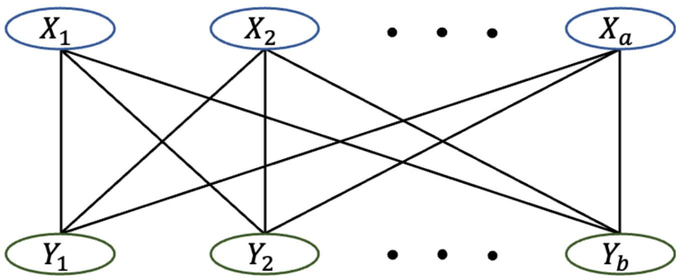


Figure 4. The graph ζ, where $X_i, Y_j \cong K_m; 1 \leq i \leq a$, $1 \leq j \leq b, m \geq 2$.

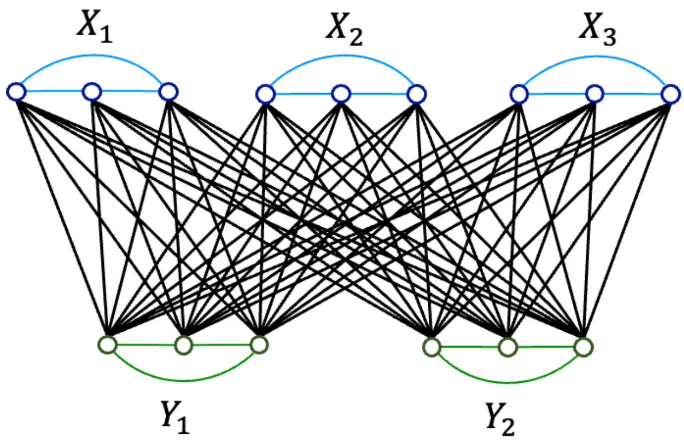


Figure 5. The graph ζ, where $X_i, Y_j \cong K_3; 1 \leq i \leq 3$, $1 \leq j \leq 2$.

By definition, $W(\zeta) = \sum_{\{\mu,\eta\} \in V(\zeta)} w_v(\mu) w_v(\eta) d_\zeta(\mu, \eta)$. To determine the Wiener index of ζ, the vertex set $\{\mu, \eta\} \subseteq V(\zeta)$ can be chosen from three cases.

- **Case 1**: $\mu \in V(X_i), \eta \in V(X_j);\ 1 \leq i, j \leq a$.
- **Case 2**: $\mu \in V(Y_i), \eta \in V(Y_j);\ 1 \leq i, j \leq b$.

Table 1. Distance based structural descriptors for strength-weighted graph ζ.

Topological Descriptor	Mathematical Expression
Wiener	$W(\zeta) = \sum_{\{\mu,\eta\}\subseteq V(\zeta)} w_v(\mu)w_v(\eta)d_\zeta(\mu,\eta)$
Szeged	$Sz_v(\zeta) = \sum_{\varepsilon=\mu\eta\in E(\zeta)} s_e(\varepsilon)n_\mu(\varepsilon)n_\eta(\varepsilon)$
Edge-Szeged	$S_{z_e}(\zeta) = \sum_{\varepsilon=\mu\eta\in E(\zeta)} s_e(\varepsilon)m_\mu(\varepsilon)m_\eta(\varepsilon)$
Edge-Vertex-Szeged	$S_{z_{ev}}(\zeta) = \frac{1}{2}\sum_{\varepsilon=\mu\eta\in E(\zeta)} s_e(\varepsilon)[n_\mu(\varepsilon)n_\eta(\varepsilon) + m_\mu(\varepsilon)n_\eta(\varepsilon)]$
Padmakar-Ivan	$P_I(\zeta) = \sum_{\varepsilon=\mu\eta\in E(\zeta)} s_e(\varepsilon)[m_\mu(\varepsilon) + m_\eta(\varepsilon)]$
Mostar	$M_o(G) = \sum_{\varepsilon=\mu\eta\in E(\zeta)} s_e(\varepsilon)\lvert n_\mu(\varepsilon) - n_\eta(\varepsilon)\rvert$
Edge-Mostar	$M_{o_e}(G) = \sum_{\varepsilon=\mu\eta\in E(\zeta)} s_e(\varepsilon)\lvert m_\mu(\varepsilon) - m_\eta(\varepsilon)\rvert$

- **Case 3**: $\mu \in V(X_i), \eta \in V(Y_j);\ 1 \leq i \leq a, 1 \leq j \leq b$.

It can be noted that, under Case 1 and Case 2, distance between the vertices μ and η will be one, when $i = j$ and it will be two, when $i \neq j$. Also, under Case 3, distance between the vertices μ and η will be one for any value of i and j.

For complete graph K_m, we have the result, $W(K_m) = \binom{m}{2}$. Hence the Wiener index for the weighted complete graphs $X_i,\ 1 \leq i \leq a$ and $Y_j,\ 1 \leq j \leq b$, which are isomorphic to K_m, can be given as,

$$W(X_i) = \binom{m}{2}\alpha^2 \quad \text{and} \quad W(Y_j) = \binom{m}{2}\beta^2.$$

By using the definition for Wiener index given in Table 1 and the observed data, the Wiener index for the given graph ζ,

$$\begin{aligned}
W(\zeta) &= \sum_{\{\mu,\eta\}\subseteq V(\zeta)} w_v(\mu)w_v(\eta)d_\zeta(\mu,\eta) \\
&= \frac{1}{2}\sum_{\mu\in V(X_p);\eta\in V(X_q)} w_v(\mu)w_v(\eta)d_\zeta(\mu,\eta) \\
&\quad + \frac{1}{2}\sum_{\mu\in V(Y_r);\eta\in V(Y_s)} w_v(\mu)w_v(\eta)d_\zeta(\mu,\eta) \\
&\quad + \frac{1}{2}\sum_{\mu\in V(X_i);\eta\in V(Y_j)} w_v(\mu)w_v(\eta)d_\zeta(\mu,\eta),
\end{aligned}$$

where $1 \leq p, q, i \leq a; 1 \leq r, s, j \leq b$.

$$\begin{aligned}
&= \frac{1}{2}\Bigg(\sum_{\mu\in V(X_p);\eta\in V(X_p)} w_v(\mu)w_v(\eta)d_\zeta(\mu,\eta) \\
&\quad + \sum_{\mu\in V(X_p);\eta\in V(X_q),p\neq q} w_v(\mu)w_v(\eta)d_\zeta(\mu,\eta)\Bigg) \\
&\quad + \frac{1}{2}\Bigg(\sum_{\mu\in V(Y_r);\eta\in V(Y_r)} w_v(\mu)w_v(\eta)d_\zeta(\mu,\eta) \\
&\quad + \sum_{\mu\in V(Y_r);\eta\in V(Y_s),r\neq s} w_v(\mu)w_v(\eta)d_\zeta(\mu,\eta)\Bigg) \\
&\quad + \frac{1}{2}\Bigg(\sum_{\mu\in V(X_i);\eta\in V(Y_j)} w_v(\mu)w_v(\eta)d_\zeta(\mu,\eta)\Bigg) \\
&= a\binom{m}{2}\alpha^2 + a(a-1)m^2\alpha^2 + b\binom{m}{2}\beta^2 \\
&\quad + b(b-1)m^2\beta^2 + abm^2\alpha\beta \\
&= m^2\left(a(a-1)\alpha^2 + b(b-1)\beta^2 + ab\alpha\beta\right) \\
&\quad + \binom{m}{2}(a\alpha^2 + b\beta^2).
\end{aligned}$$

□

3 Distance Based Structural Descriptors

The distance based structural descriptors for the benes network $BB(n), n \geq 2$, and augmented benes network, $BB^*(n), n \geq 2$ are computed using the quotient graph approach. This method is used since the regular cut method is not applicable to the augmented benes network, which is not a partial cube. Here, the technique is to convert the original graph into quotient graphs using Θ^*-classes and further, the descriptors of each quotient graph are determined and added up correspondingly in order to obtain the descriptors of original graph. This transformation reduces the original graphs into compact graphs, making the computation easier and faster.

3.1 Benes Network, BB(n)

The benes network, $BB(n)$, $n \geq 2$, has $(2n+1)2^n$ vertices and $n2^{n+2}$ edges. Using the Djoković-Winkler relation, the Θ^*-classes of $BB(n)$, $n \geq 2$ were determined and based on it, $BB(n)$ is reduced to n quotient graphs, $BB(n)/\xi_\delta$, $1 \leq \delta \leq n$. The quotient graphs are isomorphic to complete bipartite graphs, $K_{(2^{n-\delta+1}, 2^{\delta+1})}$; $n \geq 2$, $1 \leq \delta \leq n$.

For example, the quotient graphs for $BB(2)$, $BB(2)/\xi_1$, $BB(2)/\xi_2$ are isomorphic to $K_{(4,4)}$ and $K_{(2,8)}$ respectively. The Θ^*-classes ξ_1 and ξ_2 chosen for $BB(2)$ are illustrated in Figure 6. The generalised quotient graph, $BB(n)/\xi_\delta$, $1 \leq \delta \leq n$, for $BB(n)$, $n \geq 2$ is presented in Figure 7.

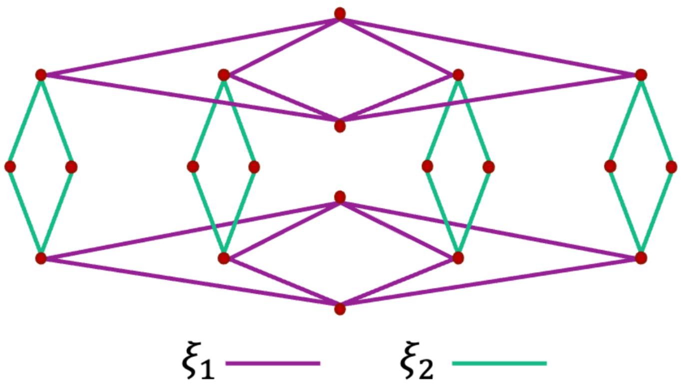
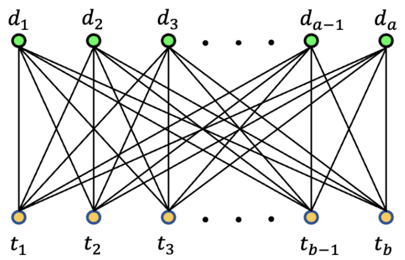


Figure 6. The Θ^*-classes ξ_1 and ξ_2 chosen for $BB(2)$.



$(w_v(d_a), s_v(d_a)) = \left(2^{n-\delta}(n-\delta+1), 2^{n-\delta+1}(n-\delta)\right), 1 \leq a \leq 2^{\delta+1}$

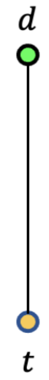
$(w_v(t_b), s_v(t_b)) = \left(2^{\delta-1}(2\delta-1), 2^{\delta+1}(\delta-1)\right), 1 \leq b \leq 2^{n-\delta+1}$

Figure 7. The quotient graph of $BB(n)$, G/ξ_δ, $1 \leq \delta \leq n$.

Theorem 2. *For $n \geq 2$,*

$$W(BB(n)) = \frac{2^n}{3}\left(50n \cdot 2^n - 69 \cdot 2^n - 9n^2 \cdot 2^n + 10n^3 \cdot 2^n + 69\right).$$

Proof. The Wiener index for $BB(n), n \geq 2$, is computed with the help of reduction theorem [14]. In this process, the quotient graphs of the benes network are converted to reduced graphs and further reduction theorem is applied to determine the result. The reduced graph of the generalised quotient graph, $G/\xi_\delta, 1 \leq \delta \leq n$, for $BB(n), n \geq 2$ is presented in Figure 8.



$$w_v(d) = 2^{n+1}(n-\delta+1),\ w_v(t) = 2^n(2\delta-1)$$

Figure 8. The reduced graph of the generalised quotient graph for $BB(n), n \geq 2$.

Using the reduction theorem (Theorem 2.1 in [15]),

$$W(BB(n)/\xi_\delta) = 2^{n+1}(n-\delta+1)(2^n)(2\delta-1)$$

$$+2\binom{2^{\delta+1}}{2}\left(2^{n-\delta}(n-\delta+1)\right)^2$$

$$+2\binom{2^{n-\delta+1}}{2}\left(2^{\delta-1}(2\delta-1)\right)^2.$$

$$W(BB(n)) = \sum_{\delta=1}^{n} W(BB(n)/\xi_\delta)$$

$$= \frac{2^n}{3}\left(50n \times 2^n - 69 \times 2^n - 9n^2 \times 2^n + 10n^3 \times 2^n + 69\right).$$

□

Theorem 3. *For $n \geq 2$,*

$$\begin{aligned} &Sz(BB(n)) \\ &= \frac{2^{n+1}}{27}\{(2608 - 1308n + 288n^2 + 36n^3)2^{2n} \\ &\quad - (2700 + 423n - 27n^2 - 18n^3)2^n + 92\}, \end{aligned}$$

$$\begin{aligned} &Sz_e(BB(n)) \\ &= 2^n\Big\{\Big(\frac{20096}{27} - \frac{2864}{9}n + \frac{112}{3}n^2 + \frac{32}{3}n^3\Big)2^{2n} \\ &\quad - \Big(792 + \frac{496}{3}n + 16n^2 - \frac{16}{3}n^3\Big)2^n + 4n + \frac{1288}{27}\Big\}, \end{aligned}$$

$$\begin{aligned} &Sz_{ev}(BB(n)) \\ &= 2^n\Big\{\Big(\frac{10636}{27} - \frac{1660}{9}n + \frac{92}{3}n^2 + \frac{16}{3}n^3\Big)2^{2n} \\ &\quad - \Big(408 + \frac{242}{3}n + 2n^2 - \frac{8}{3}n^3\Big)2^n + \frac{380}{27}\Big\}, \end{aligned}$$

$$\begin{aligned} &PI(BB(n)) \\ &= 2^n\{(16n^2 - 16n + 24)2^n - 8n - 24\}, \end{aligned}$$

$$\begin{aligned} &Mo(BB(n)) \\ &= \sum_{\delta=1}^{n} 2^{n+2}\{(2^{n-\delta}(n-\delta+1) + (2^{n-\delta+1}-1)(2^{\delta-1}(2\delta-1))) \\ &\quad - ((2^{\delta+1}-1)(2^{n-\delta}(n-\delta+1)) + (2^{\delta-1}(2\delta-1)))\}, \end{aligned}$$

$$\begin{aligned} &Mo_e(BB(n)) \\ &= \sum_{\delta=1}^{n} 2^{n+2}\{(2^{n-\delta+1}(n-\delta) + (2^{n-\delta+1}-1)(2^{\delta+1}(\delta-1)) + 2^{n-\delta+1}-1) \\ &\quad - ((2^{\delta+1}-1)(2^{n-\delta+1}(n-\delta)) + (2^{\delta+1}(\delta-1)) + 2^{\delta+1}-1)\}. \end{aligned}$$

Proof. The relationship between Szeged and Wiener indices of graphs has been investigated in [9]. The quotient graph, $BB(n)/\xi_\delta, 1 \leq \delta \leq n$ for $BB(n), n \geq 2$ is given in Figure 7. Since $BB(n)/\xi_\delta, 1 \leq \delta \leq n$ is isomorphic to complete bipartite graph, $K_{(2^{n-\delta+1}, 2^{\delta+1})}$, the cardinality of its edge set is 2^{n+2}.

For each edge $\varepsilon = \mu\eta \in BB(n)/\xi_\delta$, the neighborhood sets $N_\mu(\varepsilon|BB(n)/\xi_\delta)$, $N_\eta(\varepsilon|BB(n)/\xi_\delta)$, $M_\mu(\varepsilon|BB(n)/\xi_\delta)$ and $M_\eta(\varepsilon|BB(n)/\xi_\delta)$ and their corresponding cardinalities are determined. It is observed that, for each edge, these values remain same and are obtained as follows.

$$\begin{aligned} n_\mu(\varepsilon(BB(n)/\xi_\delta)) &= 2^{n-\delta}(n-\delta+1) \\ &\quad + \left(2^{n-\delta+1}-1\right)\left(2^{\delta-1}(2\delta-1)\right). \end{aligned}$$

$$\begin{aligned} n_\eta(\varepsilon(BB(n)/\xi_\delta)) &= \left(2^{\delta+1}-1\right)\left(2^{n-\delta}(n-\delta+1)\right) \\ &\quad + 2^{\delta-1}(2\delta-1). \end{aligned}$$

$$\begin{aligned} m_\mu(\varepsilon(BB(n)/\xi_\delta)) &= 2^{n-\delta+1}(n-\delta) \\ &\quad + \left(2^{n-\delta+1}-1\right)\left(2^{\delta+1}(\delta-1)\right) \\ &\quad + 2^{n-\delta+1}-1. \end{aligned}$$

$$\begin{aligned} m_\eta(\varepsilon(BB(n)/\xi_\delta)) &= \left(2^{\delta+1}-1\right)2^{n-\delta+1}(n-\delta) \\ &\quad + 2^{\delta+1}(\delta-1) + 2^{\delta+1}-1. \end{aligned}$$

The Szeged indices, Padmakar-Ivan index and Mostar indices of the network are computed by applying these results in the expressions from Table 1 as follows.

$$\begin{aligned} &S_{z_v}(BB(n)/\xi_\delta) \\ &= \sum_{\varepsilon=\mu\eta\in E(BB(n)/\xi_\delta)} s_e(\varepsilon)n_\mu(\varepsilon|BB(n)/\xi_\delta)n_\eta(\varepsilon|BB(n)/\xi_\delta) \\ &= 2^{n+2}\{(2^{n-\delta}(n-\delta+1) + (2^{n-\delta+1}-1)(2^{\delta-1}(2\delta-1))) \\ &\quad \times ((2^{\delta+1}-1)(2^{n-\delta}(n-\delta+1)) + (2^{\delta-1}(2\delta-1)))\}. \end{aligned}$$

$$\begin{aligned} Sz(BB(n)) &= \sum_{\delta=1}^{n} S_{z_v}(BB(n)/\xi_\delta) \\ &= \frac{2^{n+1}}{27}\{(2608 - 1308n + 288n^2 + 36n^3)2^{2n} \\ &\quad - (2700 + 423n - 27n^2 - 18n^3)2^n + 92\}. \end{aligned}$$

For Edge-Szeged index,

$$\begin{aligned} &S_{z_e}(BB(n)/\xi_\delta) \\ &= \sum_{\varepsilon=\mu\eta\in E(BB(n)/\xi_\delta)} s_e(\varepsilon)m_\mu(\varepsilon|BB(n)/\xi_\delta)m_\eta(\varepsilon|BB(n)/\xi_\delta) \\ &= 2^{n+2}\{(2^{n-\delta+1}(n-\delta) + (2^{n-\delta+1}-1)(2^{\delta+1}(\delta-1)) + 2^{n-\delta+1}-1) \\ &\quad \times ((2^{\delta+1}-1)(2^{n-\delta+1}(n-\delta)) + (2^{\delta+1}(\delta-1)) + 2^{\delta+1}-1)\}. \end{aligned}$$

$$\begin{aligned} &Sz_e(BB(n)) \\ &= \sum_{\delta=1}^{n} S_{z_e}(BB(n)/\xi_\delta) \\ &= 2^n\Big\{\Big(\frac{20096}{27} - \frac{2864}{9}n + \frac{112}{3}n^2 + \frac{32}{3}n^3\Big)2^{2n} \\ &\quad - \Big(792 + \frac{496}{3}n + 16n^2 - \frac{16}{3}n^3\Big)2^n + 4n + \frac{1288}{27}\Big\}. \end{aligned}$$

$$\begin{aligned} &Sz_{ev}(BB(n)/\xi_\delta) \\ &= \frac{1}{2}\sum_{\varepsilon=\mu\eta\in E(BB(n)/\xi_\delta)} s_e(\varepsilon)\Big[n_\mu(\varepsilon|BB(n)/\xi_\delta)m_\eta(\varepsilon|BB(n)/\xi_\delta) \\ &\quad + m_\mu(\varepsilon|BB(n)/\xi_\delta)n_\eta(\varepsilon|BB(n)/\xi_\delta)\Big] \\ &= 2^{n+2}\Big[\Big(2^{n-\delta}(n-\delta+1) + (2^{n-\delta+1}-1)(2^{\delta-1}(2\delta-1))\Big) \\ &\quad \times \Big((2^{\delta+1}-1)2^{n-\delta+1}(n-\delta) + 2^{\delta+1}(\delta-1) + 2^{\delta+1}-1\Big) \\ &\quad + \Big((2^{\delta+1}-1)2^{n-\delta}(n-\delta+1) + 2^{\delta-1}(2\delta-1)\Big) \\ &\quad \times \Big(2^{n-\delta+1}(n-\delta) + (2^{n-\delta+1}-1)2^{\delta+1}(\delta-1) + 2^{n-\delta+1}-1\Big)\Big]. \end{aligned}$$

$$\begin{aligned}&\sum_{\delta=1}^{n} S_{Z_{ev}}(BB(n)/\xi_\delta)\\&=2^n\left\{\left(\frac{10636}{27}-\frac{1660}{9}n+\frac{92}{3}n^2+\frac{16}{3}n^3\right)2^{2n}\right.\\&\quad\left.-\left(408+\frac{242}{3}n+2n^2-\frac{8}{3}n^3\right)2^n+\frac{380}{27}\right\}.\end{aligned}$$

$$\begin{aligned}&PI(BB(n)/\xi_\delta)\\&=\sum_{\varepsilon=\mu\eta\in E(BB(n)/\xi_\delta)} s_e(\varepsilon)\left[m_\mu(\varepsilon|BB(n)/\xi_\delta)+m_\eta(\varepsilon|BB(n)/\xi_\delta)\right]\\&=2^{n+2}\{(2^{n-\delta+1}(n-\delta)+(2^{n-\delta+1}-1)(2^{\delta+1}(\delta-1))+2^{n-\delta+1}-1)\\&+((2^{\delta+1}-1)(2^{n-\delta+1}(n-\delta))+(2^{\delta+1}(\delta-1))+2^{\delta+1}-1)\}.\end{aligned}$$

$$\begin{aligned}&PI(BB(n))\\&=\sum_{\delta=1}^{n} PI(BB(n)/\xi_\delta)\\&=2^n\{(16n^2-16n+24)2^n-8n-24\}.\end{aligned}$$

$$\begin{aligned}&Mo(BB(n)/\xi_\delta)\\&=\sum_{\varepsilon=\mu\eta\in E(BB(n)/\xi_\delta)} s_e(\varepsilon)\,|n_\mu(\varepsilon|BB(n)/\xi_\delta)-n_\eta(\varepsilon|BB(n)/\xi_\delta)|\\&=2^{n+2}\Big|(2^{n-\delta}(n-\delta+1)+(2^{n-\delta+1}-1)(2^{\delta-1}(2\delta-1)))\\&-((2^{\delta+1}-1)(2^{n-\delta}(n-\delta+1))+(2^{\delta-1}(2\delta-1)))\Big|.\end{aligned}$$

$$\begin{aligned}&Mo(BB(n))\\&=\sum_{\delta=1}^{n} Mo(BB(n)/\xi_\delta)\\&=\sum_{\delta=1}^{n}2^{n+2}\Big|(2^{n-\delta}(n-\delta+1)+(2^{n-\delta+1}-1)(2^{\delta-1}(2\delta-1)))\\&-((2^{\delta+1}-1)(2^{n-\delta}(n-\delta+1))+(2^{\delta-1}(2\delta-1)))\Big|.\end{aligned}$$

$$\begin{aligned}&Mo_e(BB(n)/\xi_\delta)\\&=\sum_{\varepsilon=\mu\eta\in E(BB(n)/\xi_\delta)} s_e(\varepsilon)\,|m_\mu(\varepsilon|BB(n)/\xi_\delta)-m_\eta(\varepsilon|BB(n)/\xi_\delta)|\\&=2^{n+2}\Big|(2^{n-\delta+1}(n-\delta)+(2^{n-\delta+1}-1)(2^{\delta+1}(\delta-1))+2^{n-\delta+1}-1)\\&-((2^{\delta+1}-1)(2^{n-\delta+1}(n-\delta))+(2^{\delta+1}(\delta-1))+2^{\delta+1}-1)\Big|.\end{aligned}$$

$$\begin{aligned}&Mo_e(BB(n))\\&=\sum_{\delta=1}^{n} Mo_e(BB(n)/\xi_\delta)\\&=\sum_{\delta=1}^{n}2^{n+2}\Big|(2^{n-\delta+1}(n-\delta)+(2^{n-\delta+1}-1)(2^{\delta+1}(\delta-1))+2^{n-\delta+1}-1)\\&-((2^{\delta+1}-1)(2^{n-\delta+1}(n-\delta))+(2^{\delta+1}(\delta-1))+2^{\delta+1}-1)\Big|.\end{aligned}$$

Hence,

$$\begin{aligned}&Sz_v(BB(n))\\&=\sum_{\delta=1}^{n} Sz_v(BB(n)/\xi_\delta)\\&=\frac{2^{n+1}}{27}\{(2608-1308n+288n^2+36n^3)2^{2n}\\&\quad-(2700+423n-27n^2-18n^3)2^n+92\}.\end{aligned}$$

$$\begin{aligned}&Sz_e(BB(n))\\&=\sum_{\delta=1}^{n} Sz_e(BB(n)/\xi_\delta)\\&=2^n\left\{\left(\frac{20096}{27}-\frac{2864}{9}n+\frac{112}{3}n^2+\frac{32}{3}n^3\right)2^{2n}\right.\\&\quad\left.-\left(792+\frac{496}{3}n+16n^2-\frac{16}{3}n^3\right)2^n+4n+\frac{1288}{27}\right\}.\end{aligned}$$

$$\begin{aligned}&Sz_{ev}(BB(n))\\&=\sum_{\delta=1}^{n} Sz_{ev}(BB(n)/\xi_\delta)\\&=2^n\left\{\left(\frac{10636}{27}-\frac{1660}{9}n+\frac{92}{3}n^2+\frac{16}{3}n^3\right)2^{2n}\right.\\&\quad\left.-\left(408+\frac{242}{3}n+2n^2-\frac{8}{3}n^3\right)2^n+\frac{380}{27}\right\}.\end{aligned}$$

$$\begin{aligned}PI(BB(n))&=\sum_{\delta=1}^{n} PI(BB(n)/\xi_\delta)\\&=2^n\{(16n^2-16n+24)2^n-8n-24\}.\end{aligned}$$

$$\begin{aligned}&Mo(BB(n))\\&=\sum_{\delta=1}^{n} Mo(BB(n)/\xi_\delta)\\&=\sum_{\delta=1}^{n}2^{n+2}\Big|(2^{n-\delta}(n-\delta+1)+(2^{n-\delta+1}-1)(2^{\delta-1}(2\delta-1)))\\&-((2^{\delta+1}-1)(2^{n-\delta}(n-\delta+1))+(2^{\delta-1}(2\delta-1)))\Big|.\end{aligned}$$

$$\begin{aligned}&Mo_e(BB(n))\\&=\sum_{\delta=1}^{n} Mo_e(BB(n)/\xi_\delta)\\&=\sum_{\delta=1}^{n}2^{n+2}\Big|(2^{n-\delta+1}(n-\delta)+(2^{n-\delta+1}-1)(2^{\delta+1}(\delta-1))+2^{n-\delta+1}-1)\\&-((2^{\delta+1}-1)(2^{n-\delta+1}(n-\delta))+(2^{\delta+1}(\delta-1))+2^{\delta+1}-1)\Big|.\end{aligned}$$

□

3.2 Augmented Benes Network, $BB^*(n)$

The augmented benes network, $BB^*(n), n \geq 2$, has $(2n+1)2^n$ vertices and $(12n-1)2^{n-1}$ edges. Using the Djoković-Winkler relation, the Θ^*-classes of $BB^*(n), n \geq 2$, were determined and based on those classes, $BB^*(n)$ can be reduced to n quotient graphs, $BB^*(n)/\xi_\sigma, 1 \leq \sigma \leq n$. For example, $BB^*(2)$ can have two quotient graphs, $BB^*(2)/\xi_1$, and $BB^*(2)/\xi_2$. The Θ^*-classes ξ_1 and ξ_2 chosen for $BB^*(2)$ and their quotient graphs are depicted in Figures 9, 10 and 11 respectively. The quotient graphs for $BB^*(n), n \geq 2$, can be standardised into two types, $BB^*(n)/\xi_1$, and $BB^*(n)/\xi_\sigma, 2 \leq \sigma \leq n$ as illustrated in Figures 12 and 13 respectively.

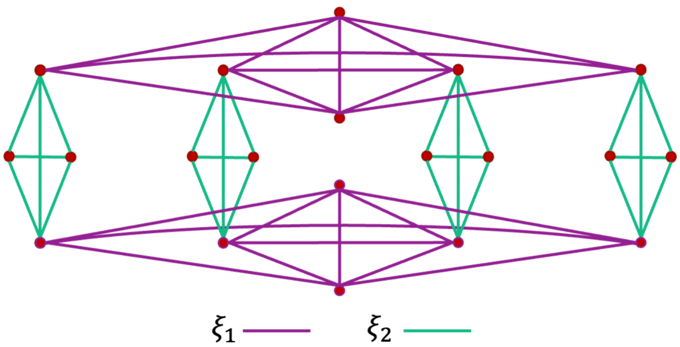


Figure 9. The Θ^*-classes ξ_1 and ξ_2 chosen for $BB^*(2)$.

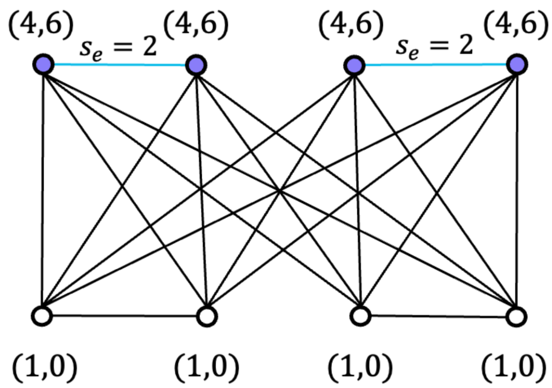


Figure 10. The quotient graph of $BB^*(2)$, $BB^*(2)/\xi_1$.

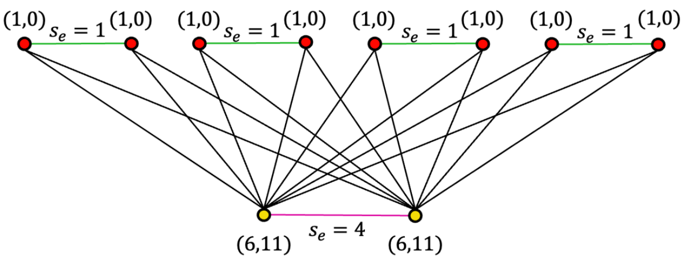


Figure 11. The quotient graph of $BB^*(2)$, $BB^*(2)/\xi_2$.

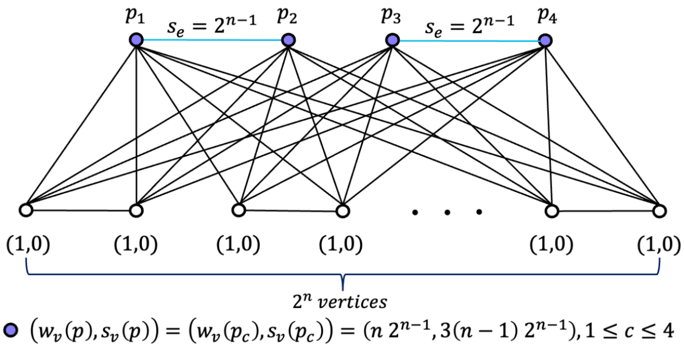


Figure 12. The first type of quotient graph for $BB^*(n)$, $BB^*(n)/\xi_1$.

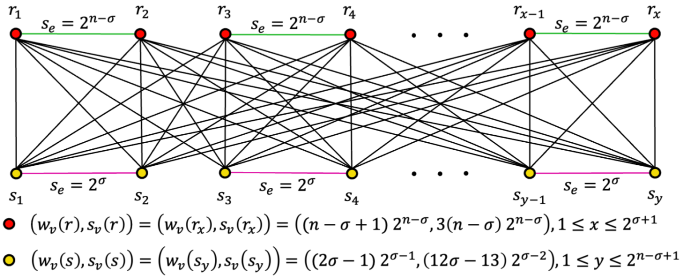


Figure 13. The second type of quotient graph for $BB^*(n)$, $BB^*(n)/\xi_\sigma, 2 \leq \sigma \leq n$.

Theorem 4. *For $n \geq 2$,*

$$W(BB^*(n)) = \frac{2^n}{6}\left((20n^3 - 36n^2 + 148n - 207)2^n + 207\right).$$

Proof. It can be observed that, the quotient graphs $BB^*(n)/\xi_1$, and $BB^*(n)/\xi_\sigma$ are isomorphic to ζ defined in Theorem 1 under specific parameters.

Case 1: Quotient graph $BB^*(n)/\xi_1$ (Figure 12).

The graph $BB^*(n)/\xi_1$ is isomorphic to ζ with $a = 2$, $b = 2^{n-1}$ and $m = 2$ defined in Theorem 1. That is, X_1, X_2 and $Y_1, Y_2, \ldots, Y_{2^{n-1}}$ are subgraphs of $BB^*(n)/\xi_1$ such that $X_i, 1 \leq i \leq 2$ and $Y_j, 1 \leq j \leq 2^{n-1}$ are isomorphic to complete graph K_2. Also $\alpha = n2^{n-1}$ and $\beta = 1$. Hence the Wiener index of graph $BB^*(n)/\xi_1$ can be determined by applying Theorem 1.

$$\begin{aligned}
&W(BB^*(n)/\xi_1) \\
&= 2^2\left(2(2-1)(n2^{n-1})^2 + 2^{n-1}(2^{n-1}-1)(1)^2\right. \\
&\quad \left.+ 2\cdot 2^{n-1}(n2^{n-1})(1)\right) + \binom{2}{2}\left(2(n2^{n-1})^2 + 2^{n-1}(1)^2\right) \\
&= 2^n\left\{\left(\frac{5}{2}n^2 + 2n + 1\right)2^n - \frac{3}{2}\right\}.
\end{aligned}$$

Case 2: Quotient graph $BB^*(n)/\xi_\sigma, 2 \leq \sigma \leq n$ (Figure 13).

The graph $BB^*(n)/\xi_\sigma$ is isomorphic to ζ defined in Theorem 1 with $a = 2^{n-\sigma-1}$, $b = 2^{\sigma-1}$ and $m = 2$. Therefore, $X_1, X_2, \ldots, X_{2^{n-\sigma-1}}$ and $Y_1, Y_2, \ldots, Y_{2^{\sigma-1}}$ are subgraphs of $BB^*(n)/\xi_\sigma$, such that $X_i \cong K_2$, $1 \leq i \leq 2^{n-\sigma-1}$ and $Y_j \cong K_2$, $1 \leq j \leq 2^{\sigma-1}$. Also $\alpha = (n-\sigma+1)2^{n-\sigma}$ and $\beta = (2\sigma-1)2^{\sigma-1}$. Hence the Wiener index of graph $BB^*(n)/\xi_\sigma$ can be determined by applying Theorem 1.

$$
\begin{aligned}
&W(BB^*(n)/\xi_\sigma)\\
&= 2^2\Big(2^{n-\sigma-1}(2^{n-\sigma-1}-1)((n-\sigma+1)2^{n-\sigma})^2\\
&\quad + 2^{\sigma-1}(2^{\sigma-1}-1)((2\sigma-1)2^{\sigma-1})^2\Big)\\
&\quad + 2^{n-\sigma-1}\cdot 2^{\sigma-1}\cdot(n-\sigma+1)2^{n-\sigma}\cdot(2\sigma-1)2^{\sigma-1}\\
&\quad + \binom{2}{2}\Big(2^{n-\sigma-1}((n-\sigma+1)2^{n-\sigma})^2\\
&\quad + 2^{\sigma-1}((2\sigma-1)2^{\sigma-1})^2\Big)\\
&= 2^{2n-\sigma}(2^{\sigma+2}-3)(n-\sigma-1)^2\\
&\quad + 2^{2n+1}(2\sigma-1)(n-\sigma+1)\\
&\quad - 2^{n-2}(2\sigma-1)^2(3\cdot 2^\sigma - 2^{n+2}).
\end{aligned}
$$

Summing over all quotient graphs,

$$
\begin{aligned}
&\sum_{\sigma=1}^{n} W(BB^*(n)/\xi_\sigma)\\
&= \frac{2^n}{6}\left((20n^3-51n^2+136n-213)2^n+216\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&W(BB^*(n))\\
&= W(BB^*(n)/\xi_1) + \sum_{\sigma=2}^{n} W(BB^*(n)/\xi_\sigma)\\
&= \frac{2^n}{6}\left((20n^3-36n^2+148n-207)2^n+207\right).
\end{aligned}
$$

□

Theorem 5. *For $n \geq 2$,*

$$
\begin{aligned}
&Sz(BB^*(n))\\
&= \frac{2^n}{54}\{(15988-7212n+882n^2+144n^3)2^{2n}\\
&\quad - (16632+3366n-270n^2-180n^3)2^n+617\},\\
&Sz_e(BB^*(n))\\
&= 2^n\bigg\{\left(\frac{30709}{12}-922n+57n^2+24n^3\right)2^{2n}\\
&\quad - \left(2756+\frac{1445}{2}n+\frac{123}{2}n^2-30n^3\right)2^n+4n+\frac{587}{3}\bigg\},\\
&Sz_{ev}(BB^*(n))\\
&= 2^n\bigg\{\left(\frac{16145}{18}-\frac{1087}{3}n+34n^2+8n^3\right)2^{2n}\\
&\quad - \left(943+\frac{909}{4}n+\frac{11}{4}n^2-10n^3\right)2^n+\frac{410}{9}\bigg\},\\
&PI(BB^*(n))\\
&= 2^n\{(73-36n+24n^2)2^n-8n-78\},\\
&Mo(BB^*(n))\\
&= 2^{n+2}|n2^{n-1}-2^n+3|\\
&\quad + \sum_{\sigma=2}^{n} 2^{n+2}\Big|(2^{n-\sigma}(n-\sigma+1)+(2\sigma-1)(2^n-2^\sigma)\\
&\quad - 2^{\sigma-1}(2\sigma-1)-2^{n-\sigma}(2^{\sigma+1}-2)(n-\sigma+1))\Big|,\\
&Mo_e(BB^*(n))\\
&= 2^{n+2}|2^{n-1}-3(n-1)2^{n-1}+2^n-6|\\
&\quad + \sum_{\sigma=2}^{n} 2^{n+2}\Big|((36n-36\sigma+16)2^{n-\sigma-2}\\
&\quad - (24n-48\sigma+26)2^{n-2}-(36\sigma-23)2^{\sigma-2})\Big|.
\end{aligned}
$$

Proof. There are two types of quotient graphs for $BB^*(n), n \geq 2$, namely $BB^*(n)/\xi_1$ and $BB^*(n)/\xi_\sigma, 2 \leq \sigma \leq n$ as depicted in Figure 12 and 13 respectively.

The first type of quotient graph, $BB^*(n)/\xi_1$ has $9 \times 2^{n-1}+2$ edges and the second type of quotient graph, $BB^*(n)/\xi_\sigma, 2 \leq \sigma \leq n$ has 2^{2n+2} edges. As in Theorem 3, the neighborhood sets and their corresponding cardinality for each edge in both quotient graphs are determined and applied in the mathematical expressions from Table 1 to compute the Szeged, Padmakar-Ivan and Mostar indices. As a result, the following outcomes are obtained.

For quotient graph $BB^*(n)/\xi_1$ (Figure 12), $w_v(p) = n2^{n-1}$, $s_v(p) = 3(n-1)2^{n-1}$.

Szeged index:

$$\begin{aligned}&Sz(BB^*(n)/\xi_1)\\&=2^{n+2}(w_v(p)+2^n-2)(2w_v(p)+1)+2^n(w_v(p))^2+2^{n-1}\\&=\frac{2^n}{4}\{(16n+9n^2)2^{2n}+(16-24n)2^n-30\}.\end{aligned}$$

Edge-Szeged index:

$$\begin{aligned}&Sz_e(BB^*(n)/\xi_1)\\&=2^{n+2}\{(s_v(p)+2^{n+1}-2)(2s_v(p)+4+2^{n-1})\}\\&\quad+2^n\{(s_v(p)+2^n)(s_v(p)+2^n)\}+2^{n-1}\{4\cdot 4\}\\&=\frac{2^n}{4}\{(81n^2-42n-19)2^{2n}+112\cdot 2^n-96\}.\end{aligned}$$

Edge-Vertex-Szeged index:

$$\begin{aligned}&Sz_{ev}(BB^*(n)/\xi_1)\\&=\frac{1}{2}\Big\{2^{n+2}\{(w_v(p)+2^n-2)(2s_v(p)+4+2^{n-1})\}\\&\quad+(2w_v(p)+1)(s_v(p)+2^{n+1}-2)\Big\}\\&\quad+2^n\{2(w_v(p)(s_v(p)+2^n)\}+2^{n-1}\{4+4\}\\&=\frac{2^n}{4}\{(27n^2+17n-20)2^{2n}+(76-36n)2^n-72\}.\end{aligned}$$

Padmakar-Ivan index:

$$\begin{aligned}&PI(BB^*(n)/\xi_1)\\&=2^{n+2}\{(s_v(p)+2^{n+1}-2)+(2s_v(p)+4+2^{n-1})\}\\&\quad+2^n\{(s_v(p)+2^n)+(s_v(p)+2^n)\}+2^{n-1}\{4+4\}\\&=3\cdot 2^n\{(7n-3)2^n+4\}.\end{aligned}$$

Mostar index:

$$\begin{aligned}&Mo(BB^*(n)/\xi_1)\\&=2^{n+2}\{|w_v(p)+2^n-2-(2w_v(p)+1)|+2\cdot 0+2^{n-1}\cdot 0\}\\&=2^{n+2}|n2^{n-1}-2^n+3|.\end{aligned}$$

Edge-Mostar index:

$$\begin{aligned}&Mo_e(BB^*(n)/\xi_1)\\&=2^{n+2}\{(s_v(p)+2^{n+1}-2)-(2s_v(p)+4+2^{n-1})\}\\&\quad+2\cdot 0+2^{n-1}\cdot 0\\&=2^{n+2}|2^{n-1}-3(n-1)2^{n-1}+2^n-6|.\end{aligned}$$

For quotient graph $BB^*(n)/\xi_\sigma, 2\leq\sigma\leq n$ (Figure 13), the similar steps used in computing the results for quotient graph $BB^*(n)/\xi_1$, are implemented, and the following results are obtained.

Szeged index:

$$\begin{aligned}&Sz(BB^*(n)/\xi_\sigma)\\&=2^{n+2}\Big\{\big(2^{n-\sigma}(n-\sigma+1)-(2^\sigma-2^n)(2\sigma-1)\big)\big(2^{\sigma-1}(2\sigma-1)\big)\\&\quad+2^{n-\sigma}(n-\sigma+1)(2^{\sigma+1}-2)\Big\}\\&\quad+2^{n-\sigma}\cdot 2^\sigma\Big\{\big(2^{n-\sigma}(n-\sigma+1)\big)\big(2^{n-\sigma}(n-\sigma+1)(2^{\sigma-1}(2\sigma-1))\big)+2^\sigma\Big\}\\&\quad+2^{n-\sigma}\Big\{\big(2^{\sigma-1}(2\sigma-1)\big)\big(2^{\sigma-1}(2\sigma-1)\big)+2^\sigma\Big\}.\end{aligned}$$

Summing over all quotient graphs,

$$\begin{aligned}&\sum_{\sigma=2}^{n}Sz(BB^*(n)/\xi_\sigma)\\&=\frac{2^n}{108}\Big\{(31976-14856n+1521n^2+288n^3)2^{2n}\\&\quad+(360n^3+540n^2-6084n-33696)2^n+2044\Big\}.\end{aligned}$$

Edge-Szeged index:

$$\begin{aligned}&Sz_e(BB^*(n)/\xi_\sigma)\\&=2^{n+2}\Big\{\big((11-12\sigma)(2^{\sigma-1}-2^{n-1})+3(n-\sigma+1)2^{n-\sigma}-1\big)\\&\quad\times\big(2^{\sigma-2}(12\sigma-13)+3(n-\sigma)2^{n-\sigma}(2^{\sigma+1}-2)\\&\quad+2^{n-\sigma}(2^\sigma-1)+2^\sigma+2^{\sigma+1}-1\big)\Big\}\\&\quad+2^\sigma\Big\{(3(n-\sigma)2^{n-\sigma}+2^{n-\sigma+1})(3(n-\sigma)2^{n-\sigma}+2^{n-\sigma}+1)+2^\sigma\Big\}\\&\quad+2^{n-\sigma}\Big\{(2^{\sigma-2}(12\sigma-13)+2^{\sigma+1})(2^{\sigma-2}(12\sigma-13)+2^{\sigma+1})\Big\}.\end{aligned}$$

Summing over all quotient graphs,

$$\begin{aligned}&\sum_{\sigma=2}^{n}Sz_e(BB^*(n)/\xi_\sigma)\\&=2^n\Bigg\{\left(\frac{15383}{6}-\frac{1823n}{2}+\frac{147n^2}{4}+24n^3\right)2^{2n}\\&\quad+\left(30n^3-\frac{123n^2}{2}-\frac{1445n}{2}-2784\right)2^n+4n+\frac{659}{3}\Bigg\}.\end{aligned}$$

Edge-Vertex-Szeged index:

$$
\begin{aligned}
& Sz_{ev}(BB^*(n)/\xi_\sigma) \\
&= \frac{1}{2}\Bigg[2^{n+2}\Big\{\big(2^{n-\sigma}(n-\sigma+1)-(2^\sigma-2^n)(2\sigma-1)\big) \\
&\quad \times \big(2^{\sigma-2}(12\sigma-13)+3(n-\sigma)2^{n-\sigma}(2^{\sigma+1}-2) \\
&\quad + 2^{n-\sigma}(2^\sigma-1)+2^\sigma+2^{\sigma+1}-1\big)\Big\} \\
&\quad + \Big(2^{\sigma-1}(2\sigma-1)+2^{n-\sigma}(n-\sigma+1)(2^{\sigma+1}-2)\Big) \\
&\quad \times \big((11-12\sigma)(2^{\sigma-1}-2^{n-1})\big)\Bigg] \\
&\quad + 2^\sigma\Big\{2\big(2^{n-\sigma}(n-\sigma+1)\big)\big(3(n-\sigma)2^{n-\sigma}+2^{n-\sigma+1}\big)+2^{n-\sigma}\Big\} \\
&\quad + 2^\sigma\Big\{2\big(2^{\sigma-1}(2\sigma-1)\big)\big(2^{\sigma-2}(12\sigma-13)+2^{\sigma+1}\big)\Big\}.
\end{aligned}
$$

Summing over all quotient graphs,

$$
\begin{aligned}
& \sum_{\sigma=2}^{n} Sz_{ev}(BB^*(n)/\xi_\sigma) \\
&= 2^n\Bigg\{\left(\frac{16235}{18}-\frac{4399n}{12}+\frac{109n^2}{4}+8n^3\right)2^{2n} \\
&\quad + \left(10n^3-\frac{11n^2}{4}-\frac{873n}{4}-962\right)2^n+\frac{572}{9}\Bigg\}.
\end{aligned}
$$

Padmakar-Ivan index:

$$
\begin{aligned}
& PI(BB^*(n)/\xi_\sigma) \\
&= 2^{n+2}\Big\{\big((11-12\sigma)(2^{\sigma-1}-2^{n-1})+3(n-\sigma+1)2^{n-\sigma}-1\big) \\
&\quad + \big(2^{\sigma-2}(12\sigma-13)+3(n-\sigma)2^{n-\sigma}(2^{\sigma+1}-2) \\
&\quad + 2^{n-\sigma}(2^\sigma-1)+2^\sigma+2^{\sigma+1}-1\big)\Big\} \\
&\quad + 2^{n-\sigma}\cdot 2^\sigma\Big\{\big(3(n-\sigma)2^{n-\sigma}+2^{n-\sigma+1}\big) \\
&\quad + \big(3(n-\sigma)2^{n-\sigma}+2^{n-\sigma+1}\big)\Big\} \\
&\quad + 2^\sigma\cdot 2^{n-\sigma}\Big\{\big(2^{\sigma-2}(12\sigma-13)+2^{\sigma+1}\big) \\
&\quad + \big(2^{\sigma-2}(12\sigma-13)+2^{\sigma+2}\big)\Big\}.
\end{aligned}
$$

Summing over all quotient graphs,

$$
\begin{aligned}
& \sum_{\sigma=2}^{n} PI(BB^*(n)/\xi_\sigma) \\
&= 2^n\big\{(24n^2-57n+82)2^n-8n-90\big\}.
\end{aligned}
$$

Mostar index:

$$
\begin{aligned}
& Mo(BB^*(n)/\xi_\sigma) \\
&= 2^{n+2}\Big\{\big(2^{n-\sigma}(n-\sigma+1)-(2^\sigma-2^n)(2^\sigma-1)\big) \\
&\quad - \big(2^{\sigma-1}(2\sigma-1)+2^{n-\sigma}(n-\sigma+1)(2^{\sigma+1}-2)\big)\Big\} \\
&\quad + 2^{n-\sigma}\cdot 2^\sigma\cdot 0+2^\sigma\cdot 2^{n-\sigma}\cdot 0.
\end{aligned}
$$

Summing over all quotient graphs,

$$
\begin{aligned}
& \sum_{\sigma=2}^{n} Mo(BB^*(n)/\xi_\sigma) \\
&= \sum_{\sigma=2}^{n} 2^{n+2}\Big|2^{n-\sigma}(n-\sigma+1)+(2\sigma-1)(2^n-2^\sigma) \\
&\quad - 2^{\sigma-1}(2\sigma-1)-2^{n-\sigma}(2^{\sigma+1}-2)(n-\sigma+1)\Big|.
\end{aligned}
$$

Edge-Mostar index:

$$
\begin{aligned}
& Mo_e(BB^*(n)/\xi_\sigma) \\
&= 2^{n+2}\Big\{\big((11-12\sigma)(2^{\sigma-1}-2^{n-1})+3(n-\sigma+1)2^{n-\sigma}-1\big) \\
&\quad - \big(2^{\sigma-2}(12\sigma-13)+3(n-\sigma)2^{n-\sigma}(2^{\sigma+1}-2) \\
&\quad + 2^{n-\sigma}(2^\sigma-1)+2^\sigma+2^{\sigma+1}-1\big)\Big\} \\
&\quad + 2^{n-\sigma}\cdot 2^\sigma\cdot 0+2^\sigma\cdot 2^{n-\sigma}\cdot 0.
\end{aligned}
$$

Summing over all quotient graphs,

$$
\begin{aligned}
& \sum_{\sigma=2}^{n} Mo_e(BB^*(n)/\xi_\sigma) \\
&= \sum_{\sigma=2}^{n} 2^{n+2}\Big|(36n-36\sigma+16)2^{n-\sigma-2} \\
&\quad - (24n-48\sigma+26)2^{n-2}-(36\sigma-23)2^{\sigma-2}\Big|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& Sz(BB^*(n)) \\
&= Sz(BB^*(n)/\xi_1)+\sum_{\sigma=2}^{n} Sz(BB^*(n)/\xi_\sigma) \\
&= \frac{2^n}{54}\Big\{(15988-7212n+882n^2+144n^3)2^{2n} \\
&\quad - (16632+3366n-270n^2-180n^3)2^n+617\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
& Sz_e(BB^*(n)) \\
&= Sz_e(BB^*(n)/\xi_1)+\sum_{\sigma=2}^{n} Sz_e(BB^*(n)/\xi_\sigma) \\
&= 2^n\Bigg\{\left(\frac{30709}{12}-922n+57n^2+24n^3\right)2^{2n} \\
&\quad - \left(2756+\frac{1445}{2}n+\frac{123}{2}n^2-30n^3\right)2^n+4n+\frac{587}{3}\Bigg\}.
\end{aligned}
$$

$$
\begin{aligned}
& Sz_{ev}(BB^*(n)) \\
&= Sz_{ev}(BB^*(n)/\xi_1) + \sum_{\sigma=2}^{n} Sz_{ev}(BB^*(n)/\xi_\sigma) \\
&= 2^n \Bigg\{ \left(\frac{16145}{18} - \frac{1087}{3} n + 34n^2 + 8n^3 \right) 2^{2n} \\
&\quad - \left(943 + \frac{909}{4} n + \frac{11}{4} n^2 - 10n^3 \right) 2^n + \frac{410}{9} \Bigg\}.
\end{aligned}
$$

$$
\begin{aligned}
& PI(BB^*(n)) \\
&= PI(BB^*(n)/\xi_1) + \sum_{\sigma=2}^{n} PI(BB^*(n)/\xi_\sigma) \\
&= 2^n \big\{ (73 - 36n + 24n^2) 2^n - 8n - 78 \big\}.
\end{aligned}
$$

$$
\begin{aligned}
& Mo(BB^*(n)) \\
&= Mo(BB^*(n)/\xi_1) + \sum_{\sigma=2}^{n} Mo(BB^*(n)/\xi_\sigma) \\
&= 2^{n+2} \big| n 2^{n-1} - 2^n + 3 \big| \\
&\quad + \sum_{\sigma=2}^{n} 2^{n+2} \Big| 2^{n-\sigma}(n - \sigma + 1) + (2\sigma - 1)(2^n - 2^\sigma) \\
&\quad - 2^{\sigma-1}(2\sigma - 1) - 2^{n-\sigma}(2^{\sigma+1} - 2)(n - \sigma + 1) \Big|.
\end{aligned}
$$

$$
\begin{aligned}
& Mo_e(BB^*(n)) \\
&= Mo_e(BB^*(n)/\xi_1) + \sum_{\sigma=2}^{n} Mo_e(BB^*(n)/\xi_\sigma) \\
&= 2^{n+2} \big| 2^{n-1} - 3(n-1) 2^{n-1} + 2^n - 6 \big| \\
&\quad + \sum_{\sigma=2}^{n} 2^{n+2} \Big| (36n - 36\sigma + 16) 2^{n-\sigma-2} \\
&\quad - (24n - 48\sigma + 26) 2^{n-2} - (36\sigma - 23) 2^{\sigma-2} \Big|.
\end{aligned}
$$

□

3.3 Numerical Analysis and Graphical Comparison

Using the analytical expressions obtained from the previous section, the structural descriptors for few fixed values of n are computed for the benes network, $BB(n)$ and augmented benes network, $BB^*(n)$. The numerical values of the descriptors for $BB(n)$ are tabulated in Tables 2 and 3 and that for $BB^*(n)$ are presented in Tables 4 and 5. Also, the numerical values are plotted graphically and are presented in Figure 14.

Comparing the numerical values of descriptors for $BB(n)$ and $BB^*(n)$, the Wiener and Szeged index for $BB^*(n)$ is less than that for $BB(n)$ for all n. On the other hand, all the remaining descriptors are high for $BB^*(n)$ compared with that for $BB(n)$ for all n. It is also observed that $Mo_e(BB^*(n)) = Mo(BB^*(n))$ for $n = 2, 3, 4$.

Table 2. Numerical values of distance based descriptors for benes network, $BB(n)$.

n	Wiener	Szeged	Edge-Szeged	Edge-Vertex-Szeged
2	492	2912	3840	3344
3	5944	69952	129824	95392
4	53872	1247872	2728576	1848128
5	412384	18711808	45143424	29116032
6	2823616	250456576	645950464	402967808

Table 3. Numerical values of distance based descriptors for benes network, $BB(n)$.

n	Padmakar-Ivan	Mostar	Edge-Mostar
2	736	192	224
3	7296	1408	1728
4	54400	8448	10624
5	350208	45568	69376
6	2059776	250880	437760

Table 4. Numerical values of distance based descriptors for augmented benes network, $BB^*(n)$.

n	Wiener	Szeged	Edge-Szeged	Edge-Vertex-Szeged
2	418	1270	6148	2752
3	5108	35628	191464	82032
4	47016	726616	4200592	1741984
5	365136	12043440	73545120	29729088
6	2531488	173578592	1105984576	438073984

4 Comparative Analysis

In this section, the topological network parameters to check the efficiency of networks are evaluated for the networks, $BB(n), n \geq 2$ and $BB^*(n), n \geq 2$. The formulas to evaluate various topological network parameters, as adopted from the framework of fixed interconnection architectures [1], are defined in Table 6.

As mentioned, $BB(n), n \geq 2$ has $(2n+1)2^n$ vertices and $n2^{n+2}$ edges and $BB^*(n), n \geq 2$ has $(2n+1)2^n$ vertices and $(12n-1)2^{n-1}$ edges. Using the formulas

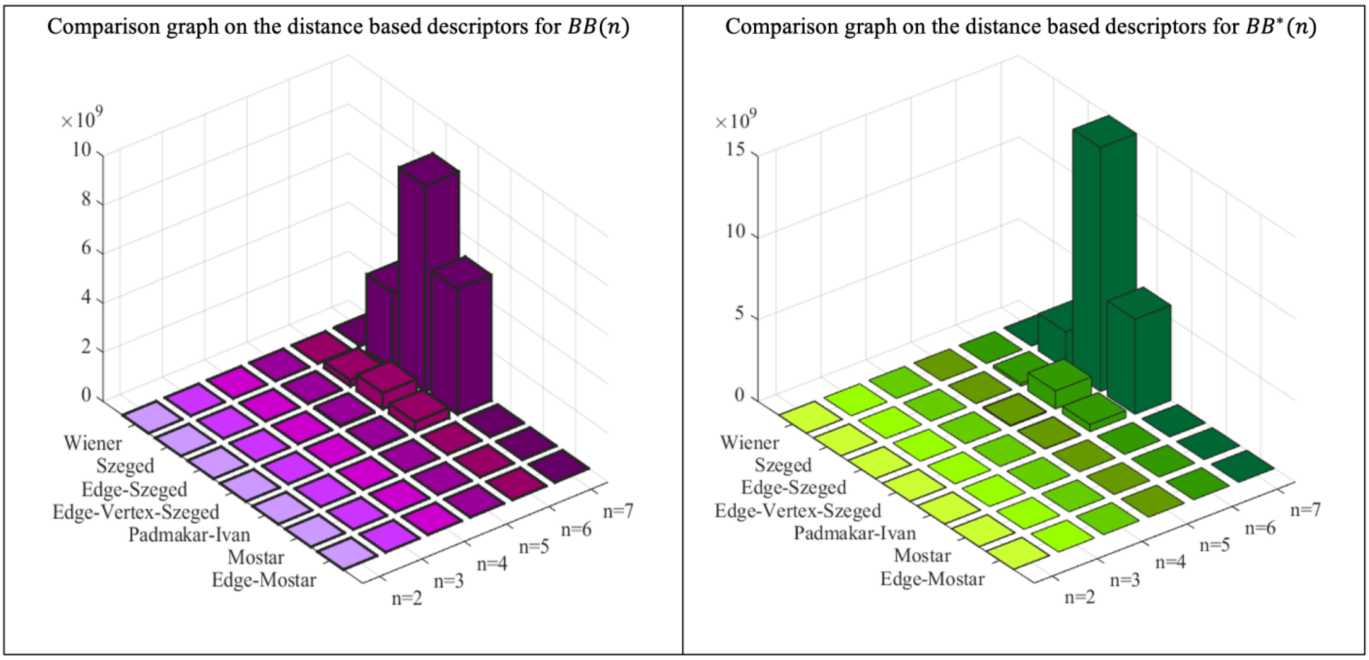


Figure 14. Comparison graph on the distance based descriptors for $BB(n)$ and $BB*(n)$.

Table 5. Numerical values of distance based descriptors for augmented benes network, $BB^*(n)$.

n	Padmakar-Ivan	Mostar	Edge-Mostar
2	1176	224	464
3	10768	1728	3808
4	78368	10624	24000
5	501056	58112	132992
6	2945152	296448	683776

Table 6. Topological network parameters of a graph ζ.

Network Parameter	Mathematical Expression
Diameter	$Dia(\zeta) = \max\{d_\zeta(\mu,\eta); \mu,\eta \in V(\zeta)\}$
Average Distance	$W(\zeta) = \sum_{\{\mu,\eta\}\subseteq V(\zeta)} w_v(\mu)w_v(\eta)d_\zeta(\mu,\eta)$
Message Traffic Density	$MTD(\zeta) = \frac{W(\zeta)\times\lvert V(\zeta)\rvert}{\lvert E(\zeta)\rvert}$
Network Throughput	$NT(\zeta) = \frac{\lvert E(\zeta)\rvert}{Dia(\zeta)}$
Graph Density	$GD(\zeta) = \frac{2\lvert E(\zeta)\rvert}{\lvert V(\zeta)\rvert(\lvert V(\zeta)\rvert - 1)}$
Total Connectivity	$TC(\zeta) = \frac{\lvert E(\zeta)\rvert}{\lvert V(\zeta)\rvert(\lvert V(\zeta)\rvert - 1)}$

given in Table 6, the parameters are computed for $BB(n)$, and $BB^*(n)$ and the results are described in Section 4.1 and Section 4.2 respectively.

4.1 Results on Benes Network, $BB(n), n \geq 2$

- **Diameter:**
  $Dia(BB(n)) = 2n$
- **Average Distance:**
  $W(BB(n)) = \frac{2^n}{3}\{(10n^3 - 9n^2 + 50n - 69)2^n + 69\}$
- **Message Traffic Density:**
  $MTD(BB(n)) = \frac{2^n(2n+1)}{12n}\{(10n^3 - 9n^2 + 50n - 69)2^n + 69\}$
- **Network Throughput:**
  $NT(BB(n)) = 2^{n+1}$
- **Graph Density:**
  $GD(BB(n)) = \frac{8n}{(2n+1)((2n+1)2^n - 1)}$
- **Total Connectivity:**
  $TC(BB(n)) = \frac{4n}{(2n+1)((2n+1)2^n - 1)}$

4.2 Results on Augmented Benes Network, $BB^*(n), n \geq 2$

- **Diameter:**
  $Dia(BB^*(n)) = 2n$
- **Average Distance:**
  $W(BB^*(n)) = \frac{2^n}{6}\{(20n^3 - 36n^2 + 148n - 207)2^n + 207\}$
- **Message Traffic Density:**

$$MTD(BB^*(n)) = \frac{2^{n+1}(2n+1)}{6(12n-1)}\{(20n^3 - 36n^2 + 148n - 207)2^n + 207\}$$

- **Network Throughput:**
$$NT(BB^*(n)) = \frac{2^{n-2}(12n-1)}{n}$$
- **Graph Density:**
$$GD(BB^*(n)) = \frac{12n-1}{(2n+1)((2n+1)2^n-1)}$$
- **Total Connectivity:**
$$TC(BB^*(n)) = \frac{12n-1}{2(2n+1)((2n+1)2^n-1)}$$

4.3 Comparative Analysis

Using the expressions obtained in Section 4.1 and Section 4.2, the numerical values of the parameters for a few fixed values of n were determined and graphically plotted. Figures 15 to 18 show the graphical representations.

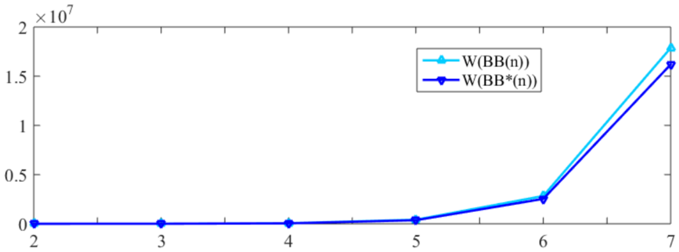


Figure 15. Comparison graph based on average distance.

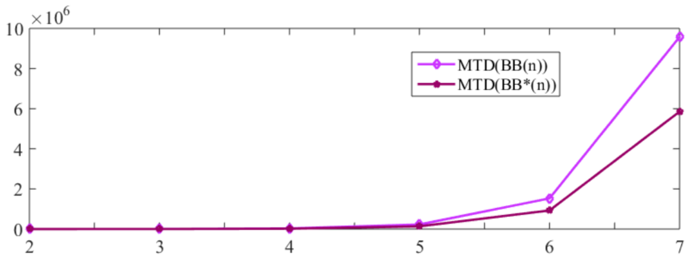


Figure 16. Comparison graph based on message traffic density.

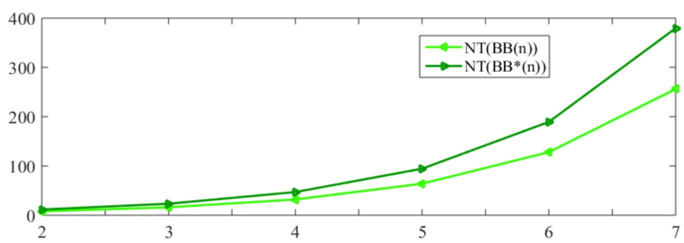


Figure 17. Comparison graph based on network throughput.

It is to be noted that, the comparison graph based on total connectivity is similar to the comparison graph

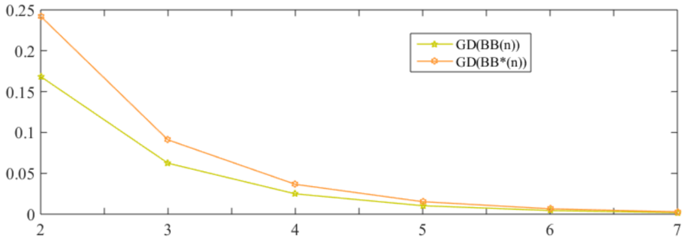


Figure 18. Comparison graph based on graph density.

based on graph density, which is given in Figure 18. From the obtained expressions as well as from the graphical presentations, it can be inferred that $BB*(n)$, has minimum average distance, low message traffic density, high network throughput, high graph density and high total connectivity, making it more efficient network than the benes network, $BB(n)$.

5 Broadcasting Algorithm

Broadcasting is an information dissemination problem in a connected graph, which is the simplest communication problem. In broadcasting process, a node or vertex of a network, called the source node or originating vertex, has a message that is disseminated from the source node to all other nodes, with the condition that each node can only transmit the message to at most one of its neighbors at a time. A spanning tree along which the message is broadcast from the source node to all other nodes is called broadcasting tree of the network. Furthest-Distance-First Protocol is a common strategy used to solve the broadcasting problem, in which, message is broadcasted from the source node to an uninformed adjacent node which leads to longest path in a tree. Broadcasting has various applications in virus spreading, internet messaging and management of distributed systems etc. Various broadcasting problems on networks and graphs have been studied extensively in the literature [3, 6].

Broadcasting Algorithm for Augmented Benes Network

In $BB^*(n)$, each node is labelled as (j,k) where j is a n-bit binary number and $0 \leq k \leq 2n$. The eccentricity of a vertex $\mu \in V(BB^*(n))$, $e(\mu) = \max\{d_{BB^*(n)}(\mu,\eta)\}$.

1. **Input:** Let S be the source node with a message to be broadcasted to all other nodes of the augmented benes network, $BB^*(n)$.
2. **Step 1:** Construct the broadcasting tree under the conditions

   (a) Source node $S(j,k)$ has either $k=0$ or $k=2n$.
   (b) Source node S has exactly three child nodes.
   (c) The broadcasting tree ends with node whose parent has only two children and the end node (j,k) is such that $k=0$ or $k=2n$.
3. **Step 2:** Apply Furthest-Distance-First Protocol to disseminate the message along the broadcasting tree.
4. **Step 3:** Stop after $e(S)+2$ steps, when n is even.
5. **Step 4:** Stop after $e(S)+3$ steps, when n is odd.

Proof of Correctness: The source node S disseminates the message according to Furthest-Distance-First protocol. Hence, S sends the message by giving priority to an uninformed adjacent node which leads to longest path in the broadcasting tree. Consider $BB^*(2)$, which can possibly have eight source nodes. The broadcasting tree generated for any selection of the source node remains isomorphic to graph T, given in Figure 19. Clearly the message dissemination ends after $e(S)+2$ steps, where $e(S)=4$. Similarly, the broadcasting tree for $BB^*(3)$ is given in Figure 20, which stops disseminating message after $e(S)+3$ steps, where $e(S)=6$.

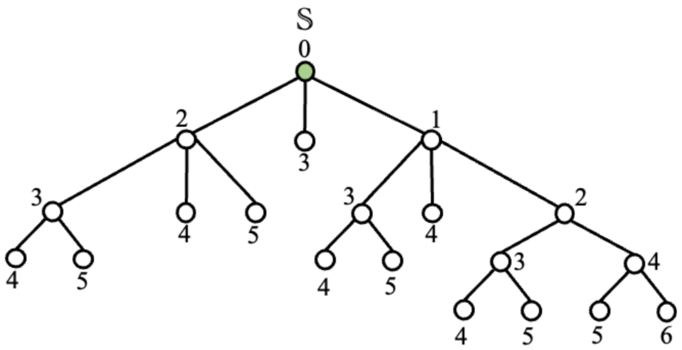


Figure 19. Broadcasting tree, T for $BB^*(2)$ with source node S.

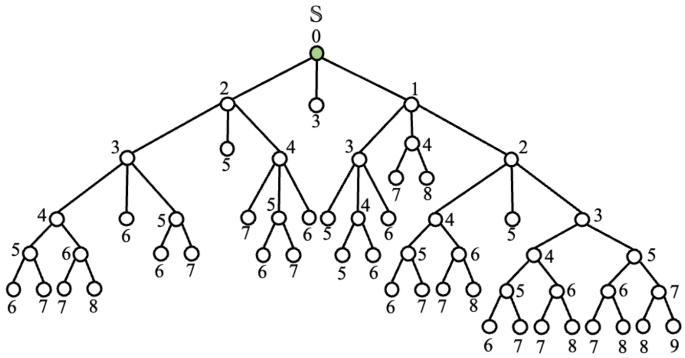


Figure 20. Broadcasting tree, T for $BB^*(3)$ with source node S.

□

6 Conclusions

In this work, a new class of interconnection network is constructed from the Benes network, called the augmented Benes network. The distance based descriptors for the n-dimensional benes network as well as for n-dimensional augmented benes network were evaluated and the analytical expressions are given in Section 3.1 and Section 3.2 respectively. Further, to recognize the efficiency of the newly derived network compared to the efficiency of benes network, the network parameters of the networks are computed and a comparative analysis was carried out. The numerical values of the topological descriptors for selected dimensions of the Benes network and augmented Benes network were computed from the analytical expressions, and the resulting graphical comparison is presented in Section 3.3. This study helps in characterising the topological properties of interconnection networks and helps to design new networks which brings advancements in technology and in field of computer science. The present work can be extended by exploring other classes of topological descriptors, such as degree-based and spectral descriptors, for the augmented Benes network. Additionally, investigating routing efficiency, fault tolerance, and scalability aspects of the proposed network can provide deeper insights into its practical applicability.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

AI Use Statement

The authors declare that no generative AI was used in the preparation of this manuscript.

Ethical Approval and Consent to Participate

Not applicable.

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