RESEARCH ARTICLE

On the Convergence and Order of An Extended 2-Point Super Class BBDF with Two Intermediate Points for Solving Stiff IVPs

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Abstract

This study investigates the order and convergence properties of an extended two-point superclass Block Backward Differentiation Formula (BBDF) incorporating two intermediate (off-step) points for the numerical solution of stiff initial value problems (IVPs). Stiff systems frequently arise in scientific and engineering applications and require numerical methods that combine stability, accuracy, and computational efficiency. While classical Backward Differentiation Formula (BDF) methods are well known for their stability, their sequential nature limits efficiency, particularly for large-scale problems. To address these limitations, the proposed framework extends the BBDF approach by introducing off-step points within a two-point superclass structure, allowing multiple solution values to be computed simultaneously. This block formulation enhances parallelism and reduces the number of integration steps required. The method is formulated in matrix form, and its theoretical properties are rigorously analyzed. The order of the method is established using a Taylor series expansion of the associated linear difference operator, and the scheme is shown to achieve fifth-order accuracy. The method's convergence is examined using the standard consistency and zero-stability criteria. The proposed scheme satisfies the necessary and sufficient conditions for the convergence of linear multistep methods. The results confirm that the proposed method provides a reliable and efficient framework for solving stiff systems.

Keywords: BBDF, convergence, order, stiff IVPs, superclass.

1 Introduction

Ordinary differential equations (ODEs) play a central role in the mathematical modeling of real-world phenomena arising in science, engineering, biology, and finance. However, in practical applications, such models are often stiff, meaning that their solutions exhibit components evolving at widely different time scales. This stiffness imposes severe restrictions on numerical solvers, particularly explicit methods, which require prohibitively small step sizes to maintain

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stability and accuracy. Stiff systems arise frequently in applications such as chemical engineering, nonlinear mechanics, and the life sciences. Stiffness is typically associated with systems whose Jacobian possesses eigenvalues with negative real parts that vary widely in magnitude, as discussed in the classical treatment of Lambert [1]. Such problems present computational challenges because many standard numerical schemes require excessively small step sizes to maintain stability. Stiff differential systems also arise in heat transfer, porous media flow, nanofluid transport, and thermal convection problems, where accurate and stable numerical solvers are essential [2–5]. Such stiff systems are equally prevalent in biological and pharmacokinetic modelling, where stability of the numerical scheme is critical for reliable simulation [6].

Among the most successful approaches for solving stiff IVPs are implicit linear multistep methods, particularly the Backward Differentiation Formula (BDF), which are well-known for their strong stability properties and ability to handle stiff systems without severe step size restrictions [7]. However, classical BDF methods are sequential in nature and incur high computational cost from repeated function evaluations and Jacobian computations. To overcome these limitations, Block Backward Differentiation Formula (BBDF) methods were developed, extending classical BDF by computing multiple solution values simultaneously within a single step. Early block formulations, including those for higher-order ODEs [8] and first-order stiff systems [9], demonstrated improved performance over classical BDF. Subsequent developments incorporated off-step points to enhance accuracy and stability, yielding further improvements for highly stiff problems [10, 11].

Over the years, several modifications of BBDF-type methods have been reported. Diagonally implicit BBDF methods structure the coefficient matrix in lower triangular form [12, 13], simplifying the solution of nonlinear systems while preserving stability properties such as A-stability. Variable step-size strategies have also been incorporated to adaptively control error and improve efficiency [14]. Beyond these, hybrid and extended superclass BBDF methods have been proposed to address specific drawbacks of traditional block schemes: hybrid approaches combine multistep and one-step techniques, while superclass formulations introduce free parameters to generate families of schemes with optimized stability regions [15, 16]. Comparative studies have further confirmed the advantages of block-type methods over classical sequential schemes [17], reflecting the ongoing effort to balance accuracy, stability, and computational efficiency.

Despite the substantial progress in the development of BBDF methods, several challenges remain. Many existing methods suffer from limitations such as reduced accuracy at large step sizes, high computational cost due to fully implicit formulations, or insufficient exploration of parameterized super-class structures. Moreover, the convergence and stability analysis of advanced BBDF variants, particularly those involving off-step points and diagonal implicitness, is still an active area of research. Motivated by these challenges, this study focuses on the order and convergence analysis of an improved BBDF-type method for solving stiff initial value problems. The present study examines the order and convergence properties of the extended two-point superclass BBDF with off-step points. A detailed derivation of the scheme is provided in [11].

The remainder of this paper is organized as follows. Section 2 presents the mathematical formulation of the model problem and the derivation of the proposed method. Section 3 discusses the theoretical analysis, including consistency, zero-stability, and convergence, and Section 4 concludes the study with discussions and future research directions.

2 Order of the Method

The scheme of the method is given:

$$
\begin{cases}
y_{n+\frac{1}{2}} = -\frac{1}{20}\frac{174\rho+1}{80\rho-3}y_{n-1} + \frac{9}{4}\frac{20\rho+1}{80\rho-3}y_n + \frac{27}{4}\frac{10\rho-1}{80\rho-3}y_{n+1} \\
\qquad - \frac{9}{5}\frac{16\rho-1}{80\rho-3}y_{n+\frac{3}{2}} + \frac{1}{4}\frac{20\rho-1}{80\rho-3}y_{n+2} + \frac{3}{80\rho-3}hf_{n+\frac{1}{2}} - \frac{3}{80\rho-3}h\rho f_{n-1}, \\
y_{n+1} = -\frac{1}{45}\frac{134\rho-2}{19\rho+2}y_{n-1} - \frac{1}{3}\frac{117\rho+4}{19\rho+2}y_n + \frac{4}{9}\frac{47\rho+16}{19\rho+2}y_{n+\frac{1}{2}} \\
\qquad + \frac{4}{15}\frac{31\rho-16}{19\rho+2}y_{n+\frac{3}{2}} - \frac{1}{9}\frac{113\rho-4}{19\rho+2}y_{n+2} + \frac{4}{19\rho+2}hf_{n+1} - \frac{4}{19\rho+2}h\rho f_{n-\frac{1}{2}}, \\
y_{n+\frac{3}{2}} = \frac{1}{4}\frac{4\rho+1}{16\rho-31}y_{n-1} + \frac{5}{4}\frac{38\rho-5}{16\rho-31}y_n - \frac{5(16\rho-5)}{16\rho-31}y_{n+\frac{3}{2}} \\
\qquad + \frac{45}{4}\frac{4\rho-5}{16\rho-31}y_{n+1} + \frac{5}{4}\frac{2\rho+5}{16\rho-31}y_{n+2} - \frac{15}{16\rho-31}hf_{n+\frac{3}{2}} + \frac{15}{16\rho-31}h\rho f_n, \\
y_{n+2} = -\frac{1}{5}\frac{\rho+4}{\rho-54}y_{n-1} + \frac{9(\rho+2)}{\rho-54}y_n + \frac{4(3\rho-16)}{\rho-54}y_{n+\frac{1}{2}} - \frac{27(\rho-4)}{\rho-54}y_{n+1} \\
\qquad + \frac{36}{5}\frac{\rho-16}{\rho-54}y_{n+\frac{3}{2}} - \frac{12}{\rho-54}hf_{n+2} + \frac{12}{\rho-54}h\rho f_{n+\frac{1}{2}}.
\end{cases}
\tag{1}
$$

The order of method (2) is determined in this section for the case $\rho = -\frac{3}{5}$. The scheme can be expressed in the form

$$
\begin{cases}
y_{n+\frac{1}{2}} = -\dfrac{517}{5100}y_{n-1} + \dfrac{33}{68}y_n + \dfrac{63}{68}y_{n+1} - \dfrac{159}{425}y_{n+\frac{3}{2}} \\
\qquad + \dfrac{13}{204}y_{n+2} - \dfrac{1}{17}hf_{n+\frac{1}{2}} - \dfrac{3}{85}hf_{n-1}, \\
y_{n+1} = -\dfrac{421}{2115}y_{n-1} - \dfrac{31}{141}y_n + \dfrac{244}{423}y_{n+\frac{1}{2}} + \dfrac{692}{705}y_{n+\frac{3}{2}} \\
\qquad - \dfrac{59}{423}y_{n+2} - \dfrac{20}{47}hf_{n+1} - \dfrac{12}{47}hf_{n-\frac{1}{2}}, \\
y_{n+\frac{3}{2}} = \dfrac{1}{116}y_{n-1} + \dfrac{695}{812}y_n - \dfrac{365}{203}y_{n+\frac{1}{2}} + \dfrac{1665}{812}y_{n+1} \\
\qquad - \dfrac{95}{812}y_{n+2} + \dfrac{75}{203}hf_{n+\frac{3}{2}} + \dfrac{45}{203}hf_n, \\
y_{n+2} = -\dfrac{17}{1365}y_{n-1} - \dfrac{3}{13}y_n + \dfrac{356}{273}y_{n+\frac{1}{2}} - \dfrac{207}{91}y_{n+1} \\
\qquad + \dfrac{996}{455}y_{n+\frac{3}{2}} + \dfrac{20}{91}hf_{n+2} + \dfrac{12}{91}hf_{n+\frac{1}{2}}.
\end{cases}
\tag{2}
$$

To facilitate the order computation, the scheme in (3) is reformulated in matrix notation as

$$
\sum_{j=0}^{1} \alpha_i Y_{m+j-1} = h \sum_{j=0}^{1} \beta_i F_{m+j-1}. \tag{3}
$$

The matrix form associated with equation (3) is:

$$
\begin{aligned}
&\begin{pmatrix} 0 & \frac{517}{5100} & 0 & -\frac{33}{68} \\ 0 & \frac{421}{2115} & 0 & \frac{31}{141} \\ 0 & -\frac{1}{116} & 0 & -\frac{695}{812} \\ 0 & -\frac{17}{1365} & 0 & \frac{3}{13} \end{pmatrix}
\begin{pmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} \\
&+ \begin{pmatrix} 1 & -\frac{63}{68} & \frac{159}{425} & -\frac{13}{204} \\ -\frac{244}{423} & 1 & -\frac{692}{705} & \frac{59}{423} \\ \frac{365}{203} & -\frac{1665}{812} & 1 & \frac{95}{812} \\ -\frac{356}{273} & \frac{207}{91} & -\frac{996}{455} & 1 \end{pmatrix}
\begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix} \\
&= h \begin{pmatrix} 0 & -\frac{3}{85} & 0 & 0 \\ 0 & 0 & -\frac{12}{47} & 0 \\ 0 & 0 & 0 & \frac{45}{203} \\ 0 & 0 & 0 & 0 \end{pmatrix}
\begin{pmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix} \\
&+ h \begin{pmatrix} -\frac{1}{17} & 0 & 0 & 0 \\ 0 & -\frac{20}{47} & 0 & 0 \\ 0 & 0 & \frac{75}{203} & 0 \\ \frac{12}{91} & 0 & 0 & \frac{20}{91} \end{pmatrix}
\begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix}.
\end{aligned}
\tag{4}
$$

where α_0^*, α_1^*, β_0^*, and β_1^* are square matrices defined by

$$
\begin{aligned}
\alpha_0^* &= \begin{pmatrix} 0 & \frac{517}{5100} & 0 & -\frac{33}{68} \\ 0 & \frac{421}{2115} & 0 & \frac{31}{141} \\ 0 & -\frac{1}{116} & 0 & -\frac{695}{812} \\ 0 & -\frac{17}{1365} & 0 & \frac{3}{13} \end{pmatrix}, \\
\alpha_1^* &= \begin{pmatrix} 1 & -\frac{63}{68} & \frac{159}{425} & -\frac{13}{204} \\ -\frac{244}{423} & 1 & -\frac{692}{705} & \frac{59}{423} \\ \frac{365}{203} & -\frac{1665}{812} & 1 & \frac{95}{812} \\ -\frac{356}{273} & \frac{207}{91} & -\frac{996}{455} & 1 \end{pmatrix}, \\
\beta_0^* &= \begin{pmatrix} 0 & -\frac{3}{85} & 0 & 0 \\ 0 & 0 & -\frac{12}{47} & 0 \\ 0 & 0 & 0 & \frac{45}{203} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\beta_1^* &= \begin{pmatrix} -\frac{1}{17} & 0 & 0 & 0 \\ 0 & -\frac{20}{47} & 0 & 0 \\ 0 & 0 & \frac{75}{203} & 0 \\ \frac{12}{91} & 0 & 0 & \frac{20}{91} \end{pmatrix}.
\end{aligned}
\tag{5}
$$

Let

$$
\alpha_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \alpha_1 = \begin{pmatrix} \frac{517}{5100} \\ \frac{421}{2115} \\ -\frac{1}{116} \\ -\frac{17}{1365} \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
$$

$$
\alpha_3 = \begin{pmatrix} -\frac{33}{68} \\ \frac{31}{141} \\ -\frac{695}{812} \\ \frac{3}{13} \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 1 \\ -\frac{244}{423} \\ \frac{365}{203} \\ -\frac{356}{273} \end{pmatrix}, \quad \alpha_5 = \begin{pmatrix} -\frac{63}{68} \\ 1 \\ -\frac{1665}{812} \\ \frac{207}{91} \end{pmatrix},
$$

$$
\alpha_6 = \begin{pmatrix} \frac{159}{425} \\ -\frac{692}{705} \\ 1 \\ -\frac{996}{455} \end{pmatrix}, \quad \alpha_7 = \begin{pmatrix} -\frac{13}{204} \\ \frac{59}{423} \\ \frac{95}{812} \\ 1 \end{pmatrix}.
$$

Similarly,

$$\beta_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} -\frac{3}{85} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 \\ -\frac{12}{47} \\ 0 \\ 0 \end{pmatrix},$$

$$\beta_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{45}{203} \\ 0 \end{pmatrix}, \quad \beta_4 = \begin{pmatrix} -\frac{1}{17} \\ 0 \\ 0 \\ \frac{12}{91} \end{pmatrix}, \quad \beta_5 = \begin{pmatrix} 0 \\ -\frac{20}{47} \\ 0 \\ 0 \end{pmatrix},$$

$$\beta_6 = \begin{pmatrix} 0 \\ 0 \\ \frac{75}{203} \\ 0 \end{pmatrix}, \quad \beta_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{20}{91} \end{pmatrix}.$$

Definition 2.1:

The order of the block method (2) is determined through its associated linear operator L, defined by

$$L\{y(x), h\} = \sum_{j=0}^{k} \left[\alpha_j y\left(x + j\frac{h}{2}\right) - h\beta_j y'\left(x + j\frac{h}{2}\right) \right]. \tag{6}$$

The difference operator (6) and the associated method (2) are considered to be of order p if

$$E_0 = E_1 = E_2 = \cdots = E_p = 0 \quad \text{and} \quad E_{p+1} \neq 0.$$

Expanding the function $y\left(x + j\frac{h}{2}\right)$ and its derivative $y'\left(x + j\frac{h}{2}\right)$ as Taylor's series about x,

$$\begin{aligned} y\left(x + j\frac{h}{2}\right) &= y(x) + j\frac{h}{2}y'(x) + \frac{\left(j\frac{h}{2}\right)^2}{2!}y''(x) \\ &\quad + \frac{\left(j\frac{h}{2}\right)^3}{3!}y'''(x) + \cdots . \end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned} y'\left(x + j\frac{h}{2}\right) &= y'(x) + j\frac{h}{2}y''(x) + \frac{\left(j\frac{h}{2}\right)^2}{2!}y'''(x) \\ &\quad + \frac{\left(j\frac{h}{2}\right)^3}{3!}y^{(4)}(x) + \cdots . \end{aligned} \tag{8}$$

Substituting (7) and (8) into (6), we obtain

$$\begin{aligned} L\{y(x), h\} &= \sum_{j=0}^{7} \alpha_j \left[y(x) + j\frac{h}{2}y'(x) + \frac{(jh/2)^2}{2!}y''(x) \right. \\ &\quad \left. + \frac{(jh/2)^3}{3!}y'''(x) + \cdots \right] \\ &\quad - h\sum_{j=0}^{7} \beta_j \left[y'(x) + j\frac{h}{2}y''(x) + \frac{(jh/2)^2}{2!}y'''(x) \right. \\ &\quad \left. + \frac{(jh/2)^3}{3!}y^{(4)}(x) + \cdots \right] \\ &= \sum_{j=0}^{7} \alpha_j y(x) + \frac{1}{2}\sum_{j=0}^{7} [j\alpha_j - 2\beta_j] h\, y'(x) \\ &\quad + \frac{1}{4}\sum_{j=0}^{7} \left[\frac{j^2}{2!}\alpha_j - 2j\beta_j \right] h^2 y''(x) \\ &\quad + \frac{1}{8}\sum_{j=0}^{7} \left[\frac{j^3}{3!}\alpha_j - \frac{2j^2}{2!}\beta_j \right] h^3 y'''(x) + \cdots . \end{aligned} \tag{9}$$

In this case,

$$\begin{aligned} E_0 &= \sum_{j=0}^{7} \alpha_j = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{517}{5100} \\ \frac{421}{2115} \\ -\frac{1}{116} \\ -\frac{17}{1365} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{33}{68} \\ \frac{31}{141} \\ -\frac{695}{812} \\ \frac{3}{13} \end{pmatrix} \\ &\quad + \begin{pmatrix} 1 \\ -\frac{244}{423} \\ \frac{365}{203} \\ -\frac{356}{273} \end{pmatrix} + \begin{pmatrix} -\frac{63}{68} \\ 1 \\ -\frac{1665}{812} \\ \frac{207}{91} \end{pmatrix} + \begin{pmatrix} \frac{159}{425} \\ -\frac{692}{705} \\ 1 \\ -\frac{996}{455} \end{pmatrix} + \begin{pmatrix} -\frac{13}{204} \\ \frac{59}{423} \\ \frac{95}{812} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{10}$$

$$\begin{aligned} E_1 &= \sum_{j=0}^{7} (j\alpha_j - 2\beta_j) \\ &= (0)\alpha_0 + (1)\alpha_1 + (2)\alpha_2 + (3)\alpha_3 + (4)\alpha_4 + (5)\alpha_5 + (6)\alpha_6 + (7)\alpha_7 \\ &\quad - 2\left(\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7\right) \\ &= (0)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (1)\begin{pmatrix} \frac{517}{5100} \\ \frac{421}{2115} \\ -\frac{1}{116} \\ -\frac{17}{1365} \end{pmatrix} + (2)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (3)\begin{pmatrix} -\frac{33}{68} \\ \frac{31}{141} \\ -\frac{695}{812} \\ \frac{3}{13} \end{pmatrix} \\ &\quad + (4)\begin{pmatrix} 1 \\ -\frac{244}{423} \\ \frac{365}{203} \\ -\frac{356}{273} \end{pmatrix} + (5)\begin{pmatrix} -\frac{63}{68} \\ 1 \\ -\frac{1665}{812} \\ \frac{207}{91} \end{pmatrix} + (6)\begin{pmatrix} \frac{159}{425} \\ -\frac{692}{705} \\ 1 \\ -\frac{996}{455} \end{pmatrix} + (7)\begin{pmatrix} -\frac{13}{204} \\ \frac{59}{423} \\ \frac{95}{812} \\ 1 \end{pmatrix} \\ &\quad - 2\left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{3}{85} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{12}{47} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{45}{203} \\ 0 \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} -\frac{1}{17} \\ 0 \\ 0 \\ \frac{12}{91} \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{20}{47} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{75}{203} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{20}{91} \end{pmatrix} \right] \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{11}$$

$$
\begin{aligned}
E_2 &= \sum_{j=0}^{7}\left(\frac{1}{2!}j^2\alpha_j - 2j\beta_j\right) \\
&= \frac{1}{2!}\Big[(0)^2\alpha_0 + (1)^2\alpha_1 + (2)^2\alpha_2 + (3)^2\alpha_3 + (4)^2\alpha_4 + (5)^2\alpha_5 + (6)^2\alpha_6 + (7)^2\alpha_7\Big] \\
&\quad - 2\Big[(0)\beta_0 + (1)\beta_1 + (2)\beta_2 + (3)\beta_3 + (4)\beta_4 + (5)\beta_5 + (6)\beta_6 + (7)\beta_7\Big] \\
&= \frac{1}{2!}\Bigg[(0)^2\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (1)^2\begin{pmatrix}\frac{517}{5100}\\ \frac{421}{2115}\\ -\frac{1}{116}\\ -\frac{17}{1365}\end{pmatrix} + (2)^2\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (3)^2\begin{pmatrix}-\frac{33}{68}\\ \frac{31}{141}\\ -\frac{695}{812}\\ \frac{3}{13}\end{pmatrix} \\
&\quad + (4)^2\begin{pmatrix}1\\ -\frac{244}{423}\\ \frac{365}{203}\\ -\frac{356}{273}\end{pmatrix} + (5)^2\begin{pmatrix}-\frac{63}{68}\\ 1\\ -\frac{1665}{812}\\ \frac{207}{91}\end{pmatrix} + (6)^2\begin{pmatrix}\frac{159}{425}\\ -\frac{692}{705}\\ 1\\ -\frac{996}{455}\end{pmatrix} + (7)^2\begin{pmatrix}-\frac{13}{204}\\ \frac{59}{423}\\ \frac{95}{812}\\ 1\end{pmatrix}\Bigg] \\
&\quad - 2\Bigg[(0)\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (1)\begin{pmatrix}-\frac{3}{85}\\0\\0\\0\end{pmatrix} + (2)\begin{pmatrix}0\\-\frac{12}{47}\\0\\0\end{pmatrix} + (3)\begin{pmatrix}0\\0\\ \frac{45}{203}\\0\end{pmatrix} \\
&\quad + (4)\begin{pmatrix}-\frac{1}{17}\\0\\0\\ \frac{12}{91}\end{pmatrix} + (5)\begin{pmatrix}0\\-\frac{20}{47}\\0\\0\end{pmatrix} + (6)\begin{pmatrix}0\\0\\ \frac{75}{203}\\0\end{pmatrix} + (7)\begin{pmatrix}0\\0\\0\\ \frac{20}{91}\end{pmatrix}\Bigg] \\
&= \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
E_3 &= \sum_{j=0}^{7}\left(\frac{1}{3!}j^3\alpha_j - 2\frac{1}{2!}j^2\beta_j\right) \\
&= \frac{1}{3!}\Bigg[(0)^3\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (1)^3\begin{pmatrix}\frac{517}{5100}\\ \frac{421}{2115}\\ -\frac{1}{116}\\ -\frac{17}{1365}\end{pmatrix} + (2)^3\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (3)^3\begin{pmatrix}-\frac{33}{68}\\ \frac{31}{141}\\ -\frac{695}{812}\\ \frac{3}{13}\end{pmatrix} \\
&\quad + (4)^3\begin{pmatrix}1\\ -\frac{244}{423}\\ \frac{365}{203}\\ -\frac{356}{273}\end{pmatrix} + (5)^3\begin{pmatrix}-\frac{63}{68}\\ 1\\ -\frac{1665}{812}\\ \frac{207}{91}\end{pmatrix} + (6)^3\begin{pmatrix}\frac{159}{425}\\ -\frac{692}{705}\\ 1\\ -\frac{996}{455}\end{pmatrix} + (7)^3\begin{pmatrix}-\frac{13}{204}\\ \frac{59}{423}\\ \frac{95}{812}\\ 1\end{pmatrix}\Bigg] \\
&\quad - \Bigg[(0)^2\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (1)^2\begin{pmatrix}-\frac{3}{85}\\0\\0\\0\end{pmatrix} + (2)^2\begin{pmatrix}0\\-\frac{12}{47}\\0\\0\end{pmatrix} + (3)^2\begin{pmatrix}0\\0\\ \frac{45}{203}\\0\end{pmatrix} \\
&\quad + (4)^2\begin{pmatrix}-\frac{1}{17}\\0\\0\\ \frac{12}{91}\end{pmatrix} + (5)^2\begin{pmatrix}0\\-\frac{20}{47}\\0\\0\end{pmatrix} + (6)^2\begin{pmatrix}0\\0\\ \frac{75}{203}\\0\end{pmatrix} + (7)^2\begin{pmatrix}0\\0\\0\\ \frac{20}{91}\end{pmatrix}\Bigg] \\
&= \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
E_4 &= \sum_{j=0}^{7}\left(\frac{1}{4!}j^4\alpha_j - 2\frac{1}{3!}j^3\beta_j\right) \\
&= \frac{1}{4!}\Bigg[(0)^4\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (1)^4\begin{pmatrix}\frac{517}{5100}\\ \frac{421}{2115}\\ -\frac{1}{116}\\ -\frac{17}{1365}\end{pmatrix} + (2)^4\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (3)^4\begin{pmatrix}-\frac{33}{68}\\ \frac{31}{141}\\ -\frac{695}{812}\\ \frac{3}{13}\end{pmatrix} \\
&\quad + (4)^4\begin{pmatrix}1\\ -\frac{244}{423}\\ \frac{365}{203}\\ -\frac{356}{273}\end{pmatrix} + (5)^4\begin{pmatrix}-\frac{63}{68}\\ 1\\ -\frac{1665}{812}\\ \frac{207}{91}\end{pmatrix} + (6)^4\begin{pmatrix}\frac{159}{425}\\ -\frac{692}{705}\\ 1\\ -\frac{996}{455}\end{pmatrix} + (7)^4\begin{pmatrix}-\frac{13}{204}\\ \frac{59}{423}\\ \frac{95}{812}\\ 1\end{pmatrix}\Bigg] \\
&\quad - \frac{2}{3!}\Bigg[(0)^3\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (1)^3\begin{pmatrix}-\frac{3}{85}\\0\\0\\0\end{pmatrix} + (2)^3\begin{pmatrix}0\\-\frac{12}{47}\\0\\0\end{pmatrix} + (3)^3\begin{pmatrix}0\\0\\ \frac{45}{203}\\0\end{pmatrix} \\
&\quad + (4)^3\begin{pmatrix}-\frac{1}{17}\\0\\0\\ \frac{12}{91}\end{pmatrix} + (5)^3\begin{pmatrix}0\\-\frac{20}{47}\\0\\0\end{pmatrix} + (6)^3\begin{pmatrix}0\\0\\ \frac{75}{203}\\0\end{pmatrix} + (7)^3\begin{pmatrix}0\\0\\0\\ \frac{20}{91}\end{pmatrix}\Bigg] \\
&= \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
E_5 &= \sum_{j=0}^{7}\left(\frac{1}{5!}j^5\alpha_j - 2\frac{1}{4!}j^4\beta_j\right) \\
&= \frac{1}{5!}\Bigg[(0)^5\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (1)^5\begin{pmatrix}\frac{517}{5100}\\ \frac{421}{2115}\\ -\frac{1}{116}\\ -\frac{17}{1365}\end{pmatrix} + (2)^5\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (3)^5\begin{pmatrix}-\frac{33}{68}\\ \frac{31}{141}\\ -\frac{695}{812}\\ \frac{3}{13}\end{pmatrix} \\
&\quad + (4)^5\begin{pmatrix}1\\ -\frac{244}{423}\\ \frac{365}{203}\\ -\frac{356}{273}\end{pmatrix} + (5)^5\begin{pmatrix}-\frac{63}{68}\\ 1\\ -\frac{1665}{812}\\ \frac{207}{91}\end{pmatrix} + (6)^5\begin{pmatrix}\frac{159}{425}\\ -\frac{692}{705}\\ 1\\ -\frac{996}{455}\end{pmatrix} + (7)^5\begin{pmatrix}-\frac{13}{204}\\ \frac{59}{423}\\ \frac{95}{812}\\ 1\end{pmatrix}\Bigg] \\
&\quad - \frac{2}{4!}\Bigg[(0)^4\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (1)^4\begin{pmatrix}-\frac{3}{85}\\0\\0\\0\end{pmatrix} + (2)^4\begin{pmatrix}0\\-\frac{12}{47}\\0\\0\end{pmatrix} + (3)^4\begin{pmatrix}0\\0\\ \frac{45}{203}\\0\end{pmatrix} \\
&\quad + (4)^4\begin{pmatrix}-\frac{1}{17}\\0\\0\\ \frac{12}{91}\end{pmatrix} + (5)^4\begin{pmatrix}0\\-\frac{20}{47}\\0\\0\end{pmatrix} + (6)^4\begin{pmatrix}0\\0\\ \frac{75}{203}\\0\end{pmatrix} + (7)^4\begin{pmatrix}0\\0\\0\\ \frac{20}{91}\end{pmatrix}\Bigg] \\
&= \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.
\end{aligned} \tag{15}
$$

$$E_6 = \sum_{j=0}^{7} \left(\frac{1}{6!} j^6 \alpha_j - 2 \frac{1}{5!} j^5 \beta_j \right)$$

$$= \frac{1}{5!} \Bigg[(0)^6 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (1)^6 \begin{pmatrix} \frac{517}{5100} \\ \frac{421}{2115} \\ -\frac{1}{116} \\ -\frac{17}{1365} \end{pmatrix} + (2)^6 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (3)^6 \begin{pmatrix} -\frac{33}{68} \\ \frac{31}{141} \\ -\frac{695}{812} \\ \frac{3}{13} \end{pmatrix}$$

$$+ (4)^6 \begin{pmatrix} 1 \\ -\frac{244}{423} \\ \frac{365}{203} \\ -\frac{356}{273} \end{pmatrix} + (5)^6 \begin{pmatrix} -\frac{63}{68} \\ 1 \\ -\frac{1665}{812} \\ \frac{207}{91} \end{pmatrix} + (6)^6 \begin{pmatrix} \frac{159}{425} \\ -\frac{692}{705} \\ 1 \\ -\frac{996}{455} \end{pmatrix}$$

$$+ (7)^6 \begin{pmatrix} -\frac{7}{116} - \frac{13}{204} \\ \frac{1}{72} - \frac{59}{423} \\ \frac{145}{492} - \frac{95}{812} \\ 1 - 1 \end{pmatrix} \Bigg]$$

$$- \frac{2}{5!} \Bigg[(0)^5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (1)^5 \begin{pmatrix} -\frac{3}{85} \\ 0 \\ 0 \\ 0 \end{pmatrix} + (2)^5 \begin{pmatrix} 0 \\ -\frac{12}{47} \\ 0 \\ 0 \end{pmatrix} + (3)^5 \begin{pmatrix} 0 \\ 0 \\ \frac{45}{203} \\ 0 \end{pmatrix}$$

$$+ (4)^5 \begin{pmatrix} -\frac{1}{17} \\ 0 \\ 0 \\ \frac{12}{91} \end{pmatrix} + (5)^5 \begin{pmatrix} 0 \\ -\frac{20}{47} \\ 0 \\ 0 \end{pmatrix} + (6)^5 \begin{pmatrix} 0 \\ 0 \\ \frac{75}{203} \\ 0 \end{pmatrix} + (7)^5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{20}{91} \end{pmatrix} \Bigg]$$

$$= \begin{pmatrix} -\frac{5}{68} \\ \frac{317}{4230} \\ \frac{77}{62500} \\ -\frac{739}{2} \end{pmatrix}. \tag{16}$$

Therefore, the method (2) is of order 5, with error constant

$$E_6 = \begin{pmatrix} -\dfrac{5}{68} \\ \dfrac{317}{4230} \\ \dfrac{77}{62500} \\ -\dfrac{739}{2} \end{pmatrix}.$$

3 Convergence of the Method

Convergence is a fundamental requirement for any reliable linear multistep scheme. In this section, the convergence of method (2) is examined. It is well established that consistency together with zero stability forms the basis for convergence of linear multistep methods (Abasi et al. [10]). For completeness, the relevant definitions associated with convergence are recalled from Ibrahim et al. [9] and stated below.

Definition 3.1 (Linear Multi-step Method)

A general linear multi-step method (LMM) takes the following form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}. \tag{17}$$

where α_j and β_j are constants and $\alpha_k \neq 0$. The coefficients α_0 and β_0 cannot both be zero at the same time. For any linear k-step method, α_k is normalized to 1.

Definition 3.2 (Linear Difference Operator L)

The linear difference operator L associated with the linear multi-step method (17) is defined by

$$L\{y(x), h\} = \sum_{j=0}^{k} \left[\alpha_j y(x + jh) - h\beta_j y'(x + jh) \right]. \tag{18}$$

where $y(x)$ is an arbitrary test function and is continuously differentiable on $[a, b]$. Expanding $y(x + jh)$ and $y'(x + jh)$ as a Taylor's series about x, and collecting common terms yields:

$$\begin{aligned} L\{y(x), h\} = c_0 y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + \cdots \\ + c_q h^q y^{(q)}(x_n) + \cdots . \end{aligned} \tag{19}$$

where c_q are constants given by

$$\begin{aligned} c_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k, \\ c_1 &= \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k), \\ &\vdots \\ c_q &= \frac{1}{q!} (\alpha_1 + 2^q \alpha_2 + \cdots + k^q \alpha_k) \\ &\quad - \frac{1}{(q-1)!} \left(\beta_1 + 2^{q-1} \beta_2 + \cdots + k^{q-1} \beta_k \right), \\ &\qquad q = 2, 3, \ldots . \end{aligned} \tag{20}$$

Definition 3.3 (Consistency)

The linear multistep method (17) is consistent provided its order satisfies $p \geq 1$. From (20), it follows that consistency holds if and only if

$$\sum_{j=0}^{k} \alpha_j = 0 \quad \text{and} \quad \sum_{j=0}^{k} j\alpha_j = \beta_j. \tag{21}$$

It also follows from equation (20) that the linear multi-step method (17) is consistent if and only if

$$\rho(1) = 0 \quad \text{and} \quad \rho'(1) = \sigma(1). \tag{22}$$

Definition 3.4 (Zero stability)

The LMM (17) is said to be zero stable if all the roots of the first characteristic polynomial have modulus less than or equal to unity, and any root with modulus equal to 1 is simple.

4 Consistency of the Method

As established in Section 2, the two-point ESBBDF with off-step points introduced by Abdulrahman and Hamisu [11] attains order five, thereby meeting, and in fact exceeding, the requirement stated in Definition 3.3. It remains to verify that the method satisfies the consistency criterion precisely when the following conditions hold:

$$\sum_{j=0}^{7} \alpha_j = 0. \tag{23}$$

$$\sum_{j=0}^{7} j\alpha_j = 2\sum_{j=0}^{7} \beta_j. \tag{24}$$

where the α_j's and β_j's are as previously defined.

The first condition, which is equation (23), becomes

$$\begin{aligned}\sum_{j=0}^{7} \alpha_j &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 \\ &= \begin{pmatrix}0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}\frac{517}{5100}\\ \frac{421}{2115}\\ -\frac{1}{116}\\ -\frac{17}{1365}\end{pmatrix} + \begin{pmatrix}0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}-\frac{33}{68}\\ \frac{31}{141}\\ -\frac{695}{812}\\ \frac{3}{13}\end{pmatrix} \\ &+ \begin{pmatrix}1\\ -\frac{244}{423}\\ \frac{365}{203}\\ -\frac{356}{273}\end{pmatrix} + \begin{pmatrix}-\frac{63}{68}\\ 1\\ -\frac{1665}{812}\\ \frac{207}{91}\end{pmatrix} + \begin{pmatrix}\frac{159}{425}\\ -\frac{692}{705}\\ 1\\ -\frac{996}{455}\end{pmatrix} + \begin{pmatrix}-\frac{13}{204}\\ \frac{59}{423}\\ \frac{95}{812}\\ 1\end{pmatrix} \\ &= \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.\end{aligned} \tag{25}$$

Similarly, the second condition, that is (24), also becomes

$$\begin{aligned}\sum_{j=0}^{7}(j\alpha_j) &= (0)\alpha_0 + (1)\alpha_1 + (2)\alpha_2 + (3)\alpha_3 + (4)\alpha_4 + (5)\alpha_5 + (6)\alpha_6 + (7)\alpha_7 \\ &= (0)\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (1)\begin{pmatrix}\frac{517}{5100}\\ \frac{421}{2115}\\ -\frac{1}{116}\\ -\frac{17}{1365}\end{pmatrix} + (2)\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + (3)\begin{pmatrix}-\frac{33}{68}\\ \frac{31}{141}\\ -\frac{695}{812}\\ \frac{3}{13}\end{pmatrix} \\ &+ (4)\begin{pmatrix}1\\ -\frac{244}{423}\\ \frac{365}{203}\\ -\frac{356}{273}\end{pmatrix} + (5)\begin{pmatrix}-\frac{63}{68}\\ 1\\ -\frac{1665}{812}\\ \frac{207}{91}\end{pmatrix} \\ &+ (6)\begin{pmatrix}\frac{159}{425}\\ -\frac{692}{705}\\ 1\\ -\frac{996}{455}\end{pmatrix} + (7)\begin{pmatrix}-\frac{13}{204}\\ \frac{59}{423}\\ \frac{95}{812}\\ 1\end{pmatrix} \\ &= \begin{pmatrix}-\frac{16}{85}\\ -\frac{64}{47}\\ \frac{240}{203}\\ \frac{64}{91}\end{pmatrix}.\end{aligned} \tag{26}$$

$$\begin{aligned}\sum_{j=0}^{7} 2\beta_j &= 2\left(\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7\right) \\ &= 2\left[\begin{pmatrix}0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}-\frac{3}{85}\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\ -\frac{12}{47}\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\ \frac{45}{203}\\0\end{pmatrix}\right. \\ &\left. + \begin{pmatrix}-\frac{1}{17}\\0\\0\\ \frac{12}{91}\end{pmatrix} + \begin{pmatrix}0\\ -\frac{20}{47}\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\ \frac{75}{203}\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\ \frac{20}{91}\end{pmatrix}\right] \\ &= \begin{pmatrix}-\frac{16}{85}\\ -\frac{64}{47}\\ \frac{240}{203}\\ \frac{64}{91}\end{pmatrix}.\end{aligned} \tag{27}$$

Hence,

$$\sum_{j=0}^{7} j\alpha_j = 2\sum_{j=0}^{7} \beta_j.$$

Thus, the second condition, that is (24), is satisfied. The consistency conditions are therefore satisfied. Hence, the method (2) is consistent.

Consistency ensures that the method locally reproduces the differential equation's Taylor expansion up to the required order and is a necessary condition (together with zero-stability) for the scheme's global convergence.

5 Zero Stability

To show that the method is zero stable, consider the stability polynomial of the method (2) below:

$$\begin{aligned}R(t,\bar{h}) = &-\frac{4225784}{221398905}t\bar{h} - \frac{235684}{14759927}t\bar{h}^2 - \frac{16150556}{221398905}t^2\bar{h}^2 - \frac{1443944}{51092055}t^2\bar{h} \\ &+ \frac{1280768}{94885245}t^2 - \frac{154556}{44279781}t^3\bar{h}^3 - \frac{79514332}{664196715}t^3\bar{h}^2 + \frac{21351704}{221398905}t^3\bar{h} \\ &- \frac{200604}{73799635}t^2\bar{h}^3 - \frac{18828}{43411155}t\bar{h}^3 + \frac{30000}{14759927}t^4\bar{h}^4 + \frac{382300}{14759927}t^4\bar{h}^3 \\ &- \frac{5157956}{44279781}t^4\bar{h}^2 + \frac{7319656}{39070395}t^4\bar{h} - \frac{3888}{14759927}t\bar{h}^4 - \frac{69554944}{664196715}t^4 \\ &- \frac{4660736}{51092055}t^3 = 0.\end{aligned} \tag{28}$$

To show that the method (2) is zero stable, we substitute $\bar{h} = 0$ in equation (28) to obtain the following first characteristic polynomial as:

$$\frac{104211}{398315}t^2 - \frac{457902}{398315}t^3 + \frac{353691}{398315}t^4 = 0. \tag{29}$$

Factoring equation (29), we obtain the roots of the first characteristic polynomial as:

$$t = 0 \quad (\text{multiplicity } 2), \quad t = \frac{11579}{39299}, \quad t = 1. \tag{30}$$

All roots satisfy $|t| \leq 1$, and the root $t = 1$ is simple. Therefore, by Definition 3.4, method (2) is zero stable. Since it is both zero stable and consistent, it thus converges.

Zero stability is the second necessary condition for convergence. When combined with consistency, it implies that the global error tends to zero as the step size approaches 0, so the computed solution converges to the true solution.

6 Conclusion

The analysis in this study establishes the order and convergence of the extended 2-point superclass BBDF with two intermediate off-step points as a reliable high-order scheme for the numerical integration of stiff initial value problems. The proof of consistency, zero-stability, and convergence confirms the mathematical soundness of the method and provides a rigorous foundation for its practical implementation. The block structure of the method enables the simultaneous computation of multiple solution values, making it computationally attractive for large-scale stiff systems and parallel computing environments. Such characteristics are particularly relevant in applications involving chemical kinetics, electrical circuit simulation, control systems, and biological models, where stiff differential equations frequently arise. Despite these advantages, the present study is limited to fixed step-size analysis and theoretical convergence properties. Future work may focus on variable step-size implementations, adaptive error control strategies, stability region optimization, and diagonally implicit formulations to further improve computational efficiency and robustness for highly nonlinear stiff systems.

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References

[1] Scholz, S. (1974). [Review of the book Computational Methods in Ordinary Differential Equations, by J. D. Lambert]. ZAMM - *Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik, 54*(7), 523. [CrossRef]

[2] Nasarudin, A. A., Ibrahim, Z. B., & Rosali, H. (2020). On the integration of stiff ODEs using block backward differentiation formulas of order six. *Symmetry, 12*(6), 952. [CrossRef]

[3] Mohd Ijam, H., & Ibrahim, Z. B. (2019). Diagonally implicit block backward differentiation formula with optimal stability properties for stiff ordinary differential equations. *Symmetry, 11*(11), 1342. [CrossRef]

[4] Zawawi, I. S. M., Ibrahim, Z. B., Ismail, F., & Majid, Z. A. (2012). Diagonally implicit block backward differentiation formulas for solving ordinary differential equations. *International journal of mathematics and mathematical sciences, 2012*(1), 767328. [CrossRef]

[5] Mohd Zawawi, I. S., Ibrahim, Z. B., & Othman, K. I. (2015). Derivation of diagonally implicit block backward differentiation formulas for solving stiff initial value problems. *Mathematical problems in engineering, 2015*(1), 179231. [CrossRef]

[6] Mohd Ijam, H., Ibrahim, Z. B., Abdul Majid, Z., & Senu, N. (2020). Stability analysis of a diagonally implicit scheme of block backward differentiation formula for stiff pharmacokinetics models. *Advances in difference equations, 2020*(1), 400. [CrossRef]

[7] Curtiss, C. F., & Hirschfelder, J. O. (1952). Integration of stiff equations. *Proceedings of the national academy of sciences, 38*(3), 235-243. [CrossRef]

[8] Ola Fatunla, S. (1991). Block methods for second order ODEs. *International journal of computer mathematics, 41*(1-2), 55-63. [CrossRef]

[9] Ibrahim, Z. B., Othman, K. I., & Suleiman, M. (2007). Implicit r-point block backward differentiation formula for solving first-order stiff ODEs. *Applied Mathematics and Computation, 186*(1), 558-565. [CrossRef]

[10] Abasi, N., Suleiman, M., Abbasi, N., & Musa, H. (2014). 2-point block BDF method with off-step points for solving stiff ODEs. *Journal of soft computing and Applications, 2014*, 1-15. [CrossRef]

[11] Abdulrahman, A., & Hamisu, M. (2025). An Improved Extended 2-Point Super Class of Block Backward Differentiation Formula for Solving Stiff Initial Value Problems. *Journal of Science and Technology, 17*(1), 11-22.

https://journal.uthm.edu.my/index.php/JST/article/view/20763

[12] Jana Aksah, S., Ibrahim, Z. B., & Mohd Zawawi, I. S. (2019). Stability analysis of singly diagonally implicit block backward differentiation formulas for stiff ordinary differential equations. *Mathematics*, *7*(2), 211. [CrossRef]

[13] Jana Aksah, S., & Ibrahim, Z. B. (2019). Singly diagonally implicit block backward differentiation formulas for HIV infection of CD4+T cells. *Symmetry*, *11*(5), 625. [CrossRef]

[14] Abubakar, A. U., & Chandrasekaran, E. (2026). A new variable step-size hybrid block method for the numerical solution of ordinary differential equations. *Journal of Interdisciplinary Mathematics*, *29*(4), 987-1000. [CrossRef]

[15] Ibrahim, Z. B., & Nasarudin, A. A. (2020). A class of hybrid multistep block methods with a–stability for the numerical solution of stiff ordinary differential equations. *Mathematics*, *8*(6), 914. [CrossRef]

[16] Musa, H., & Alhassan, B. (2025, December). Fully implicit 2-point super class of block extended backward differentiation formula for solving stiff ordinary differential equations. In *AIP Conference Proceedings* (Vol. 3338, No. 1, p. 040002). AIP Publishing LLC. [CrossRef]

[17] Yatim, S. A. M., Ibrahim, Z. B., Othman, K. I., & Suleiman, M. B. (2011). A quantitative comparison of numerical method for solving stiff ordinary differential equations. *Mathematical Problems in Engineering*, *2011*(1), 193691. [CrossRef]

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