



Toward a Theory of Granular Digital Twins: Mathematical Foundations for Adaptive Digital Twin Systems

Jinghan Li ^{1,*}

¹ Lincoln University College, Petaling Jaya 47301, Selangor, Malaysia

Abstract

Digital Twin (DT) technology models and synchronizes physical systems through virtual counterparts. Existing mathematical formulations of DTs concentrate on physical-virtual synchronization and say little about how representations of the same system at different levels of detail should be related. This paper builds a mathematical foundation for Granular Digital Twins (GDTs) by combining Granular Computing with DT theory. We introduce information granules, the Granular Representation Mapping, the Granular Twin Space, and the Granular Digital Twin, and then construct a Granular Twin Algebra whose aggregation, refinement, and adaptive transformation operators act on an ordered family of granular representations. The structure is analyzed through lattice properties, Granular Twin Homomorphisms, adaptive granular evolution, and granular equilibrium. When the representation family is a complete lattice and the adaptive transformation is order-preserving, the

Knaster-Tarski theorem guarantees that the set of granular equilibria is non-empty and is itself a complete lattice; each equilibrium is invariant under repeated adaptive transformation. These results are existence and invariance results only; uniqueness, convergence, and dynamical stability are not claimed. The framework moves DT theory from state-oriented synchronization toward adaptive granular representation and gives a formal basis for organizing hierarchical information in DT systems.

Keywords: digital twin, granular computing, information granules, granular twin algebra, lattice theory, fixed-point theory, adaptive systems.

1 Introduction

Digital Twin (DT) technology represents, monitors, and analyzes physical systems through dynamically connected virtual representations. A DT links a physical system to its virtual counterpart through sensing, data acquisition, modeling, simulation, and synchronization [1–3]. The concept was first framed as a virtual counterpart of a physical product and was later developed for complex engineered systems, notably in aerospace [4]. DT research has since spread



Submitted: 10 August 2026

Revised: 29 September 2026

Accepted: 11 October 2026

Published: 19 October 2026

Vol. 2, No. 4, 2026.

10.62762/JAM.2026.667211

*Corresponding author:

✉ Jinghan Li

jinghan.phdscholar@lincoln.edu.my

Citation

Li, J. (2026). Toward a Theory of Granular Digital Twins: Mathematical Foundations for Adaptive Digital Twin Systems. *ICCK Journal of Applied Mathematics*, 2(4), 302–314.



© 2026 by the Author. Published by Institute of Central Computation and Knowledge. This is an open access article under the CC BY license (<https://creativecommons.org/licenses/by/4.0/>).

from manufacturing to infrastructure, smart cities, healthcare, energy, and other domains [2, 3, 5–10].

As DT systems have grown more complex, DT modeling has moved beyond a single physical-virtual correspondence. Studies of DT modeling, enabling technologies, and system characterization show that one physical system may be described by several models at different levels of detail, depending on the application [2, 3, 5, 6, 8–10]. This raises a mathematical question that the literature has not treated formally: how should multiple representations of the same physical system be organized and related, particularly when the level of granularity has to change with information needs or computational constraints? More concretely, if a system is described at component, subsystem, and system levels, how are these descriptions related, how can refinement and aggregation be characterized within one structure, and how can a change of granularity be modeled without losing the relations among representations?

These questions are close to the core of Granular Computing. Information granulation treats complex information through groups, classes, regions, and other structured units instead of isolated elements [11–16], which supports abstraction, approximation, and multi-level organization of information. Fuzzy sets, rough sets, and related theories offer different formal routes to granulation [11, 13, 14, 17–22].

Existing DT studies provide modeling approaches, enabling technologies, and architectures for integrating heterogeneous data and models [5, 9, 10]. Granular Computing provides formal ways of organizing information at different granularities [11, 12, 15, 16]. Rough set theory and lattice theory supply tools for analyzing information structures and the relations between levels of abstraction [17, 21–23], and fixed-point results for ordered structures are relevant to equilibrium questions [24, 25]. Recent work has extended granular computing to concept-lattice-based learning, to transformations between granular and numeric representations, and to approximation by fuzzy granules [26–28]. What has not been done is to organize the representations of a physical system into an explicit ordered structure and to analyze adaptive transformations on that structure. This paper addresses that gap by studying refinement order, algebraic operations, adaptive transformation, and equilibrium within one framework for Granular Digital Twins (GDTs).

The framework first separates the physical system from its physical state universe and builds a family of information granules on the state universe. A refinement relation orders the granules by resolution. On this basis we define the Granular Representation Mapping, the Granular Twin Space, and the Granular Digital Twin. We then introduce the Granular Twin Algebra, in which aggregation and refinement are the join and meet of the order (when they exist) and an adaptive transformation operator changes the granular representation. Because the adaptive operator acts on the representation layer and not on the physical system, representation change can be studied separately from physical evolution.

Let the adaptive transformation be an operator $\Gamma : \mathcal{G}(X) \rightarrow \mathcal{G}(X)$, where $\mathcal{G}(X)$ is the family of granular representations built on the physical state universe X . If Γ is order-preserving and $(\mathcal{G}(X), \preceq)$ is a complete lattice, the existence of a granular equilibrium becomes the fixed-point problem

$$\Gamma(G^*) = G^*, \quad (1)$$

where G^* is a representation left unchanged by the adaptive transformation. This equilibrium concerns the representation layer, not the physical equilibrium of the system. The contribution of the paper is therefore not hierarchical or multiscale representation as such, but a formal order-theoretic structure that connects granular representation, algebraic operations, adaptive transformation, and fixed-point equilibrium.

The main contributions are as follows.

- (1) *Formal granular representation for DT systems.* The physical system is distinguished from its physical state universe, and information granules are constructed on the state universe, giving a formal basis for describing one system at several levels of abstraction.
- (2) *An ordered Granular Twin Algebra.* The family of granular representations is equipped with a refinement relation, with aggregation and refinement operations where they exist, and with an adaptive transformation operator for changes in resolution.
- (3) *Structural analysis.* Lattice conditions, order-preserving maps and Granular Twin Homomorphisms, and the monotonicity of adaptive evolution are characterized, with each result stated under explicit assumptions.

- (4) *Fixed-point formulation of granular equilibrium.* For a complete lattice and an order-preserving adaptive transformation, the Knaster-Tarski theorem [25] yields the existence of granular equilibria, which are invariant under repeated adaptive transformation.

The scope is deliberately mathematical. The framework does not give an implementation architecture, and it does not by itself establish physical fidelity, computational optimality, uniqueness of equilibrium, convergence rate, or dynamical stability; these depend on the specific system, granulation mechanism, information measure, and adaptive transformation and are left to later theoretical and empirical work.

Section 2 reviews DT research, Granular Computing, order and lattice theory, and fixed-point theory, and positions the framework. Section 3 defines the physical system, physical state universe, information granules, Granular Representation Mapping, Granular Twin Space, and Granular Digital Twin. Section 4 develops the Granular Twin Algebra. Section 5 studies lattice structure, Granular Twin Homomorphisms, adaptive evolution, and granular equilibrium. Section 6 discusses scope and limitations, and Section 7 concludes.

2 Related Work and Mathematical Background

2.1 Digital Twin Theory

DT research has moved from representing and monitoring physical entities toward integrated frameworks for sensing, simulation, prediction, data integration, and decision support [1–3, 5–10]. Early work stressed the relation between a physical system and its virtual counterpart and its use in mitigating undesirable emergent behavior in complex systems [4]; later studies broadened DT modeling through system integration, computational methods, and application-specific architectures in manufacturing, civil engineering, and other fields [2, 8–10]. More recent studies examine multi-scale DT frameworks, which use representations at several resolutions, and semantic DTs, which use ontologies and structured knowledge to organize heterogeneous information [8–10].

Mathematically, a DT involves a physical system, its admissible states, and one or more virtual representations. Let P denote a physical system and X its physical state universe, that is, the set of admissible states that P may attain or that may be represented for

P . Thus $P \neq X$: P is the physical entity, and X is the mathematical domain of its possible states. A virtual representation is a mapping

$$\Phi : X \rightarrow V, \quad (2)$$

where V is a virtual representation space, and a DT can be written as

$$\text{DT} = (P, X, V, \Phi). \quad (3)$$

This formulation prescribes no architecture; it separates the physical system, its state domain, and its virtual representation so that granular constructions can be added without conflating states with representations.

Systematic reviews, modeling-oriented frameworks, enabling-technology surveys, and civil-engineering applications [3, 5–9] stress the need to integrate heterogeneous data, models, and domain information in DT systems [5, 8, 9] and describe DT architectures across application contexts [3, 6, 7]. The question addressed here is narrower. We do not claim that hierarchical or adaptive representations are missing from the DT literature. We ask how multiple representations of one physical system, ordered by abstraction or resolution, can be related through an explicit mathematical order, and how aggregation, refinement, and adaptive transformation can be defined on that order. To answer it, granular representation is treated as a mathematical object in its own right.

2.2 Hierarchical, Semantic, and Multiscale Digital Twin Representations

DT systems increasingly use representations at several spatial, temporal, functional, and organizational scales, so that each component or process can be analyzed at the resolution a task requires. A manufacturing DT may represent a component, a subsystem, and a whole production line; an urban DT may represent buildings, districts, transport networks, and the city. These levels are related by aggregation, decomposition, or abstraction. Semantic DTs add another dimension: ontologies, metadata, and knowledge graphs describe entities, attributes, and relations, focusing on meaning rather than numerical state. Adaptive DTs, finally, change their models, configurations, or computational mechanisms in response to new observations, environmental conditions, uncertainty, or operational requirements, so DT representations need not be fixed at a single level.

In the present framework these perspectives share one mathematical object. Let $\mathcal{G}(X)$ denote a family of admissible information granules built on X ; each $G \in \mathcal{G}(X)$ represents a particular granularity or abstraction of the state information. A relation $\preceq$ expresses refinement or abstraction between granules; its precise meaning depends on the granulation mechanism and has to be stated explicitly. In short, hierarchical and multiscale representation concerns the level at which a system is represented, semantic representation concerns what is represented, adaptive representation concerns how representations change, and the present framework concerns how granular representations are ordered and how transformations among them can be analyzed.

2.3 Granular Computing and Information Granulation

Granular Computing grew out of information granulation, in which complex information is organized into groups, classes, regions, or other higher-level entities rather than treated element by element [11–16]. Formal approaches include fuzzy information granulation [11], rough set theory [17, 18, 20], and lattice-based formulations of information structures [19, 21–23]. The granularity chosen depends on the similarity, proximity, equivalence, uncertainty, or functional relation used in the representation.

For a similarity-based formulation, let $\rho : X \times X \rightarrow [0, 1]$ be a similarity function on X , a larger value of $\rho(x_1, x_2)$ indicating greater similarity. For a threshold $\lambda \in [0, 1]$, the granule centered at $c \in X$ is the set of states whose similarity to c is at least λ (formally defined in Section 3.2). Other mechanisms—equivalence relations, fuzzy relations, rough approximations, intervals, and knowledge-based grouping—may be used instead.

The properties of the resulting granules therefore depend on ρ , λ , and the construction mechanism, and the relation between granularity and information content is not universally monotone without a specified information measure. If an information measure I is order-reversing (antitone) with respect to the refinement relation, then

$$G_1 \preceq G_2 \Rightarrow I(G_1) \geq I(G_2), \quad (4)$$

where G_1 is the finer representation under the adopted convention. This is a property of the chosen measure, not of granular representations in general.

The family $\mathcal{G}(X)$ together with $\preceq$ forms the ordered granular representation structure $(\mathcal{G}(X), \preceq)$, which is used throughout the paper. The progression is

$$X \longrightarrow \mathcal{G}(X) \longrightarrow (\mathcal{G}(X), \preceq) : \quad (5)$$

the state universe is turned into a family of granular representations, which is then ordered by refinement.

2.4 Ordered Structures and Lattice Theory

Order theory provides the language for relating granular representations [23]. The relation $\preceq$ on $\mathcal{G}(X)$ is assumed to be a partial order, that is, for all $G, G_1, G_2, G_3 \in \mathcal{G}(X)$:

- reflexivity: $G \preceq G$;
- antisymmetry: $G_1 \preceq G_2$ and $G_2 \preceq G_1$ imply $G_1 = G_2$;
- transitivity: $G_1 \preceq G_2$ and $G_2 \preceq G_3$ imply $G_1 \preceq G_3$.

Then $(\mathcal{G}(X), \preceq)$ is a partially ordered set (poset). A poset is not automatically a lattice. If every pair $G_1, G_2 \in \mathcal{G}(X)$ has a least upper bound $G_1 \vee G_2$ and a greatest lower bound $G_1 \wedge G_2$ in $\mathcal{G}(X)$, then $(\mathcal{G}(X), \preceq, \vee, \wedge)$ is a lattice [23]. Under the convention that finer representations lie lower in the order, the join $\vee$ models aggregation into a coarser common representation and the meet $\wedge$ models a common refinement.

The order $\preceq$ gives the basic hierarchy; the existence of joins and meets is an additional algebraic condition. We therefore do not assume that every granular representation space is a lattice, and lattice structure is introduced only when the bound conditions hold.

2.5 Fixed-Point Theory and Adaptive Transformations

Adaptive DTs may change their representation as system conditions, information requirements, or modeling objectives change. We model this by an adaptive transformation $\Gamma : \mathcal{G}(X) \rightarrow \mathcal{G}(X)$ acting on the granular representation layer, not on the physical system P : $\Gamma(G)$ is the representation that replaces G . A representation G^* is a granular equilibrium if $\Gamma(G^*) = G^*$, i.e. if it is invariant under the adaptive transformation.

Existence of such invariant representations is studied under the order-preserving (monotone) condition

$$G_1 \preceq G_2 \Rightarrow \Gamma(G_1) \preceq \Gamma(G_2). \quad (6)$$

This is a structural assumption, not a property of all adaptive DT mechanisms. It is natural when

the adaptive mechanism is designed to keep the refinement hierarchy during updates, but it may fail under non-monotone optimization, discontinuous model switching, competing constraints, or update criteria that are not aligned with the refinement relation.

If $(\mathcal{G}(X), \preceq)$ is a complete lattice and Γ satisfies (6), the Knaster-Tarski fixed-point theorem [25] applies: the fixed-point set

$$\text{Fix}(\Gamma) = \{G \in \mathcal{G}(X) : \Gamma(G) = G\} \quad (7)$$

is non-empty and is itself a complete lattice, so at least one granular equilibrium exists. Fixed-point existence says nothing about uniqueness, convergence rate, or Lyapunov, asymptotic, or structural stability; these require further assumptions. Fixed-point theory is widely used in existence and equilibrium analysis [24, 25]; here it is applied to adaptive transformations of granular representations.

2.6 Positioning of the Present Framework

The ingredients of the framework come from established traditions. DT research supplies physical-virtual correspondence, synchronization, simulation, semantic representation, and adaptive modeling; Granular Computing supplies information abstraction and multi-level representation; order and lattice theory supply the formal structure of ordered elements; and fixed-point theory supplies tools for invariant elements of mappings. The framework does not introduce these concepts individually. It organizes them around the problem of adaptive granular representation in DT systems. The main mathematical objects are:

- (1) the physical system P and its physical state universe X ;
- (2) the family of information granules $\mathcal{G}(X)$;
- (3) the refinement relation $\preceq$ and the ordered structure $(\mathcal{G}(X), \preceq)$;
- (4) the join $\vee$ and meet $\wedge$, when they exist;
- (5) the Granular Representation Mapping Φ_G ;
- (6) the adaptive transformation Γ ;
- (7) the structure-preserving mapping h between granular structures;
- (8) the granular equilibrium G^* and the fixed-point set $\text{Fix}(\Gamma)$.

Table 1 summarizes the notation. These objects are developed along the path

$$\begin{aligned} X &\rightarrow \mathcal{G}(X) \rightarrow (\mathcal{G}(X), \preceq) \\ &\rightarrow (\mathcal{G}(X), \preceq, \vee, \wedge) \rightarrow \Gamma \rightarrow G^*. \end{aligned} \quad (8)$$

The aim is not to replace hierarchical, semantic, multiscale, or adaptive DT approaches, but to give an explicit structure in which their representation-level aspects can be analyzed.

Table 1. Mathematical notation used in the framework.

Symbol	Definition
P	Physical system
X	Physical state universe associated with P
V	Virtual representation space
Φ	Physical-to-virtual mapping, $\Phi : X \rightarrow V$
ρ	Similarity function, $\rho : X \times X \rightarrow [0, 1]$
λ	Granulation threshold
$G_\lambda(c)$	Similarity-based information granule centered at c
$\mathcal{G}(X)$	Family of admissible information granules on X
$\preceq$	Refinement relation among granules
$(\mathcal{G}(X), \preceq)$	Ordered granular representation structure
$\vee, \wedge$	Join (aggregation) and meet (refinement), when they exist
Φ_G	Granular Representation Mapping, $\Phi_G : X \rightarrow 2^{\mathcal{G}(X)}$
GT	Granular Twin Space
GDT	Granular Digital Twin
$\mathfrak{GT}$	Granular Twin Algebra
I	Information measure, $I : \mathcal{G}(X) \rightarrow \mathbb{R}$ (optional)
Γ	Adaptive transformation, $\Gamma : \mathcal{G}(X) \rightarrow \mathcal{G}(X)$
h	Order-preserving map / Granular Twin Homomorphism between granular structures
G^*	Granular equilibrium, $\Gamma(G^*) = G^*$
$\text{Fix}(\Gamma)$	Fixed-point set of Γ

3 Mathematical Foundation of Granular Digital Twins

This section defines information granules, the granular representation structure, and the relation between physical states and their representations. Figure 1 gives an overview of the framework.

3.1 Physical System and Physical State Universe

As in Section 2, P denotes the physical system, X its physical state universe, V the virtual representation

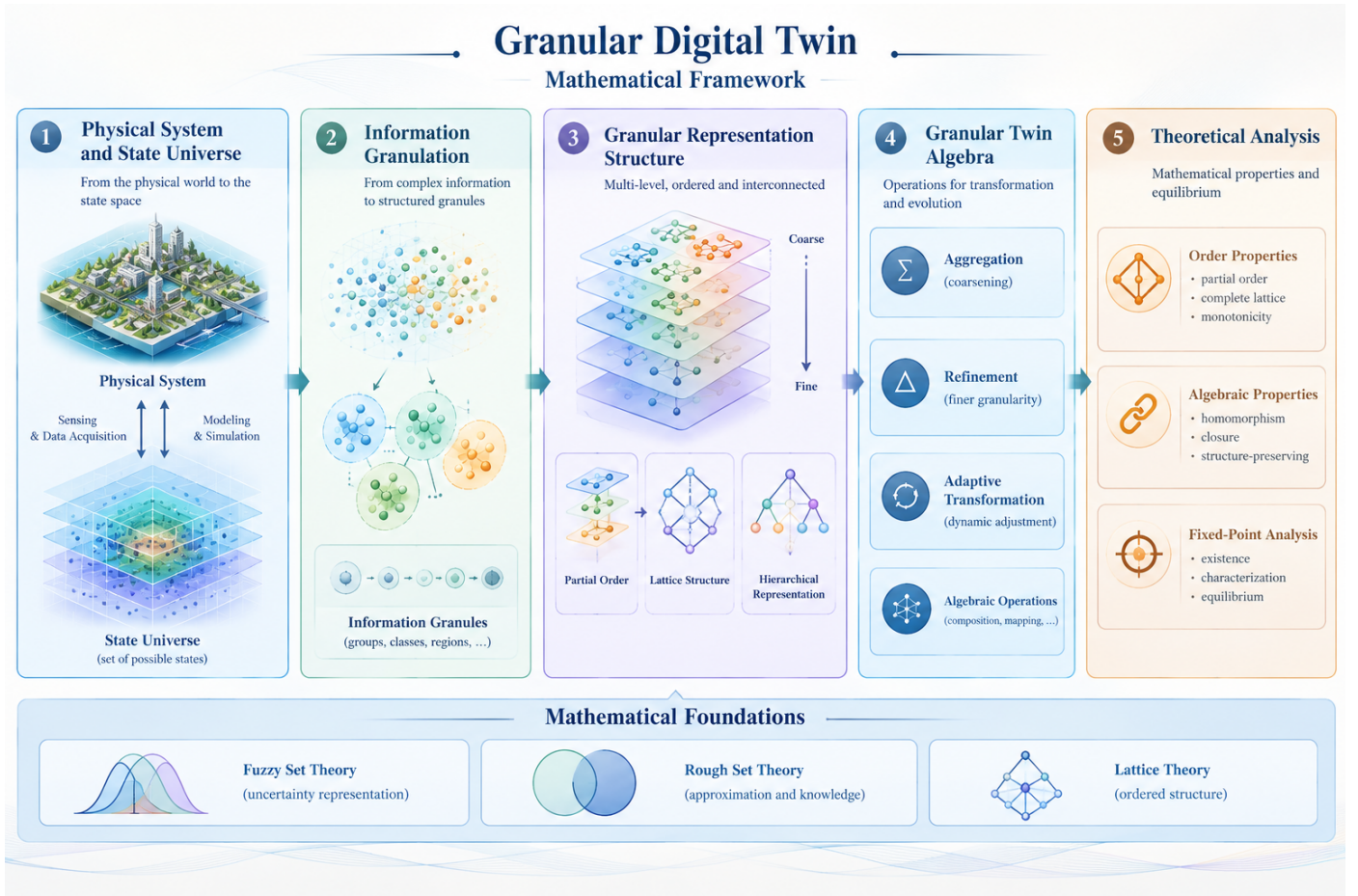


Figure 1. Conceptual framework of Granular Digital Twins.

space, and $\Phi : X \rightarrow V$ the physical-to-virtual mapping, with $\Phi(x)$ the virtual representation of a state $x \in X$; a DT is the tuple (3). Granular representation does not replace any of these components. It adds a representation layer that organizes information about X at several levels of abstraction. The constructions in this section therefore operate on representations of X , not on P , and the adaptive transformation introduced later acts on these granular representations while P remains the object being represented.

3.2 Similarity-Based Information Granules

Let $X \neq \emptyset$. An information granule is a collection of elements of X grouped by similarity, proximity, equivalence, semantic relation, or another granulation criterion. We adopt similarity-based granulation as one explicit realization; the ordered framework that follows does not depend on this choice.

Let $\rho : X \times X \rightarrow [0, 1]$ be a similarity function on X . For a representative element $c \in X$ and a threshold $\lambda \in [0, 1]$, the similarity-based information granule is

$$G_\lambda(c) = \{x \in X \mid \rho(x, c) \geq \lambda\}, \quad (9)$$

and the family of admissible granules is

$$\mathcal{G}(X) = \{G_\lambda(c) \mid c \in X, \lambda \in [0, 1]\}. \quad (10)$$

A larger λ imposes a stricter similarity requirement and yields a smaller granule; a smaller λ admits a broader set of states. Granules generated by different thresholds or centers are not, however, automatically ordered by refinement; the refinement relation has to be specified separately.

Remark 1 (Generality of the granulation mechanism) *Fuzzy, rough, interval, equivalence-based, and other uncertainty-aware granules [11, 13, 17, 20] can be used by replacing (9) while keeping the family notation $\mathcal{G}(X)$ and the relation $\preceq$. The theory concerns the organization of the resulting granular representations, not the particular choice of ρ .*

3.3 Granular Representation Structure

Definition 1 (Granular refinement relation) *For $G_1, G_2 \in \mathcal{G}(X)$, $G_1 \preceq G_2$ means that G_1 refines G_2 , i.e. G_1 represents the corresponding information at a resolution at least as fine as G_2 under the adopted granulation scheme.*

The relation $\preceq$ is assumed to be reflexive, antisymmetric, and transitive (Section 2).

With these properties $(\mathcal{G}(X), \preceq)$ is a partially ordered granular representation structure, the basic ordered structure of the paper. Note the difference between $\mathcal{G}(X)$, the family of granules, and $(\mathcal{G}(X), \preceq)$, the same family with an explicit refinement relation, which carries structural information that the set alone does not.

For set-valued granules, $\preceq$ may be instantiated by inclusion,

$$G_1 \preceq G_2 \iff G_1 \subseteq G_2, \quad (11)$$

which is one realization, not a requirement; other granulation mechanisms may need other definitions of refinement.

3.4 Granular Hierarchy and Representation Resolution

Suppose $G^{(1)}, G^{(2)}, \dots, G^{(k)} \in \mathcal{G}(X)$ satisfy

$$G^{(1)} \preceq G^{(2)} \preceq \dots \preceq G^{(k)}. \quad (12)$$

Under our convention $G^{(1)}$ is the finest and $G^{(k)}$ the coarsest level of this chain, which describes several abstraction levels of the same physical information—for example, component, subsystem, and system levels. The levels need not match a particular application hierarchy; their relation is set by the granulation mechanism and the refinement relation.

Property 1 (Hierarchical representation) *Let $x \in X$, and suppose the admissible granular representations associated with x include a chain (12). Then these granules form a hierarchical representation of the corresponding physical information in $(\mathcal{G}(X), \preceq)$.*

This follows directly from the refinement relation. It does not require all elements of $\mathcal{G}(X)$ to be comparable, so $(\mathcal{G}(X), \preceq)$ may contain both chains and incomparable representations.

3.5 Granular Representation Mapping

Definition 2 (Granular Representation Mapping) *The granular representation mapping is the set-valued map*

$$\Phi_{\mathcal{G}} : X \rightarrow 2^{\mathcal{G}(X)}, \quad \Phi_{\mathcal{G}}(x) = \{G \in \mathcal{G}(X) \mid x \in G\}, \quad (13)$$

where $2^{\mathcal{G}(X)}$ is the power set of $\mathcal{G}(X)$.

$\Phi_{\mathcal{G}}(x)$ collects the admissible granules containing x . A state need not have a unique granular representation: $\Phi_{\mathcal{G}}(x)$ may, for example, contain a chain (12), in which case x is represented at several levels of granularity.

Property 2 (Representation coverage) *If $\bigcup_{G \in \mathcal{G}(X)} G = X$, then $\Phi_{\mathcal{G}}(x) \neq \emptyset$ for every $x \in X$.*

Indeed, every x lies in at least one granule. If ρ is reflexive ($\rho(x, x) = 1$ for all x), then $x \in G_{\lambda}(x)$ for every λ , so the coverage condition holds automatically for the family (9). Coverage should not be confused with lattice completeness: it does not imply that $(\mathcal{G}(X), \preceq)$ is a complete lattice.

3.6 Granular Twin Space

Definition 3 (Granular Twin Space) *The Granular Twin Space is*

$$GT = (X, \mathcal{G}(X), \preceq, \Phi_{\mathcal{G}}), \quad (14)$$

where X is the physical state universe, $\mathcal{G}(X)$ the family of admissible granules, $\preceq$ the refinement relation, and $\Phi_{\mathcal{G}}$ the granular representation mapping.

In GT, several representations of a physical state coexist and are organized by refinement. No lattice structure is assumed at this stage; it is introduced in Section 5 only when every pair of granules has the required bounds.

Remark 2 (Granular Twin Space and conventional DT) *The conventional DT (3) describes the physical-to-virtual representation $\Phi : X \rightarrow V$, whereas GT describes the association $\Phi_{\mathcal{G}} : X \rightarrow 2^{\mathcal{G}(X)}$ between states and their granular representations. The granular layer supplements, rather than replaces, the physical-virtual correspondence.*

3.7 Granular Digital Twin

Definition 4 (Granular Digital Twin) *A Granular Digital Twin is the tuple*

$$GDT = (P, X, V, \Phi, GT, \Gamma), \quad (15)$$

where (P, X, V, Φ) is the conventional DT (3), GT is the Granular Twin Space (14), and $\Gamma : \mathcal{G}(X) \rightarrow \mathcal{G}(X)$ is the adaptive transformation acting on the granular representation family.

Because Γ acts on the representation layer, a change in granular representation does not by definition imply a physical change in P ; physical evolution and representation evolution are kept separate.

3.8 Representation Resolution and Information Measures

The refinement relation orders representations but does not define a numerical measure of information. If an information measure $I : \mathcal{G}(X) \rightarrow \mathbb{R}$ is introduced,

its relation to refinement must be specified. For an antitone measure,

$$G_1 \preceq G_2 \Rightarrow I(G_1) \geq I(G_2), \quad (16)$$

so a finer representation receives at least as much information. Other measures—of uncertainty, complexity, distinguishability, compression, or semantic richness—may relate to granularity differently. The framework therefore rests on $(\mathcal{G}(X), \preceq)$ and requires no specific I ; a measure can be added when an application needs one.

3.9 Summary of the Representation Layer

The constructions of this section give the progression

$$X \rightarrow \mathcal{G}(X) \rightarrow (\mathcal{G}(X), \preceq) \rightarrow \text{GT} \rightarrow \text{GDT}, \quad (17)$$

in which the conventional DT (P, X, V, Φ) with $\Phi : X \rightarrow V$ is extended by the similarity-based granules (9), the refinement relation, the mapping Φ_G , and the adaptive transformation $\Gamma : \mathcal{G}(X) \rightarrow \mathcal{G}(X)$. Nothing here assumes that $(\mathcal{G}(X), \preceq)$ is a lattice or that Γ has a fixed point. Section 4 introduces joins and meets under their existence conditions, and Section 5 adds the completeness and monotonicity assumptions needed for equilibrium.

4 Granular Twin Algebra

The Granular Twin Space organizes granular representations by refinement. To combine, refine, and adaptively update them, three operations are introduced: aggregation and refinement, which are order-based, and adaptive transformation, which models change within the representation layer. The resulting structure, the Granular Twin Algebra, is an ordered adaptive structure rather than an algebra in the strict universal-algebraic sense.

4.1 Aggregation Operator

When two representations are combined, it is natural to look for the least representation coarser than both.

Definition 5 (Aggregation operator) *Let $G_1, G_2 \in \mathcal{G}(X)$. If G_1 and G_2 have a least upper bound in $\mathcal{G}(X)$ with respect to $\preceq$, their aggregation is*

$$G_1 \vee G_2 = \text{lub}_{\preceq}(G_1, G_2), \quad (18)$$

that is, $G_1 \preceq G_1 \vee G_2$, $G_2 \preceq G_1 \vee G_2$, and for every $G \in \mathcal{G}(X)$,

$$G_1 \preceq G, G_2 \preceq G \Rightarrow G_1 \vee G_2 \preceq G. \quad (19)$$

With finer representations ordered below coarser ones, the join is the least common coarser representation. Its existence does not follow from the partial order: the least upper bound must belong to $\mathcal{G}(X)$. Aggregation is therefore a partial operation in general (see Section 5.1).

4.2 Refinement Operator

The dual operation identifies the greatest lower bound of two representations, i.e. the most detailed common information shared by both.

Definition 6 (Refinement operator) *Let $G_1, G_2 \in \mathcal{G}(X)$. If G_1 and G_2 have a greatest lower bound in $\mathcal{G}(X)$ with respect to $\preceq$, their refinement is*

$$G_1 \wedge G_2 = \text{glb}_{\preceq}(G_1, G_2), \quad (20)$$

that is, $G_1 \wedge G_2 \preceq G_1$, $G_1 \wedge G_2 \preceq G_2$, and for every $G \in \mathcal{G}(X)$,

$$G \preceq G_1, G \preceq G_2 \Rightarrow G \preceq G_1 \wedge G_2. \quad (21)$$

Refinement does not create new physical information; it is a more detailed organization of what the granular framework already represents. Like the join, the meet is in general a partial operation and becomes total only under the lattice condition of Section 5.1.

4.3 Adaptive Transformation Operator

Aggregation and refinement describe static relations among representations. They do not describe how a representation changes when the observation context, computational objective, or representation requirement changes.

Definition 7 (Adaptive transformation operator) *An adaptive transformation operator is a map $\Gamma : \mathcal{G}(X) \rightarrow \mathcal{G}(X)$; for $G \in \mathcal{G}(X)$, $\Gamma(G) = G' \in \mathcal{G}(X)$ is the updated granular representation. In particular, Γ is closed on $\mathcal{G}(X)$.*

Γ describes the evolution of the representation, $G \mapsto \Gamma(G)$, not a physical transformation of P ; recall that P , X , and $\mathcal{G}(X)$ are distinct objects. It can model changes in resolution, abstraction level, or granulation mechanism; its concrete form depends on the application. Closure keeps the update inside the admissible family but implies nothing about convergence or stability.

4.4 Granular Twin Algebra

Definition 8 (Granular Twin Algebra) *Let GT be a Granular Twin Space (14). The corresponding Granular Twin Algebra is*

$$\mathfrak{GT} = (\mathcal{G}(X), \preceq, \vee, \wedge, \Gamma), \quad (22)$$

where $\vee$ and $\wedge$ are defined whenever the corresponding bounds exist (Definitions 5 and 6) and Γ is the adaptive transformation. The symbol $\mathfrak{GT}$ denotes the algebraic structure and does not replace GT.

$\mathfrak{GT}$ combines three aspects of granular representation: ordered organization ($\preceq$), combination and specialization ($\vee$, $\wedge$), and adaptive evolution (Γ). It does not presuppose that $\mathcal{G}(X)$ is a lattice, and it imposes no algebraic laws beyond those carried by the order and the operations; “algebra” here refers to the integration of objects, order, operations, and adaptation in one structure.

Remark 3 (Roles of h and Γ) *The map h introduced in Section 5.2 relates different granular structures, whereas Γ transforms representations within one family. The two maps play different roles and should not be conflated.*

4.5 Basic Properties

We record the well-definedness and order-preservation properties used later; lattice structure and fixed-point existence are treated conditionally in Section 5.

Proposition 1 (Uniqueness of aggregation and refinement) *Let $G_1, G_2 \in \mathcal{G}(X)$. If their least upper bound exists, it is unique; likewise for their greatest lower bound.*

Proof. Let U_1 and U_2 both be least upper bounds of G_1 and G_2 . Since U_2 is an upper bound and U_1 is the least upper bound, $U_1 \preceq U_2$; symmetrically $U_2 \preceq U_1$. By antisymmetry, $U_1 = U_2$. The argument for greatest lower bounds is dual. $\square$

The proposition gives uniqueness conditional on existence; existence of $\vee$ and $\wedge$ for every pair is exactly the lattice condition of Section 5.1.

Assumption 1 (Order preservation of adaptive transformation) *For all $G_1, G_2 \in \mathcal{G}(X)$, $G_1 \preceq G_2$ implies $\Gamma(G_1) \preceq \Gamma(G_2)$.*

Assumption 1 suits adaptive mechanisms that change resolution while keeping the hierarchy encoded by $\preceq$: if G_1 is finer than G_2 , the update should keep $\Gamma(G_1)$ finer than $\Gamma(G_2)$. It can fail for non-monotone optimization criteria, discontinuous model switching, competing constraints, or update rules that do not respect the chosen notion of refinement. It is thus a sufficient structural condition for the order-theoretic analysis, not a universal property of adaptive DTs.

Remark 4 (Role of order preservation) *Under Assumption 1, Γ is monotone by definition. Proposition 4*

in Section 5.3 restates this for later reference; its role is to supply the hypothesis of the fixed-point analysis rather than to add new content.

4.6 Structural Characteristics of the Granular Twin Algebra

In summary, $\mathfrak{GT}$ is closed under Γ , and under $\vee$ and $\wedge$ whenever the corresponding bounds exist in $\mathcal{G}(X)$; it is hierarchically organized by $\preceq$, which need not make all representations comparable; $\vee$ and $\wedge$ give the least common upper and greatest common lower representations; and Γ induces the representation dynamics

$$G_{t+1} = \Gamma(G_t), \quad (23)$$

which describes the representation layer, not the physical system. Assumption 1 connects this dynamics to the order. Section 5 examines when $(\mathcal{G}(X), \preceq)$ is a lattice, how structure-preserving maps between granular structures are defined, and how monotone adaptive transformations can be analyzed by fixed-point theory.

5 Theoretical Properties of Granular Twin Algebra

This section separates what follows directly from the definitions from what requires further assumptions. A partial order does not by itself yield a lattice, and an order-preserving Γ does not by itself yield convergence or stability; each result is stated with its conditions.

5.1 Lattice Structure of Granular Representations

Definition 9 (Granular representation lattice) *$(\mathcal{G}(X), \preceq)$ is a lattice if every pair $G_1, G_2 \in \mathcal{G}(X)$ has a least upper bound and a greatest lower bound in $\mathcal{G}(X)$; then $G_1 \vee G_2$, $G_1 \wedge G_2 \in \mathcal{G}(X)$ satisfy Definitions 5 and 6 [23].*

Proposition 2 (Characterization of the lattice structure) *If every pair of elements of the poset $(\mathcal{G}(X), \preceq)$ has a least upper bound and a greatest lower bound in $\mathcal{G}(X)$, then $(\mathcal{G}(X), \preceq, \vee, \wedge)$ is a lattice.*

Proof. By hypothesis $G_1 \vee G_2$ and $G_1 \wedge G_2$ exist in $\mathcal{G}(X)$ for every pair, which is the definition of a lattice for a poset [23]. $\square$

The proposition is conditional: whether all bounds exist depends on how $\mathcal{G}(X)$ is built and on the refinement relation. A family may contain two elements with no least upper bound in the family even though one exists in a larger collection of representations; the family is then not a lattice. Lattice

structure must therefore be proved for a specific family or imposed as a modeling condition.

5.2 Structure-Preserving Mappings Between Granular Twin Structures

Different DT systems may use different state universes, granulation mechanisms, or hierarchies. Comparing them requires maps that preserve the relevant structure. Unlike Γ , which acts within one family, the map h relates two granular structures.

Definition 10 (Granular Twin Homomorphism) *Let $(\mathcal{G}(X_1), \preceq_1)$ and $(\mathcal{G}(X_2), \preceq_2)$ be granular representation structures. A map $h : \mathcal{G}(X_1) \rightarrow \mathcal{G}(X_2)$ is order-preserving if, for all $G_1, G_2 \in \mathcal{G}(X_1)$,*

$$G_1 \preceq_1 G_2 \Rightarrow h(G_1) \preceq_2 h(G_2). \quad (24)$$

If both structures are lattices, h is a lattice homomorphism (Granular Twin Homomorphism) if, in addition,

$$h(G_1 \vee_1 G_2) = h(G_1) \vee_2 h(G_2), \quad (25)$$

$$h(G_1 \wedge_1 G_2) = h(G_1) \wedge_2 h(G_2), \quad (26)$$

for all $G_1, G_2 \in \mathcal{G}(X_1)$, where $\vee_i, \wedge_i$ are the lattice operations of the i -th structure.

Proposition 3 (Preservation of representation order) *If h is order-preserving and $G_1 \preceq_1 G_2$, then $h(G_1) \preceq_2 h(G_2)$.*

Proof. Immediate from the definition. $\square$

Order preservation does not imply that h preserves joins and meets; that requires the homomorphism conditions. Even a lattice homomorphism need not preserve semantic attributes, physical quantities, or application-specific meanings, which would require further assumptions. The purpose of h is to formalize the preservation of specified mathematical relations, not to assert equivalence between heterogeneous DT systems.

5.3 Adaptive Transformation and Order Preservation

Proposition 4 (Monotonicity of adaptive evolution) *Under Assumption 1, Γ is a monotone self-map of $(\mathcal{G}(X), \preceq)$.*

Proof. For arbitrary $G_1, G_2 \in \mathcal{G}(X)$, Assumption 1 gives $G_1 \preceq G_2 \Rightarrow \Gamma(G_1) \preceq \Gamma(G_2)$, which is the definition of monotonicity with respect to $\preceq$. $\square$

Monotonicity means that Γ preserves the order between any two inputs; it does not say that $\Gamma(G)$ is finer, coarser, or more informative than G . The results below hold only under this assumption.

5.4 Granular Equilibrium and Fixed-Point Existence

Definition 11 (Granular equilibrium) *$G^* \in \mathcal{G}(X)$ is a granular equilibrium of Γ if $\Gamma(G^*) = G^*$, i.e. if it is a fixed point of Γ .*

Assumption 2 (Complete granular lattice) *$(\mathcal{G}(X), \preceq)$ is a complete lattice and $\Gamma : \mathcal{G}(X) \rightarrow \mathcal{G}(X)$ satisfies Assumption 1.*

Completeness means that every subset of $\mathcal{G}(X)$, including the empty set, has a least upper bound and a greatest lower bound in $\mathcal{G}(X)$ [23]; in particular $\mathcal{G}(X)$ has a least and a greatest element.

Theorem 1 (Existence of granular equilibrium) *Under Assumption 2, the set of granular equilibria*

$$\text{Fix}(\Gamma) = \{G \in \mathcal{G}(X) \mid \Gamma(G) = G\} \quad (27)$$

is non-empty and, with the order inherited from $\preceq$, is a complete lattice.

Proof. By Assumption 2, Γ is a monotone self-map of the complete lattice $(\mathcal{G}(X), \preceq)$. By the Knaster-Tarski theorem [25], the fixed-point set of such a map is non-empty and is a complete lattice under the inherited order. Hence there is $G^* \in \mathcal{G}(X)$ with $\Gamma(G^*) = G^*$, which is a granular equilibrium. $\square$

Theorem 1 guarantees at least one invariant representation. It does not give uniqueness, convergence from an arbitrary initial representation, a convergence rate, or stability, all of which require further analysis. Completeness is a genuine restriction—a lattice need not be complete—and must be verified before the theorem is applied.

5.5 Invariance of Granular Equilibrium

Corollary 1 (Invariance under repeated adaptive transformation) *If $\Gamma(G^*) = G^*$, then $\Gamma^n(G^*) = G^*$ for all $n \geq 1$.*

Proof. For $n = 1$ this is the hypothesis. If $\Gamma^n(G^*) = G^*$, then $\Gamma^{n+1}(G^*) = \Gamma(\Gamma^n(G^*)) = \Gamma(G^*) = G^*$. The claim follows by induction. $\square$

The corollary is an invariance statement, not a stability statement: it says nothing about representations that start near, but not at, a fixed point. Whether a granular equilibrium is attracting, asymptotically stable, or robust to perturbation remains open.

5.6 Implications and Scope of the Results

Under the stated conditions, the results show that (i) refinement orders granular representations, and joins and meets exist for every pair exactly when the structure is a lattice; (ii) order-preserving maps h keep the hierarchy between granular structures, while preservation of joins and meets needs the stronger homomorphism conditions; (iii) Γ preserves the order under Assumption 1; and (iv) a granular equilibrium exists when the structure is a complete lattice and Γ is monotone, and it is invariant once reached. Their applicability depends on the granulation mechanism, the definition of refinement, the existence of the required bounds, and the monotonicity of Γ . Physical fidelity, empirical performance, and the suitability of a granulation mechanism for an application are outside these results and require application-specific modeling and validation.

6 Discussion

6.1 A Representation-Oriented Perspective

The framework follows the progression

$$\begin{aligned} X &\rightarrow \mathcal{G}(X) \rightarrow (\mathcal{G}(X), \preceq) \\ &\rightarrow (\mathcal{G}(X), \preceq, \vee, \wedge, \Gamma) \rightarrow \text{Fix}(\Gamma), \end{aligned} \quad (28)$$

which links the construction of representations with their order-theoretic organization and fixed-point analysis. Here $\vee$ and $\wedge$ exist only when the required bounds do, and equilibrium analysis needs completeness and monotonicity. Compared with work on other aspects of physical-virtual interaction, the framework concentrates on how granular representations are organized and transformed; it does not assume that DT research lacks hierarchical, multiscale, semantic, or adaptive capabilities.

6.2 Relationship with Existing Mathematical Theories

Granular Computing supplies the construction of granules [11, 12, 15, 16]; order theory formalizes refinement; lattice theory characterizes aggregation and refinement when bounds exist [19, 23]; and fixed-point theory gives conditions for granular equilibrium [24, 25]. The contribution lies in connecting these within one representation framework. Several distinctions matter: a partial order need not be a lattice, a lattice need not be complete, an order-preserving h need not preserve lattice operations, and existence of an equilibrium does not imply uniqueness, convergence, or stability.

6.3 Limitations and Future Research

First, $\mathcal{G}(X)$ depends on the granulation mechanism, the similarity function, and the refinement relation, and different choices give different ordered structures; conditions under which these structures are (complete) lattices should be established. Second, the equilibrium result depends on completeness and on the monotonicity of Γ ; sufficient conditions for uniqueness, convergence from arbitrary initial representations, and stability remain to be found. Third, the framework does not address physical fidelity or empirical performance; integrating physical dynamics, observation mechanisms, and application-specific validation is needed to assess its practical value. Finally, topological and metric structures could support continuity and convergence analysis, and fuzzy, rough, and interval-based granules could extend the framework to other forms of uncertainty [13, 19, 28].

7 Conclusion

This paper developed a mathematical framework for granular representations in DT systems. Separating the physical system P , its state universe X , and the family of admissible granules $\mathcal{G}(X)$ gives a basis for describing representation granularity, refinement relations, and adaptive transformations. The Granular Twin Algebra organizes granular representations through an order, lattice operations where bounds exist, and an adaptive transformation operator, while the map h formalizes order preservation between representation systems and, under stronger conditions, preservation of lattice operations.

The main result concerns granular equilibrium: when $(\mathcal{G}(X), \preceq)$ is a complete lattice and Γ is order-preserving, the Knaster-Tarski theorem guarantees that $\text{Fix}(\Gamma)$ is non-empty and is itself a complete lattice. Uniqueness, convergence from arbitrary initial representations, and dynamical stability are not established. The structural results depend on the granulation mechanism and the refinement relation, and the framework has not yet been validated on a physical system. Future work should identify application-specific conditions for the assumptions, derive results on uniqueness, convergence, and stability, and test the framework in practical DT applications. In this sense, the work is a starting point for further mathematical development rather than a complete theory of DT behavior.

Data Availability Statement

No data were generated or analyzed in this study.

Funding

This work was supported without any funding.

Conflicts of Interest

The author declares no conflicts of interest.

AI Use Statement

The author declares that no generative AI was used in the preparation of this manuscript.

Ethical Approval and Consent to Participate

Not applicable. This study is purely theoretical and did not involve human participants, animals, personal data, or identifiable individual information.

References

- [1] Barricelli, B. R., Casiraghi, E., & Fogli, D. (2019). A survey on digital twin: Definitions, characteristics, applications, and design implications. *IEEE Access*, 7, 167653-167671. [CrossRef]
- [2] Tao, F., Zhang, H., Liu, A., & Nee, A. Y. (2018). Digital twin in industry: State-of-the-art. *IEEE Transactions on industrial informatics*, 15(4), 2405-2415. [CrossRef]
- [3] Fuller, A., Fan, Z., Day, C., & Barlow, C. (2020). Digital twin: Enabling technologies, challenges and open research. *IEEE Access*, 8, 108952-108971. [CrossRef]
- [4] Grieves, M., & Vickers, J. (2016). Digital twin: Mitigating unpredictable, undesirable emergent behavior in complex systems. In *Transdisciplinary perspectives on complex systems: New findings and approaches* (pp. 85-113). Cham: Springer International Publishing. [CrossRef]
- [5] Rasheed, A., San, O., & Kvamsdal, T. (2020). Digital twin: Values, challenges and enablers from a modeling perspective. *IEEE Access*, 8, 21980-22012. [CrossRef]
- [6] Jones, D., Snider, C., Nassehi, A., Yon, J., & Hicks, B. (2020). Characterising the digital twin: A systematic literature review. *CIRP Journal of Manufacturing Science and Technology*, 29, 36-52. [CrossRef]
- [7] Semeraro, C., Lezoche, M., Panetto, H., & Dassisti, M. (2021). Digital twin paradigm: A systematic literature review. *Computers in Industry*, 130, 103469. [CrossRef]
- [8] Jiang, F., Ma, L., Broyd, T., & Chen, K. (2021). Digital twin and its implementations in the civil engineering sector. *Automation in Construction*, 130, 103838. [CrossRef]
- [9] Qi, Q., Tao, F., Hu, T., Anwer, N., Liu, A., Wei, Y., Wang, L., & Nee, A. Y. C. (2021). Enabling technologies and tools for digital twin. *Journal of Manufacturing Systems*, 58, 3-21. [CrossRef]
- [10] Tao, F., Xiao, B., Qi, Q., Cheng, J., & Ji, P. (2022). Digital twin modeling. *Journal of Manufacturing Systems*, 64, 372-389. [CrossRef]
- [11] Zadeh, L. A. (1997). Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic. *Fuzzy Sets and Systems*, 90(2), 111-127. [CrossRef]
- [12] Yao, J. T., Vasilakos, A. V., & Pedrycz, W. (2013). Granular computing: perspectives and challenges. *IEEE transactions on cybernetics*, 43(6), 1977-1989. [CrossRef]
- [13] Skowron, A., & Stepaniuk, J. (2001). Information granules: Towards foundations of granular computing. *International Journal of Intelligent Systems*, 16(1), 57-85. [CrossRef]
- [14] Yao, Y., & Zhong, N. (2007). Granular computing. In *Wiley Encyclopedia of Computer Science and Engineering*. Wiley. [CrossRef]
- [15] Yao, Y. (2016). A triarchic theory of granular computing. *Granular Computing*, 1(2), 145-157. [CrossRef]
- [16] Pedrycz, W., Skowron, A., & Kreinovich, V. (Eds.). (2008). *Handbook of Granular Computing*. Wiley. [CrossRef]
- [17] Pawlak, Z. (1982). Rough sets. *International Journal of Computer & Information Sciences*, 11(5), 341-356. [CrossRef]
- [18] Pawlak, Z. (1991). *Rough Sets: Theoretical Aspects of Reasoning about Data*. Kluwer Academic Publishers. [CrossRef]
- [19] Järvinen, J. (2007). Lattice theory for rough sets. In *Transactions on Rough Sets VI: Commemorating the Life and Work of Zdzisław Pawlak, Part I* (pp. 400-498). Berlin, Heidelberg: Springer Berlin Heidelberg. [CrossRef]
- [20] Pawlak, Z., Polkowski, L., & Skowron, A. (2007). Rough set theory. In *Wiley Encyclopedia of Computer Science and Engineering*. Wiley. [CrossRef]
- [21] Estaji, A. A., Hooshmandasl, M. R., & Davvaz, B. (2012). Rough set theory applied to lattice theory. *Information Sciences*, 200, 108-122. [CrossRef]
- [22] Li, X., Yi, H., & Liu, S. (2016). Rough sets and matroids from a lattice-theoretic viewpoint. *Information Sciences*, 342, 37-52. [CrossRef]
- [23] Davey, B. A., & Priestley, H. A. (2002). *Introduction to Lattices and Order* (2nd ed.). Cambridge University Press. [CrossRef]
- [24] Schröder, B. S. (2012). The fixed point property for ordered sets. *Arabian Journal of Mathematics*, 1(4), 529-547. [CrossRef]
- [25] Tarski, A. (1955). A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5(2), 285-309. [CrossRef]

- [26] Yang, X. B., Qian, Y. H., & Yang, J. Y. (2012). Hierarchical structures on multigranulation spaces. *Journal of Computer Science and Technology*, 27(6), 1169-1183. [[CrossRef](#)]
- [27] Qian, Y., Zhang, H., Li, F., Hu, Q., & Liang, J. (2014). Set-based granular computing: a lattice model. *International Journal of Approximate Reasoning*, 55(3), 834-852. [[CrossRef](#)]
- [28] Chiaselotti, G., Ciucci, D., Gentile, T., & Infusino, F. G. (2016). The granular partition lattice of an information table. *Information Sciences*, 373, 57-78. [[CrossRef](#)]



Jinghan Li is currently a Ph.D. candidate in Management at Lincoln University College, Malaysia. He received the M.Econ. degree from Hiroshima University, Japan, and the B.A. degree from Tiangong University, China. His research interests include artificial intelligence, digital economy, digital twins, sustainable development, environmental risk management, and geo-energy applications. (Email: jinghan.phdscholar@lincoln.edu.my)