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RESEARCH ARTICLE

Limit Cycles for Continuous Piecewise Linear Systems with Three Zones Having Two Degenerate Subsystems

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Abstract

This paper deals with continuous piecewise linear differential systems with three zones separated by two parallel straight lines (for short, CPWL3). The number of limit cycles of CPWL3 systems with two degenerate subsystems is not clear. In the paper, we provide a complete study on the maximum number of limit cycles by geometric techniques when the continuous piecewise linear systems with three zones have two degenerate subsystems. During our analysis, we also detect some bifurcation phenomena, such as boundary equilibrium bifurcation, scabbard bifurcation, grazing bifurcation, heteroclinic bifurcation and Hopf bifurcation.

Keywords: limit cycle, piecewise linear systems, bifurcation.

1 Introduction and main results

Limit cycle problem is one of the most important problems in the qualitative theory of differential systems, which is known as the Hilbert's 16th problem. This problem is unsolved even for the quadratic polynomial differential systems, many scholars pay attention to the special situations, such as Liénard systems see [1, 2], Z_2-equivariant systems see [3, 4], Hamiltonian systems see [5, 6].

During the past two decades, there are many works concerned with the dynamics of the piecewise linear differential systems (for short, PWL systems) which occur naturally in mechanics, electronics, economy and so on [7, 8]. PWL systems seem to have almost the same dynamics of nonlinear differential systems, such as limit cycles, homoclinic and heteroclinic orbits and so on.

There exist many results on limit cycles for the PWL system with two zones separated by a straight line(for short, PWL2 system). Freire et al. [9] proved that the continuous PWL2 system has at most a limit cycle. Han and Zhang [10] revealed the discontinuous PWL2 system can have two limit cycles via Hopf bifurcation. Freire et al. [11] presented a Liénard-like canonical form for discontinuous PWL2 system with seven parameters. According to the type of equilibria of the subsystems, PWL2 system can be classified into 15 types: FF, FC, FN, FN', $FS, CC, CN, CN', CS, NN, NN', NS, N'N', N'S$ and SS, where F, S, C, N, N' denote focus, saddle, center, node with different eigenvalues and node with equal eigenvalues, respectively. According to the known

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results, such as [13–16], Llibre [12] summarized the maximum number of known limit cycles of these types was the shown in Table 1.

Table 1. The known results on the lower bounds for the maximum number of limit cycles of the discontinuous PWL2 system.

	F	C	N	N'	S
F	3	2	3	3	3
C	-	0	1	1	1
N	-	-	2	2	2
N'	-	-	-	2	2
S	-	-	-	-	2

Recently, Chen and Liu [18] showed that PWL system with type $N'N'$ has at most one limit cycle, and the upper bound can be reached. Carmona et al. [17] proved that the maximum number of limit cycles of PWL2 system is less than or equal to 8. Combined with Table 1, the maximum number of limit cycles of PWL2 system is probably three.

Lots of works showed that the shape of the switch monifold affects the number of limit cycles. Denote the PWL system with two zones, where the switch line is the positive x-axis and the positive y-axis, by PWLxy system. Liang et al. [19] proved that PWLxy system has at least three limit cycles. Sun and Du [20] showed that PWLxy system of type NN can have four limit cycles. Zhao et al. [21] proved that PWLxy of type SC has at least three limit cycles. Alves et al. [22] showed that PWLxy system of type CC can have two limit cycles and presented the possible bifurcation for a particular PWLxy system. Wei et al. [23] proved that PWLxy system of type SS has at least five limit cycles. The authors [24] proved the existence of PWL with two zones having four, five, six and seven limit cycles. The authors [25] showed that PWL system with two zones can have n limit cycles for any given positive integer n. Gasull [26] proposed to research the maximum number $\mathcal{L}_n$ of limit cycles of PWL system with two zones separated by a branch of an algebraic curve of degree n. Andrade et al. [27] proved that $\mathcal{L}_2 \geq 4$, $\mathcal{L}_3 \geq 8$, $\mathcal{L}_n \geq 7$ for $n \geq 4$ even, and $\mathcal{L}_n \geq 9$ for $n \geq 4$ odd.

Let $\mathbf{z} = (x,y)^\top$, R_l, R_c and R_r are open nonoverlapping regions separated by two parallel straight lines Σ_l and Σ_r, and $R_l \cup \Sigma_l \cup R_c \cup \Sigma_r \cup R_r = R^2$. The general piecewise linear differential systems with three zones (for short, PWL3 systems) separated by two parallel lines can be written as

$$\dot{\mathbf{z}} = \begin{cases} A_l\mathbf{z} + b_l, & \mathbf{z} \in R_l, \\ A_c\mathbf{z} + b_c, & \mathbf{z} \in R_c, \\ A_r\mathbf{z} + b_r, & \mathbf{z} \in R_r, \end{cases} \tag{1.1}$$

where $A_k = (a_{ij}^k)_{2\times 2}$ are 2×2 constant matrices, and $b_k = (b_1^k, b_2^k)^\top$ are constant vectors of R^2, $k = l, c, r$.

Many scholars are interested in the number of limit cycles of the system (1.1). Berbache et al. [28] showed that system (1.1) of type boundary focus-Hamiltonian linear saddle-boundary focus has at most four crossing limit cycles, and system (1.1) of type boundary focus-Hamiltonian linear saddle-boundary center has at most three crossing limit cycles. Via the first-order Melnikov function, considering a particular system (1.1) with two centers and a double heteroclinic loop, Liu et al. [29] proved that 2 limit cycles can appear near two centers and the double heteroclinic loop, respectively. Li et al. [30] showed that system (1.1) with either boundary focus-center-boundary focus or boundary focus-center-center types has at most three limit cycles. Xiong et al. [31] obtained that system (1.1) has at least seven limit cycles via Melnikov method.

As far as we know, the maximum number of limit cycles for a general discontinuous PWL3 system (1.1) is unknown and there are more than one hundred cases according to the type of equilibria of the three subsystems. We call that system is degenerate if the determinant of its linear part vanishes. In the present paper, we focus on the continuous PWL3 system with two degenerate subsystems.

According to [32], the general continuous PWL3 system (1.1) can be transformed into

$$\begin{cases} \frac{dx}{dt} = & m_1x + n_1y + c_1 + \alpha_1 |k_1x + k_2y + d_1| \\ & +\alpha_2 |k_1x + k_2y + d_2|, \\ \frac{dy}{dt} = & m_2x + n_2y + c_2 + \beta_1 |k_1x + k_2y + d_1| \\ & +\beta_2 |k_1x + k_2y + d_2|, \end{cases} \tag{1.2}$$

where $|\alpha_1| + |\beta_1| > 0, |\alpha_2| + |\beta_2| > 0, |k_1| + |k_2| > 0, d_1 \neq d_2$. If $n_1k_1^2 - m_1k_1k_2 - m_2k_2^2 + n_2k_1k_2 \neq 0$, then system (1.2) can be written as Lienaréd form:

$$\frac{dx}{dt} = F(x) - y, \quad \frac{dy}{dt} = g(x) - \alpha, \tag{1.3}$$

where

$$F(x) = \begin{cases} t_r(x-1) + t_c, & x > 1, \\ t_cx, & -1 \leq x \leq 1, \\ t_l(x+1) - t_c, & x < -1, \end{cases}$$

Table 2. The maximum number of known limit cycles of continuous PWL3 system (1.3).

FSF	FNF, FNN′, FNN, FN′F, FN′N′, FN′N, NFN, FFF, FFN, FFN′, FDF, FSS, FDS, FDD, FFN, FNN′, FFC, NFS, SFS	NND, NN′D, NFD, NFN′, NNN, NNN′, NN′N′, NN′N, DND, DFD, DN′D, FND, FN′D, FFD, N′ND, N′N′D, N′FD, N′FN, N′FN′, N′NN, N′NN′, N′N′N′, N′N′N, SNS, FDN, DFF, FCF, FCN, FCS, NCN, NSN, SCS, SN′S, NCN, NCS, FDN′, DFN′	SDS, SCS, DNN, N′CN′, FCF, CSC, N′SN′, DSD, NSN, N′CN′, DCD, SDN′, SDN, SSN′, SSN, DN′D, SSS, NSN′
3	2	1	0

$$g(x) = \begin{cases} d_r(x-1)+d_c, & x>1, \\ d_c x, & -1 \le x \le 1, \\ d_l(x+1)-d_c, & x<-1. \end{cases}$$

Limit cycles of system (1.3) have been studied by many scholars. When the subsystem of system (1.3) is degenerate, the result of two limit cycles is obtained in [33]. For the general case, more results with two limit cycles are found in [35, 36]. There are examples of system (1.3) having three limit cycles, see [32, 37]. We denote the degenerate subsystem of system (1.3) by D. According to the known results, the maximum number of known limit cycles of system (1.3) is given in Table 2.

We do not know the maximal number of limit cycles of system (1.3) with two degenerate subsystems. In this paper, we study the number of limit cycles of system (1.3) with two degenerate subsystems. For system (1.3) with two degenerate subsystems, these types $FDD, CDD, SDD, NDD, N'DD$ can be reduced to these types DDF, DDC, DDS, DDN, DDN'. Therefore, we only need to consider these types DDF, DDC, DDS, DDN, DDN'. Our main results are as follows.

Theorem 1.1. *System (1.3) of types $DND, DN'D$, and DFD has at most a limit cycle.*

Theorem 1.2. *System (1.3) of types DCD, DSD, DDN, DDS, and DDN' does not have limit cycles.*

Theorem 1.3. *System (1.3) of type DDF has at most two limit cycles and the number is reached.*

Theorem 1.4. *System (1.3) of type DDC has at most a limit cycle.*

Remark 1.1. *For PWL3 system with two degenerate subsystems, we find the following bifurcation phenomena: System (1.3) of types $DND, DN'D$, and DFD has boundary equilibrium bifurcation, scabbard bifurcation, grazing bifurcation, heteroclinic bifurcation and Hopf bifurcation. System (1.3) of types DDF and DDC has boundary equilibrium bifurcation, scabbard bifurcation, grazing bifurcation and Hopf bifurcation.*

Let

$$\phi(x) = \begin{cases} t_r, & x>1, \\ t_c, & -1 \le x \le 1, \\ t_l, & x<-1. \end{cases}$$

According to [34], for any limit cycle Γ of system (1.3), if

$$\oint_\Gamma \phi(x)dt < 0 (>0), \tag{1.4}$$

then Γ is stable (unstable). If the possible limit cycle Γ satisfies expression (1.4), then system (1.3) has at most one limit cycle. Based on the above result, two methods we employed in this paper are as follows:

(1) When $t_l < 0, t_c > 0, t_r > 0$, assume that system (1.3) has one limit cycle Γ, with Coppel transformation $(x,y) \to (w,y)$, where $w = F(x)$, we try to compare the values of $\oint_{\Gamma^*} \phi(x)dt$ and $\oint_{\Gamma^{**}} \phi(x)dt$, and then we determine whether $\oint_\Gamma \phi(x)dt$ is less than zero. Here Γ^* (Γ^{**}) is the part of Γ restricted to $x < -1$ ($x > -1$). Using the same ideas, we can deal with the case $t_l < 0, t_c \le 0, t_r > 0$ and the case $t_l > 0, t_r < 0$.

(2) When $t_l t_r > 0$, assume that Γ_1 and Γ_2 are the innermost two limit cycles of system (1.3), we try to prove $\oint_{\Gamma_1} \phi(x)dt \le \oint_{\Gamma_2} \phi(x)dt$ or $\oint_{\Gamma_2} \phi(x)dt \le \oint_{\Gamma_1} \phi(x)dt$. If one of the above expression holds, then system (1.3) has at most two limit cycles.

The above methods are also effective for system (1.3) with other types.

This paper is organized as follows. We study the limit cycle of types $DND, DFD, DN'D$ of system (1.3) in

Section 2. The nonexistence of limit cycles of types DCD, DSD, DDN, DDS, DDN' is given in Section 3. We investigate the limit cycle of types DDF and DDC type of system (1.3) in Section 4 and Section 5 respectively. In Section 6, we give some examples of these types to illustrate the maximum number of limit cycles can be reached.

2 Proof of Theorem 1.1

Let ϵ satisfies $0 < \epsilon \ll 1$. We use ϵ many times.

According to Lemma 5.7 in [34], we can prove the following lemma.

Lemma 2.1. *The amplitude of a stable (unstable) limit cycle of system (1.3) is increasing (decreasing) as one of t_l, t_c, t_r increases.*

From the terminology in [34], a limit cycle involving two(three) zones is called a *small limit cycle*(*large limit cycle*). In this section, we assume that the left and right subsystems of system (1.2) are degenerate, and the central subsystem is either node or focus types.

Set

$$\begin{aligned}\mathcal{G}_1 &:= \{(\alpha, d_l, d_c, d_r, t_l, t_c, t_r) \in R^7 : \\ &\quad d_l = 0, d_c > 0, t_c \neq 0, t_r > 0, d_r = 0\},\\ \mathcal{G}_2 &:= \{(\alpha, d_l, d_c, d_r, t_l, t_c, t_r) \in R^7 : \\ &\quad d_l = 0, d_c > 0, t_c \neq 0, t_r = 0, d_r = 0\},\\ \mathcal{G}_3 &:= \{(\alpha, d_l, d_c, d_r, t_l, t_c, t_r) \in R^7 : \\ &\quad d_l = 0, d_c > 0, t_c \neq 0, t_r < 0, d_r = 0\}.\end{aligned}$$

In the parameter regions $\mathcal{G}_1$-$\mathcal{G}_3$, with the transformation $(x, y) \to (x - 1, y - t_c)$, the left and central subsystems of system (1.3) can be transformed as

$$\frac{dx}{dt} = \hat{F}(x) - y, \quad \frac{dy}{dt} = \hat{g}(x) - (d_c + \alpha), \tag{2.5}$$

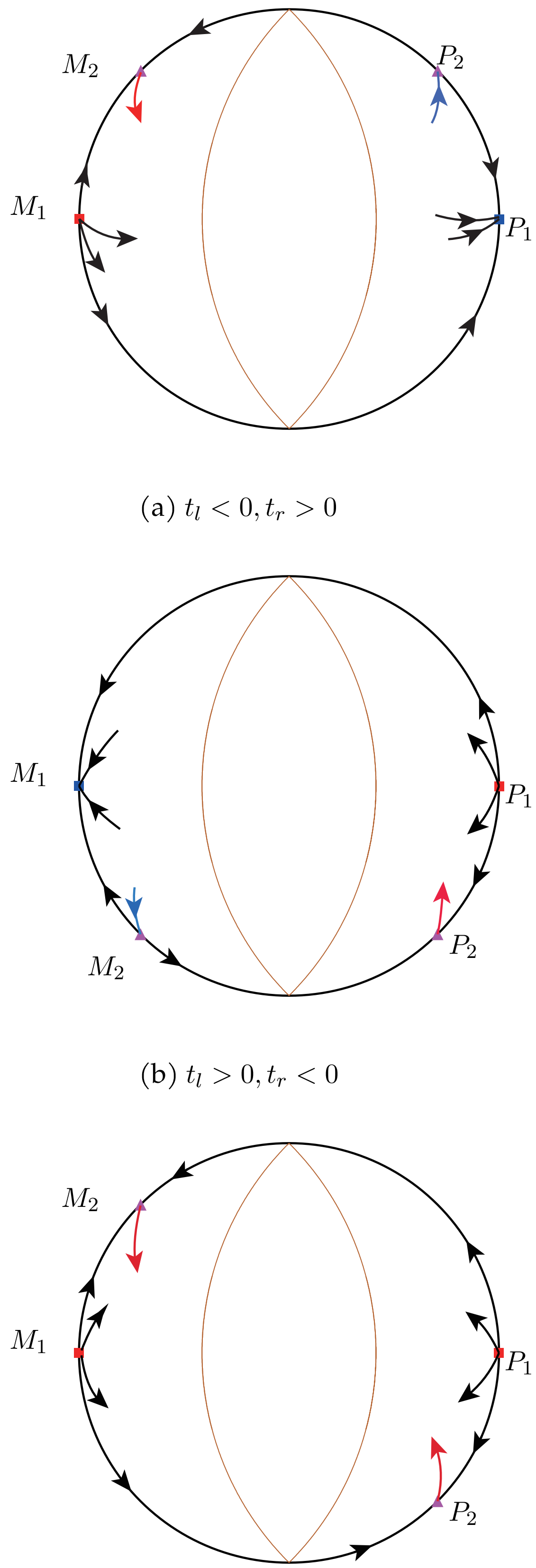
where

$$\hat{F}(x) = \begin{cases} t_c x, & 0 \le x \le 2, \\ t_l x, & x < 0, \end{cases}$$

$$\hat{g}(x) = \begin{cases} d_c x, & 0 \le x \le 2, \\ 0, & x < 0. \end{cases}$$

By Theorem 4 in [33], we study the dynamics of the left and central subsystems of system (1.3).

Lemma 2.2. *The infinite equilibria of system (1.3) in the Poincaré disc of system (1.3) are showed in Figure 1 if $t_l t_r < 0, -d_c < \alpha < d_c$ or $t_l < 0, t_r < 0, -d_c < \alpha < d_c$.*



(a) $t_l < 0, t_r > 0$

(b) $t_l > 0, t_r < 0$

(c) $t_l < 0, t_r < 0$

Figure 1. The infinite equilibria in the Poincaré disc of system (1.3) with $t_l t_r < 0, -d_c < \alpha < d_c$ or $t_l < 0, t_r < 0, -d_c < \alpha < d_c$.

Proof. We apply Poincaré compactification (see Chapter 5 in [38]) to prove the results.

In the chart U_1, perform the Poincaré transform $x = \frac{1}{v}$, $y = \frac{u}{v}$, $v > 0$, then the right subsystem of system (1.3) becomes

$$\begin{cases} \dot{u} = u^2 + (t_r - t_c)\,uv - t_r u + (d_c - \alpha)\,v, \\ \dot{v} = -t_r v + (t_r - t_c)\,v^2 + uv. \end{cases} \tag{2.6}$$

System (2.6) has two equilibria $P_1(0,0)$ and $P_2(t_r, 0)$ on $v = 0$. The eigenvalues of P_1 are $-t_r$ and $-t_r$, so P_1 is a stable node for $t_r > 0$ and an unstable node for $t_r < 0$. The local structure of P_1 is shown in Figure 1(a) for $t_r > 0$ and in Figure 1(b)–(c) for $t_r < 0$. The eigenvalues of P_2 are t_r and 0. Translating P_2 to the origin O by the transformation

$$u = u_1 - \frac{t_r(t_r - t_c) + d_c - \alpha}{t_r} v_1 + t_r, \qquad v = v_1,$$

system (2.6) is changed into

$$\begin{cases} \dot{u}_1 = t_r u_1 - \frac{d_c - \alpha}{t_r} u_1 v_1 + u_1^2, \\ \dot{v}_1 = u_1 v_1 - \frac{d_c - \alpha}{t_r} v_1^2. \end{cases} \tag{2.7}$$

Substituting $u_1 = 0$ into the second equation of system (2.7), we have $\dot{v}_1 = -\frac{d_c - \alpha}{t_r} v_1^2$. According to Theorem 7.1 of Chapter 2 in [38], the origin of system (2.7) is a saddle–node. The local structure of P_2 (saddle–node) is shown in Figure 1(a) for $t_r > 0$ and in Figure 1(b)–(c) for $t_r < 0$.

In the chart V_1, make the Poincaré transform $x = \frac{1}{v}$, $y = \frac{u}{v}$, $v < 0$, then the left subsystem of system (1.3) is transformed into

$$\begin{cases} \dot{u} = -(d_c + \alpha)\,v - t_l u + (t_c - t_l)\,uv + u^2, \\ \dot{v} = -t_l v + (t_c - t_l)\,v^2 + uv. \end{cases} \tag{2.8}$$

System (2.8) has two equilibria $M_1(0,0)$ and $M_2(t_l, 0)$ on $v = 0$. The eigenvalues of M_1 are $-t_l$ and $-t_l$, so M_1 is a stable node for $t_l > 0$ and an unstable node for $t_l < 0$. The local structure of M_1 is shown in Figure 1(b) for $t_l > 0$ and in Figure 1(a), (c) for $t_l < 0$. The eigenvalues of M_2 are t_l and 0. Similar to the analysis in the chart U_1, M_2 is a saddle–node. Its local structure is shown in Figure 1(b) for $t_l > 0$ and in Figure 1(a), (c) for $t_l < 0$.

In the charts U_2 and V_2, by the transformation $x = \frac{u}{v}, y = \frac{1}{v}$, for $-1 \le x \le 1$ system (1.3) can be transformed as

$$\begin{cases} \dot{u} = -d_c u^2 + t_c u + \alpha uv - 1, \\ \dot{v} = -d_c uv + \alpha v^2. \end{cases} \tag{2.9}$$

It is easy to verify that there is no infinite equilibria of system (1.3) in the the charts U_2 and V_2. By the aforementioned analysis, we know that the infinite equilibria of system (1.3) in the Poincaré disc, as shown in Figure 1. □

If system (1.3) admits a limit cycle Γ, then the interior region of Γ should contain an equilibrium. For $\frac{\alpha}{d_c} < -1$ or $\frac{\alpha}{d_c} > 1$, system (1.3) does not admit limit cycles since system (1.3) has no equilibria. Hence we only consider the case $-1 \le \frac{\alpha}{d_c} \le 1$ in the following. For $-1 \le \frac{\alpha}{d_c} \le 1$, system (1.3) has a equilibrium $E_c(\frac{\alpha}{d_c}, \frac{t_c \alpha}{d_c})$: E_c is a focus for $t_c^2 - 4d_c < 0$, a node with equal eigenvalues for $t_c^2 - 4d_c = 0$, and a node with different eigenvalues for $t_c^2 - 4d_c > 0$.

Lemma 2.3. *In the parameter region $\mathcal{G}_1$, system (1.3) admits at most a limit cycle.*

Proof. In the parameter region $\mathcal{G}_1$, we have $d_l = 0, d_c > 0, t_c \neq 0, t_r > 0, d_r = 0$. We divide the proof into two cases.

Case 1: $t_l < 0$. By the transformation $(x, y) \to (x + \frac{\alpha}{d_c}, y + \frac{t_c \alpha}{d_c})$, system (1.3) can be changed into

$$\frac{dx}{dt} = \check{F}(x) - y, \qquad \frac{dy}{dt} = \check{g}(x), \tag{2.10}$$

where

$$\check{F}(x) = \begin{cases} t_r x + t_c - t_r + \frac{t_r \alpha}{d_c} - \frac{t_c \alpha}{d_c}, & x > 1 - \frac{\alpha}{d_c}, \\ t_c x, & -1 - \frac{\alpha}{d_c} \le x \le 1 - \frac{\alpha}{d_c}, \\ t_l x + t_l - t_c + \frac{t_l \alpha}{d_c} - \frac{t_c \alpha}{d_c}, & x < -1 - \frac{\alpha}{d_c}, \end{cases}$$

and

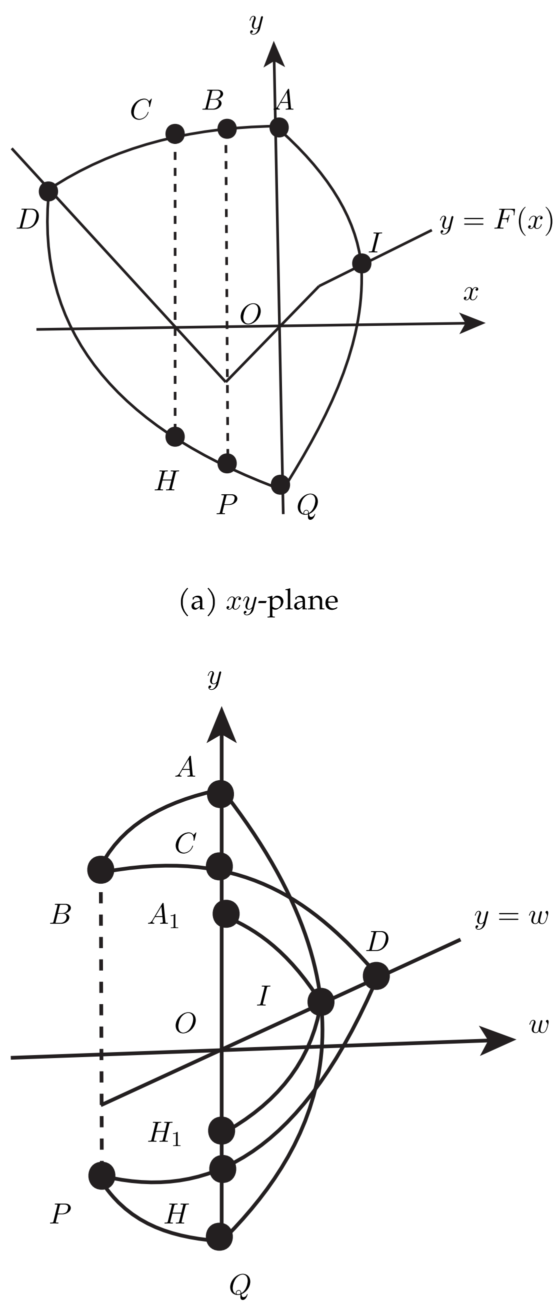
$$\check{g}(x) = \begin{cases} d_c - \alpha, & x > 1 - \frac{\alpha}{d_c}, \\ d_c x, & -1 - \frac{\alpha}{d_c} \le x \le 1 - \frac{\alpha}{d_c}, \\ -d_c - \alpha, & x < -1 - \frac{\alpha}{d_c}. \end{cases}$$

For convenience, set $E(x, y) = \frac{y^2}{2} + \int_0^x \check{g}(x)dx$.

For $t_c > 0$, assume that system (1.3) has a large limit cycle $\Gamma^0_{\mathcal{G}_1}$, which is shown in Figure 2, where $A, B, C, D, H, P, Q, I \in \Gamma^0_{\mathcal{G}_1}$. Let $E(x, y) = \frac{y^2}{2} + \int_0^x \check{g}(x)dx$. By the sign of $\check{g}(x)$, we have $y_A > y_B > y_C > y_D$ and $y_H > y_P > y_Q$. We claim $y_D > 0$. In fact, if $y_D \le 0$, then

$$\frac{dE}{dt} = \check{F}(x)\check{g}(x) \ge 0 (\not\equiv 0)$$

and $0 = \int_{\Gamma^0_{\mathcal{G}_1}} dE = \int_{\Gamma^0_{\mathcal{G}_1}} \check{F}(x)\check{g}(x)dt > 0$. This is a contradiction.



(a) xy-plane

(b) wy-plane

Figure 2. $\Gamma^0_{\mathcal{G}_1}$.

Let $w = w(x) = \check{F}(x)$, and $\check{x}_1(w)$ ($\check{x}_2(w)$) be the branch of the inverse of $w(x)$ for $x \geq -1$ ($x < -1$). With the transformation $(x, y) \to (w, y)$, system (2.10) can be transformed into:

$$\frac{dy}{dw} = \frac{\check{\tau}_i(w)}{w - y}, \quad i = 1, 2, \tag{2.11}$$

where

$$\check{\tau}_1(w) := \check{\tau}(\check{x}_1(w)) = \begin{cases} \frac{d_c - \alpha}{t_r}, & w > t_c - \frac{t_c\alpha}{d_c}, \\ \frac{d_c w}{t_c^2}, & -t_c - \frac{t_c\alpha}{d_c} \leq w \leq t_c - \frac{t_c\alpha}{d_c} \end{cases}$$

and

$$\check{\tau}_2(w) := \check{\tau}(\check{x}_2(w)) = \frac{-d_c - \alpha}{t_l}, w \geq -t_c - \frac{t_c\alpha}{d_c}.$$

It follows that

$$\check{\tau}_1(w) - \check{\tau}_2(w) = \begin{cases} d_c\left(\frac{1}{t_r} + \frac{1}{t_l}\right) + \alpha\left(\frac{1}{t_l} - \frac{1}{t_r}\right), & w > t_c - \frac{t_c\alpha}{d_c}, \\ \frac{d_c w}{t_c^2} + \frac{d_c + \alpha}{t_l}, & -t_c - \frac{t_c\alpha}{d_c} \leq w \leq t_c - \frac{t_c\alpha}{d_c}. \end{cases}$$

The equation $\check{\tau}_1(w) - \check{\tau}_2(w) = 0$ has the unique solution $w_1 = -\frac{t_c^2(d_c+\alpha)}{d_c t_l}$ and $w_1 \in \left[-t_c - \frac{t_c\alpha}{d_c}, t_c - \frac{t_c\alpha}{d_c}\right]$.

Suppose that each orbit segment of $\Gamma^0_{\mathcal{G}_1}$ on the xy plane is transformed into their image arc segments with the same notations on the half wy plane respectively, as exhibited in Figure 2(b). Denote $y = y_1(w)$, $y_2(w)$, $z_1(w)$ and $z_2(w)$ by the orbit segments $\widetilde{IAB}, \widetilde{BCD}, \widetilde{PQI}, \widetilde{DHP}$ in the wy-plane, respectively. Set $u(w) = y_1(w) - y_2(w)$ to obtain

$$\frac{du}{dw} = \frac{\check{\tau}_1(w)}{w - y_1(w)} - \frac{\check{\tau}_2(w)}{w - y_2(w)} \triangleq q_1(w)u + q_2(w), \tag{2.12}$$

where $q_1(w) = \frac{\check{\tau}_1(w)}{(w-y_1(w))(w-y_2(w))}$ and $q_2(w) = \frac{\check{\tau}_1(w) - \check{\tau}_2(w)}{w - y_2(w)}$. From the Variation of Constants Formula, it yields that

$$u(w) = \exp\left(\int_0^w q_1(s)ds\right) U(w),$$

where $U(w) := u_0 + \int_0^w q_2(s) \exp\left(\int_s^0 q_1(\varsigma)d\varsigma\right) ds$ with $u_0 = y_A - y_C > 0$. $U'(w) > 0$ for $w < w_1$ and $U'(w) < 0$ for $w > w_1$ since $U'(w) = q_2(w)\exp(\int_w^0 q_1(\varsigma)d\varsigma)$ and $\check{\tau}_1(-t_c - \frac{t_c\alpha}{d_c}) - \tau_2(-t_c - \frac{t_c\alpha}{d_c}) < 0$. Therefore, the equation $u(w) = y_1(w) - y_2(w) = 0$ has at most a solution for $w \geq -t_c - \frac{t_c\alpha}{d_c}$ since $U(0) = u_0 > 0$. Similarly, the equation $v(w) = z_1(w) - z_2(w) = 0$ has at most a solution for $w \geq -t_c - \frac{t_c\alpha}{d_c}$ too.

It follows from Green formula that

$$\begin{aligned} 0 &= \oint_{\Gamma^0_{\mathcal{G}_1}} \check{g}(x)dx - (\check{F}(x) - y)dy \\ &= -\iint_{\Delta} \check{F}'(x)dxdy \\ &= \iint_{\Delta_2} dwdy - \iint_{\Delta_1} dwdy, \end{aligned}$$

where Δ is the interior region surrounded by $\Gamma^0_{\mathcal{G}_1}$ in Figure 2(a), Δ_1 is the region surrounded by the $\overline{BP}$ and $\widetilde{PIB}$, Δ_2 is the region surrounded by the $\overline{PB}$ and $\widetilde{BDP}$ in Figure 2(b).

In the following, we call $y_D > y_I$. If $y_D \leq y_I$, then $\iint_{\Delta_2} dwdy - \iint_{\Delta_1} dwdy < 0$. This is a contradiction.

Let $k = \frac{y_I}{y_D}$, with the scaling $(w, y) \to (kw, ky)$, the second equation of system (2.11) is changed into

$$\frac{dy}{dw} = \frac{\check{\tau}_2(kw)}{k(w - y)}. \tag{2.13}$$

The orbit segment $\widetilde{CDH}$ of equation (2.11) is changed into the orbit segment $\widetilde{A_1IH_1}$ of equation (2.13), and

$$\begin{aligned}\int_{\widetilde{CDH}} \frac{1}{w-y}dw &= \int_{\widetilde{A_1IH_1}} \frac{1}{k(w-y)}d(kw) \\ &= \int_{\widetilde{A_1IH_1}} \frac{1}{w-y}dw.\end{aligned}$$

Denote $y_3(w)$ and $z_3(w)$ by the orbit segments $\widetilde{A_1I}$ and $\widetilde{IH_1}$ respectively, recall that $y_1(w)$ and $z_1(w)$ represent the orbit segments $\widetilde{IA}$ and $\widetilde{QI}$, then $y_3(w) < y_1(w), z_1(w) < z_3(w)$ for $0 = w_A \le w \le w_I$. By calculation, we have

$$\begin{aligned}&\int_{\widetilde{QIA}} \frac{dw}{w-y} + \int_{\widetilde{CDH}} \frac{dw}{w-y} \\ =&\int_{\widetilde{QIA}} \frac{dw}{w-y} + \int_{\widetilde{A_1I_1H_1}} \frac{dw}{w-y} \\ =&\int_{\widetilde{IA}} \frac{dw}{w-y_1(w)} + \int_{\widetilde{A_1I}} \frac{dw}{w-y_3(w)} \\ &+ \int_{\widetilde{QI}} \frac{dw}{w-z_1(w)} + \int_{\widetilde{IH_1}} \frac{dw}{w-z_3(w)} \\ =&\int_{x_A}^{x_I} \frac{y_3(w)-y_1(w)}{(w-y_1(w))\,(w-y_3(w))}dw \\ &+ \int_{x_{H_1}}^{x_I} \frac{z_1(w)-z_3(w)}{(w-z_1(w))\,(w-z_3(w))}dw < 0.\end{aligned}$$

From $y_1(w) > y_2(w), z_1(w) < z_2(w)$ for $w_B \le w \le w_A = 0$, it yields that

$$\begin{aligned}&\int_{\widetilde{AB}} \frac{1}{w-y_1(w)}dw + \int_{\widetilde{BC}} \frac{1}{w-y_2(w)}dw = \\ &\int_{w_B}^{w_A} \frac{y_2(w)-y_1(w)}{(w-y_1(w))\,(w-y_2(w))}dw < 0.\end{aligned}$$

Similarly,

$$\int_{\widetilde{HP}} \frac{1}{w-z_1(w)}dw + \int_{\widetilde{PQ}} \frac{1}{w-z_2(w)}dw < 0.$$

Therefore,

$$\begin{aligned}\int_{\Gamma^0_{\mathcal{G}_1}} \check{F}'(x)dt =& \int_{\widetilde{IAB}} \frac{1}{w-y}dw + \int_{\widetilde{BCD}} \frac{1}{w-y}dw \\ &+ \int_{\widetilde{CDH}} \frac{dw}{w-y} + \int_{\widetilde{HPQ}} \frac{1}{w-y}dw \\ &+ \int_{\widetilde{QIA}} \frac{dw}{w-y} < 0.\end{aligned}$$

By $\int_{\Gamma^0_{\mathcal{G}_1}} F'(x)dt = \int_{\Gamma^0_{\mathcal{G}_1}} \check{F}'(x)dt$, it follows that that system (1.3) has at most a large limit cycle. From Theorem 4 in [33], system (1.3) admits at most a small limit cycle. If system (1.3) has at least two limit cycles, where $\Gamma^1_{\mathcal{G}_1}$ and $\Gamma^2_{\mathcal{G}_1}$ are the innermost two limit cycles, then $\Gamma^2_{\mathcal{G}_1}$ is internal unstable and large, which implying $\int_{\Gamma^2_{\mathcal{G}_1}} F'(x)dt \ge 0$. This is a contradiction. Hence system (1.3) has at most a limit cycle. Similarly, we can prove that system (1.3) admits at most a limit cycle for $t_c < 0$.

Case 2: $t_l \ge 0$**.** We claim that system (1.3) does not have no limit cycles for $t_c > 0$. If system (1.3) has a limit cycle $\Gamma^3_{\mathcal{G}_1}$, then we have $\int_{\Gamma^3_{\mathcal{G}_1}} F'(x)dt > 0$, which implies Γ is unstable. However, $\Gamma^3_{\mathcal{G}_1}$ is semi-stable or stable since system (1.3) has an unstable equilibrium E_c, which imply $\int_{\Gamma^3_{\mathcal{G}_1}} F'(x)dt \le 0$. This lead to a contradiction.

In the following we consider the case $t_c < 0$. Assume that system (1.3) has two large limit cycles $\Gamma^4_{\mathcal{G}_1}$ and $\Gamma^5_{\mathcal{G}_1}$, which are illustrated in Figure 3, where $A_i, B_i, C_i, D_i \in \Gamma^{i+3}_{\mathcal{G}_1}, i = 1, 2$.

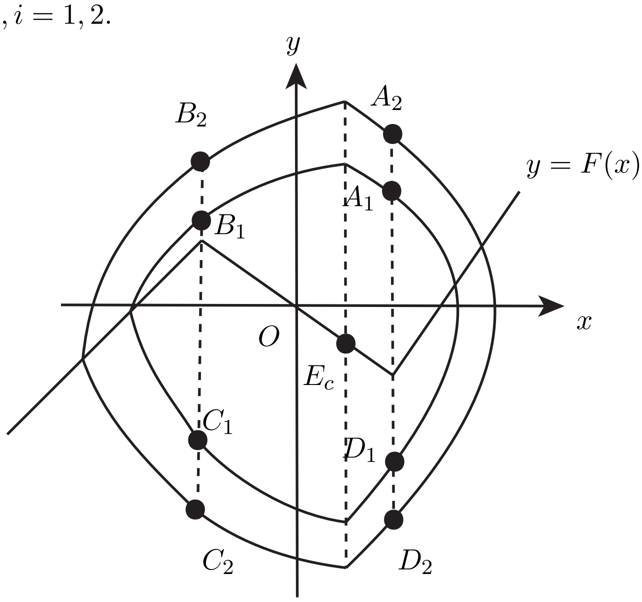


Figure 3. $\Gamma^4_{\mathcal{G}_1}$ and $\Gamma^5_{\mathcal{G}_1}$.

Denote $y = y_i(x)$ by the orbit segment $\widetilde{A_iB_i}$ for $i = 1, 2$. Obviously $y_1(x) < y_2(x)$. We obtain that

$$\begin{aligned}&\int_{\widetilde{A_2B_2}} F'(x)dt - \int_{\widetilde{A_1B_1}} F'(x)dt \\ =&\int_{x_{A_2}}^{x_{B_2}} \frac{t_c}{F(x)-y_2(x)}dx - \int_{x_{A_1}}^{x_{B_1}} \frac{t_c}{F(x)-y_1(x)}dx \\ =&\int_{x_{A_1}}^{x_{B_1}} \frac{t_c\,(y_2(x)-y_1(x))}{(F(x)-y_2(x))\,(F(x)-y_1(x))}dx > 0,\end{aligned}$$

i.e.,

$$\int_{\widetilde{A_2B_2}} F'(x)dt > \int_{\widetilde{A_1B_1}} F'(x)dt. \tag{2.14}$$

Similarly, we have

$$\int_{\widetilde{C_2D_2}} F'(x)dt > \int_{\widetilde{C_1D_1}} F'(x)dt. \tag{2.15}$$

For $t_l > 0$, assume that the orbit starting from $B_1(B_2)$ needs time $t_1(t_2)$ to intersects the y-axis first. Note that $y = \alpha t + C_0$ for $x < -1(C_0$ is a constant). From $|\overline{B_1C_1}| < |\overline{B_2C_2}|$, it follows that $t_1 < t_2$ and

$$\int_{\widetilde{B_2C_2}} F'(x)dt > \int_{\widetilde{B_1C_1}} F'(x)dt. \tag{2.16}$$

Similarly, we have

$$\int_{\widetilde{D_2A_2}} F'(x)dt > \int_{\widetilde{D_1A_1}} F'(x)dt. \tag{2.17}$$

From (2.14)-(2.17), it yields that

$$\int_{\Gamma^5_{\mathcal{G}_1}} F'(x)dt > \int_{\Gamma^4_{\mathcal{G}_1}} F'(x)dt. \tag{2.18}$$

If $t_l = 0$, then

$$\int_{\widetilde{B_2C_2}} F'(x)dt = \int_{\widetilde{B_1C_1}} F'(x)dt = 0. \tag{2.19}$$

The inequality (2.18) holds for $t_l = 0$ since the inequalities (2.14),(2.15) and (2.17) hold for $t_l = 0$. If system (1.3) has two limit cycles γ_1 and γ_2, then γ_1 is semi-stable or unstable since E_c is stable. If γ_1 is semi-stable, then fixing $t_l, t_r, d_l, d_c, d_r, \alpha$, γ_1 becomes an unstable limit cycle γ_{11} and a stable limit cycle γ_{12} as $t_c \to t_c - \epsilon$, where γ_{11} lies in the interior region surrounded by γ_{12}, that is, $\int_{\gamma_{12}} F'(x)dt < \int_{\gamma_{11}} F'(x)dt$, which is contradiction. If γ_1 is unstable, then γ_2 is internal stable, i.e., $\int_{\gamma_2} F'(x)dt \leq 0$. Then we have $\int_{\gamma_2} F'(x)dt < \int_{\gamma_1} F'(x)dt$. This is a contradiction. Therefore, system (1.3) admits at most a limit cycle. □

Lemma 2.4. *In the parameter region $\mathcal{G}_2$, system* (1.3) *admits at most a limit cycle.*

Proof. In the parameter region $\mathcal{G}_2$, we have $d_l = 0, d_c > 0, t_c \neq 0, t_r = 0, d_r = 0$. We divide the proof into two cases.

Case 1: $t_l < 0$. For $t_c < 0$, we claim that system (1.3) has no limit cycles. In fact, if system (1.3) has a limit cycle $\Gamma^0_{\mathcal{G}_2}$, then $\int_{\Gamma^0_{\mathcal{G}_2}} F'(x)dt < 0$ implies $\Gamma^0_{\mathcal{G}_2}$ is stable. This is a contradiction since system (1.3) has a stable equilibrium.

In the following we consider $t_c > 0$. Assume that system (1.3) has two large limit cycles $\Gamma^1_{\mathcal{G}_2}$ and $\Gamma^2_{\mathcal{G}_2}$, which is illustrated in Figure 4, where $A_{i+2}, B_{i+2}, C_{i+2}, D_{i+2} \in \Gamma^i_{\mathcal{G}_2}, i = 1, 2$.

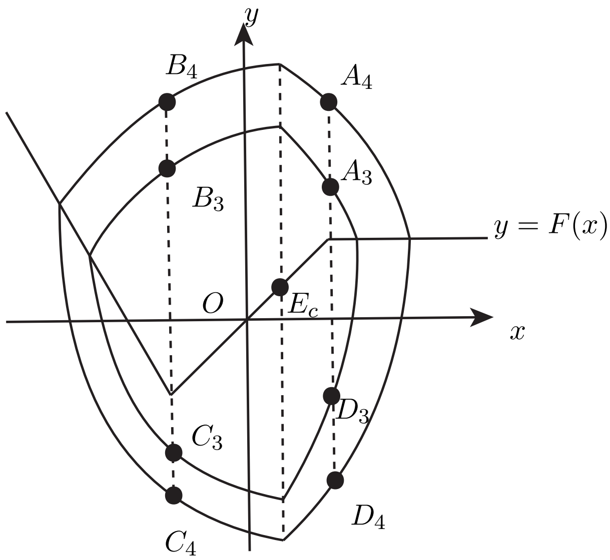


Figure 4. $\Gamma^1_{\mathcal{G}_2}$ and $\Gamma^2_{\mathcal{G}_2}$.

Denote $y = y_{i+2}(x)$ by the orbit segment $\widetilde{A_{i+2}B_{i+2}}$ for $i = 1, 2$, then $y_3(x) < y_4(x)$. Therefore,

$$\int_{\widetilde{A_4B_4}} F'(x)dt - \int_{\widetilde{A_3B_3}} F'(x)dt = \int_{x_{A_4}}^{x_{B_4}} \frac{t_c}{F(x) - y_4(x)}dx - \int_{x_{A_3}}^{x_{B_3}} \frac{t_c}{F(x) - y_3(x)}dx = \int_{x_{A_3}}^{x_{B_3}} \frac{t_c\,(y_4(x) - y_3(x))}{(F(x) - y_4(x))\,(F(x) - y_3(x))}dx < 0,$$

i.e,

$$\int_{\widetilde{A_4B_4}} F'(x)dt < \int_{\widetilde{A_3B_3}} F'(x)dt. \tag{2.20}$$

Similarly, we have

$$\int_{\widetilde{C_4D_4}} F'(x)dt < \int_{\widetilde{C_3D_3}} F'(x)dt. \tag{2.21}$$

Similar to the proof of Lemma 2.3, we can prove

$$\int_{\widetilde{B_4C_4}} F'(x)dt < \int_{\widetilde{B_3C_3}} F'(x)dt, \tag{2.22}$$

and

$$\int_{\widetilde{D_4A_4}} F'(x)dt = \int_{\widetilde{D_3A_3}} F'(x)dt = 0. \tag{2.23}$$

From (2.20)-(2.23), we obtain

$$\int_{\Gamma^2_{\mathcal{G}_2}} F'(x)dt < \int_{\Gamma^1_{\mathcal{G}_2}} F'(x)dt.$$

For $t_c > 0$, the central subsystem of system (1.3) has an unstable equilibrium, thus $\Gamma^1_{\mathcal{G}_2}$ is either semi-stable or stable. If $\Gamma^1_{\mathcal{G}_2}$ is semi-stable, then $\Gamma^1_{\mathcal{G}_2}$ becomes a stable limit cycle $\Gamma^{11}_{\mathcal{G}_2}$ and an unstable limit cycle $\Gamma^{12}_{\mathcal{G}_2}$ as $t_c \to t_c - \epsilon$, where Γ_{11} lies in the interior region surrounded by $\Gamma^{12}_{\mathcal{G}_2}$, which implies $\int_{\Gamma^{12}_{\mathcal{G}_2}} F'(x)dt > \int_{\Gamma^{11}_{\mathcal{G}_2}} F'(x)dt$. This is impossible. Similarly, if $\Gamma^1_{\mathcal{G}_2}$ is stable, then $\Gamma^2_{\mathcal{G}_2}$ is internal unstable, i.e., $\int_{\Gamma^2_{\mathcal{G}_2}} F'(x)dt \geq 0 > \int_{\Gamma^1_{\mathcal{G}_2}} F'(x)dt$. This is a contradiction. Therefore, system (1.3) admits at most a limit cycle.

Case 2: $t_l \geq 0$. For $t_l > 0$ and $t_c < 0$, similar to Lemma 2.3, we can prove that system (1.3) admits at most a limit cycle. For $t_l = 0$ and $t_c < 0$, assume that system (1.3) has a limit cycle $\Gamma^3_{\mathcal{G}_2}$, then $\int_{\Gamma^3_{\mathcal{G}_2}} F'(x)dt < 0$. However, the central subsystem of system (1.3) has a stable equilibrium, thus $\Gamma^3_{\mathcal{G}_2}$ is internally unstable, which implies $\int_{\Gamma^3_{\mathcal{G}_2}} F'(x)dt \geq 0$. This is a contradiction. Therefore, system (1.3) does not admit limit cycles.

For $t_l \geq 0$ and $t_c > 0$, from Theorem 4 in [33], system (1.3) doesn't admit small limit cycles. If system (1.3) has a large limit cycle $\Gamma^4_{\mathcal{G}_2}$, then $\int_{\Gamma^4_{\mathcal{G}_2}} F'(x)dt > 0$. However, E_c is an unstable equilibrium, thus $\Gamma^4_{\mathcal{G}_2}$ is internal stable, which implies $\int_{\Gamma^4_{\mathcal{G}_2}} F'(x)dt \leq 0$. This is a contradiction. Therefore, system (1.3) does not admit limit cycles. □

Lemma 2.5. *In the parameter region $\mathcal{G}_3$, system* (1.3) *admits at most a limit cycle.*

Proof. In the parameter region $\mathcal{G}_3$, we have $d_l = 0, d_c > 0, t_c \neq 0, t_r < 0, d_r = 0$. We divide the proof into two cases.

Case 1: $t_l \leq 0$. For $t_c < 0$, if system (1.3) has a limit cycle $\Gamma^0_{\mathcal{G}_3}$, then $\int_{\Gamma^0_{\mathcal{G}_3}} F'(x)dt < 0$. System (1.3) has a stable equilibrium E_c for $t_c < 0$, thus $\Gamma^0_{\mathcal{G}_3}$ is semi-unstable or unstable, i.e, $\int_{\Gamma^0_{\mathcal{G}_3}} F'(x)dt \geq 0$. This is a contradiction. Therefore, system (1.3) has no limit cycles.

For $t_c > 0$, suppose that system (1.3) has at least two limit cycles, where $\Gamma^1_{\mathcal{G}_3}$ and $\Gamma^2_{\mathcal{G}_3}$ are the innermost two limit cycles and $\Gamma^1_{\mathcal{G}_3}$ lies in the interior region surrounded by $\Gamma^2_{\mathcal{G}_3}$. Similar to the proof of Lemma 2.3, we have $\int_{\Gamma^1_{\mathcal{G}_3}} F'(x)dt > \int_{\Gamma^2_{\mathcal{G}_3}} F'(x)dt$. The central subsystem of system (1.3) has an unstable equilibrium E_c, thus $\Gamma^1_{\mathcal{G}_3}$ is stable or semi-stable. We can assume $\Gamma^1_{\mathcal{G}_3}$ is stable. Otherwise, we can perturb such the stable limit cycle from semi-stable limit cycle. Thereby we have $\int_{\Gamma^1_{\mathcal{G}_3}} F'(x)dt < 0$. From $\int_{\Gamma^2_{\mathcal{G}_3}} F'(x)dt < 0$, it yields that $\Gamma^2_{\mathcal{G}_3}$ is stable. This is a contradiction. Therefore, system (1.3) has at most a limit cycle.

Case 2: $t_l > 0$. For $t_c > 0$, assume that system (1.3) has a large limit cycle $\Gamma^3_{\mathcal{G}_3}$, as shown in Figure 5, where $A_5, B_5, C_5, D_5, I_5, J_5, H_5, Q_5 \in \Gamma^3_{\mathcal{G}_3}$. Let $w = F(x)$ and $\grave{x}_1(w)(\grave{x}_2(w))$ be the inverse function of $w(x)$ for $x > 1(x \leq 1)$. With Coppel transformation $(x, y) \to (w, y)$, system (2.10) can be transformed into the following system,

$$\frac{dy}{dw} = \frac{\grave{\tau}_i(w)}{w - y}, \quad i = 1, 2, \tag{2.24}$$

where

$$\grave{\tau}_1(w) := \grave{\tau}(\grave{x}_1(w)) = \frac{d_c - \alpha}{t_r}, \quad w < t_c - \frac{t_c\alpha}{d_c},$$

$$\grave{\tau}_2(w) := \grave{\tau}(\grave{x}_1(w)) = \begin{cases} \frac{d_c w}{t_c^2}, & -t_c - \frac{t_c\alpha}{d_c} \leq w \leq t_c - \frac{t_c\alpha}{d_c}, \\ \frac{-d_c - \alpha}{t_l}, & w < -t_c - \frac{t_c\alpha}{d_c}. \end{cases}$$

By computation, we have

$$\grave{\tau}_2(w) - \grave{\tau}_1(w) = \begin{cases} \frac{d_c w}{t_c^2} - \frac{d_c - \alpha}{t_r}, & -t_c - \frac{t_c\alpha}{d_c} \leq w \leq t_c - \frac{t_c\alpha}{d_c}, \\ \frac{-d_c - \alpha}{t_l} - \frac{d_c - \alpha}{t_r}, & w < -t_c - \frac{t_c\alpha}{d_c}. \end{cases}$$

Thus the equation $\grave{\tau}_2(w) - \grave{\tau}_1(w) = 0$ has at most a solution.

Suppose that each orbit segment of $\Gamma^3_{\mathcal{G}_3}$ on the xy plane is transformed into their image arc segments with the same notations on the half wy plane respectively, as showed in Figure 5. Denote $(x_P, y_P)((w_P, y_P))$ by the coordinate of the point P in the $xy(wy)$ plane, where $P \in \{A_5, B_5, C_5, D_5, I_5, J_5, H_5, Q_5\}$. By the sign of $\breve{g}(x)$ in system (2.10), we have $y_{B_5} > y_{A_5} > y_{Q_5}$ and $y_{D_5} < y_{I_5} < y_{J_5}$. Let $h_1(w)$, $h_2(w)$, $p_1(w)$ and $p_2(w)$ represent the orbit segments $\widehat{H_5Q_5A_5}$, $\widehat{A_5B_5C_5}$, $\widehat{I_5J_5H_5}$ and $\widehat{C_5D_5I_5}$, respectively. Similar to the proof of Lemma 2.3, we can prove that the equations $h_1(w) - h_2(w) = 0$ and $p_1(w) - p_2(w) = 0$ have at most a solution, and $y_{H_5} < y_{C_5}$.

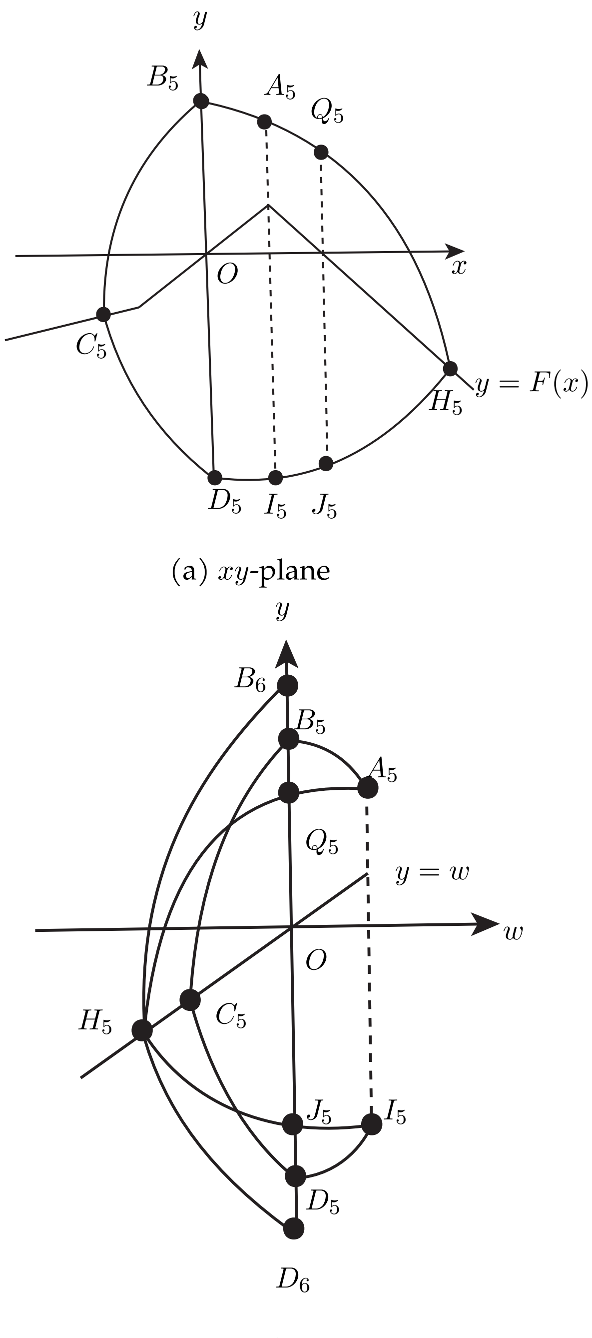


Figure 5. $\Gamma^3_{\mathcal{G}_3}$.

From $h_1(w) < h_2(w)$ for $0 = w_{B_5} \le w \le w_{A_5}$, it follows that

$$\begin{aligned}\int_{\widetilde{Q_5A_5B_5}} F'(x)dt &= \int_{\widetilde{Q_5A_5}} F'(x)dt + \int_{\widetilde{A_5B_5}} F'(x)dt \\ &= \int_{\widetilde{Q_5A_5}} \frac{dw}{w-y} + \int_{\widetilde{A_5B_5}} \frac{dw}{w-y} \\ &= \int_{w_{A_5}}^{w_{B_5}} \frac{dw}{w-h_2(w)} - \int_{w_{A_5}}^{w_{B_5}} \frac{dw}{w-h_1(w)} \\ &= \int_{w_{A_5}}^{w_{B_5}} \frac{h_2(w)-h_1(w)}{(w-h_2(w))(w-h_1(w))} dw < 0.\end{aligned} \tag{2.25}$$

Similarly, we have

$$\int_{\widetilde{D_5I_5J_5}} F'(x)dt < 0. \tag{2.26}$$

With the scaling $(w, y) \to (kw, ky)$, the second equation of system (2.11) is transformed into

$$\frac{dy}{dw} = \frac{\check{\tau}_2(kw)}{k(w-y)}, \tag{2.27}$$

where $k = \frac{w_{H_5}}{w_{C_5}}$. Therefore, $\widetilde{B_5C_5D_5}$ of system (2.11) is changed into $\widetilde{B_6H_5D_6}$ of system (2.27), that is,

$$\int_{\widetilde{B_5C_5D_5}} \frac{1}{w-y} dw = \int_{\widetilde{B_6H_5D_6}} \frac{1}{w-y} dw. \tag{2.28}$$

Denote $h_3(w)$ and $p_3(w)$ by the orbit segment $\widetilde{B_6H_5}$ and $\widetilde{H_5D_6}$ respectively. Then $h_1(w) < h_3(w)$ and $p_3(w) < p_1(w)$ for $w_{H_5} \le w \le w_{Q_5} = 0$. Therefore, we have

$$\begin{aligned}&\int_{\widetilde{B_5Q_5D_5}} F(x)dt + \int_{\widetilde{J_5H_5Q_5}} F'(x)dt \\ =& \int_{\widetilde{B_5C_5D_5}} \frac{dw}{w-y} + \int_{\widetilde{J_5H_5Q_5}} \frac{dw}{w-y} \\ =& \int_{\widetilde{B_6H_5D_6}} \frac{dw}{w-y} + \int_{\widetilde{J_5H_5Q_5}} \frac{dw}{w-y} \\ =& \int_{w_{H_5}}^{w_{Q_5}} \frac{dw}{w-h_1(w)} - \int_{w_{H_5}}^{w_{Q_5}} \frac{dw}{w-h_3(w)} \\ &+ \int_{w_{H_5}}^{w_{Q_5}} \frac{dw}{w-p_3(w)} - \int_{w_{H_5}}^{w_{Q_5}} \frac{dw}{w-p_1(w)} \\ =& \int_{w_{H_5}}^{w_{Q_5}} \frac{h_1(w)-h_3(w)}{(w-h_1(w))(w-h_3(w))} dw \\ &+ \int_{w_{H_5}}^{w_{Q_5}} \frac{p_3(w)-p_1(w)}{(w-p_3(w))(w-p_1(w))} dw < 0,\end{aligned}$$

that is,

$$\int_{\widetilde{B_5Q_5D_5}} F(x)dt + \int_{\widetilde{J_5H_5Q_5}} F'(x)dt < 0. \tag{2.29}$$

From (2.25), (2.26) and (2.29), we have

$$\int_{\Gamma^3_{\mathcal{G}_3}} F'(x)dt < 0. \tag{2.30}$$

For $t_c < 0$, if system (1.3) has a limit cycle $\Gamma^3_{\mathcal{G}_3}$, we similarly prove that the inequality (2.30) holds.

For $t_c > 0$, if system (1.3) has at least two limit cycles, where $\Gamma^4_{\mathcal{G}_3}$ and $\Gamma^5_{\mathcal{G}_3}$ are the innermost two limit cycles, and $\Gamma^4_{\mathcal{G}_3}$ lies in the interior region surrounded by $\Gamma^5_{\mathcal{G}_3}$, then $\Gamma^4_{\mathcal{G}_3}$ is semi-stable or stable. From Theorem 4 in [33], system (1.3) has no small semi-stable limit cycle. If $\Gamma^4_{\mathcal{G}_3}$ is semi-stable, then it is large and $\int_{\Gamma^4_{\mathcal{G}_3}} F'(x)dt = 0$. This is a contradiction. If $\Gamma^4_{\mathcal{G}_3}$ is stable, then $\Gamma^5_{\mathcal{G}_3}$ is large and internal unstable, that is, $\int_{\Gamma^5_{\mathcal{G}_3}} F'(x)dt \geq 0$, which is also a contradiction. Therefore, system (1.3) has at most a limit cycle.

For $t_c < 0$, assume that system (1.3) has a small limit cycle $\Gamma^6_{\mathcal{G}_3}$, then $\Gamma^6_{\mathcal{G}_3}$ is unstable. By Lemma 2.1 and Theorem 4 in [33], the amplitute of $\Gamma^6_{\mathcal{G}_3}$ is increasing as t_c decreases and there exists t_c^* such that $\Gamma^6_{\mathcal{G}_3}$ involve three zones as $t_c = t_c^* - \epsilon$. Therefore, we have $\int_{\Gamma^6_{\mathcal{G}_3}} F'(x)dt > 0$ as $t_c = t_c^* - \epsilon$. This is a contradiction. Hence, system (1.3) has no limit cycles. □

By Lemma 2.2, all orbits are positively bounded as $t_l < 0, t_r < 0$. Note that the central subsystem of system (1.3) has an unstable equilibrium for $t_c > 0$. Therefore, system (1.3) has a limit cycle for $t_l < 0, t_c > 0, t_r < 0$. In Section 6, we give three numerical examples.

3 Proof of Theorem 1.2

When system (1.3) has two denegerate subsystems, if the left or right subsystem of system (1.3) has a node or saddle, then system (1.3) has no limit cycles since the node or saddle has the invariant line. That is to say, system (1.3) for types DDN, DDS and DDN' has no limit cycles. If the central subsystem of system (1.3) is not degenerate for $\frac{\alpha}{d_c} < -1$ or $\frac{\alpha}{d_c} > 1$, system (1.3) has no equilibria, then system (1.3) has no limit cycles. For $\alpha = |d_c|$, system (1.3) has a singular line with infinite length, thus system (1.3) has no limit cycles. Hence we only consider the case $-1 < \frac{\alpha}{d_c} < 1$ in the rest part of this section.

From Chapter 3 in [38], the sum of the indices of all equilibria surrounded by a limit cycle is 1, and the index of a saddle is 1. If system (1.3) of type DSD has a limit cycle, then the index of this limit cycle is -1. This is a contradiction. Therefore, system (1.3) of DSD type has no limit cycles. In the following, we only deal with the type DCD. Set

$$\mathcal{N} := \{(\alpha, d_l, d_c, d_r, t_l, t_c, t_r) \in R^7 : \\ d_l = 0, d_c > 0, d_r = 0, t_c = 0\}.$$

Lemma 3.1. *In the parameter region $\mathcal{N}$, system (1.3) has no limit cycles.*

Proof. For $t_c = 0$, from Theorem 4 in [33], system (1.3) has no small limit cycles. From

$$\begin{vmatrix} -y & g(x) - \alpha + yf(x) \\ \frac{\partial(-y)}{\partial t_c} & \frac{\partial(g(x) - \alpha + yf(x))}{\partial t_c} \end{vmatrix} = \begin{cases} -y^2 \leq 0, & -1 \leq x \leq 1, \\ 0, & other, \end{cases}$$

we have that the vector field $(-y, g(x) - \alpha + yf(x))$ is rotated with respect to t_c. For $t_l t_r < 0$, if system (1.3) has a large limit cycle Γ_1, then it is the hyperbolic large limit cycle. Otherwise, Γ_1 becomes two limit cycles as $t_c = \epsilon$ or $t_c = -\epsilon$. This is a contradiction with the uniqueness of limit cycle of system (1.3) as $t_c \neq 0$.

By Lemma 2.2, the infinite equilibria of system (1.3) do not change when t_c changes. If Γ_1 is stable, then system (1.3) has two limit cycles as $t_c = -\epsilon$ from Poincaré–Bendixson Theorem, which contradicts the uniqueness of limit cycle of system (1.3) as $t_c < 0$. Therefore, system (1.3) has no limit cycles. Similarly, we can prove that system (1.3) has no limit cycles when Γ_1 is unstable.

For $t_l t_r \geq 0$, assume that system (2.10) has a large limit cycle Γ_2. Since system (2.10) is topologically equivalent to system (1.3), it suffices to study the properties of limit cycles of system (2.10) to obtain the corresponding results of system (1.3). If $t_l > 0, t_r \geq 0$ or $t_l = 0, t_r > 0$, then $\check{F}(x)\breve{g}(x) \geq 0$ and $\check{F}(x)\breve{g}(x) \not\equiv 0$, and $0 = \int_{\Gamma_2} dE = \int_{\Gamma_2} \check{F}(x)\breve{g}(x)dt > 0$. This is a contradiction. Therefore, system (2.10) has no limit cycles. Similarly, we can prove that system (2.10) has no limit cycles for $t_l \leq 0, t_r < 0$ or $t_l < 0, t_r = 0$. If $t_l = 0, t_r = 0$ and system (2.10) has a large limit cycle Γ_3, then Γ_3 is semi-stable. Fix $t_l, t_r, d_l, d_c, d_r, \alpha$, then system (2.10) has two limit cycles for $t_c = \epsilon$ or $t_c = -\epsilon$, where ϵ is a sufficiently positive number. This is a contradiction to the fact that system (2.10) has at most a limit cycle as $t_c \neq 0$ by Lemma 2.3. Therefore, system (1.3) has no limit cycles. □

By Lemma 3.1, system (1.3) for DCD type has no limit cycles. This completes the proof of Theorem 1.2.

4 Proof of Theorem 1.3

In this section we consider the case DDF. Recently, we [39] proved system (1.3) has at most two limit cycles in the following parameter region

$$\mathcal{H} := \{(\alpha, d_l, d_c, d_r, t_l, t_c, t_r) \in R^7 : \\ d_l = 0, d_c = 0, t_r > 0, t_r^2 - d_r < 0\}.$$

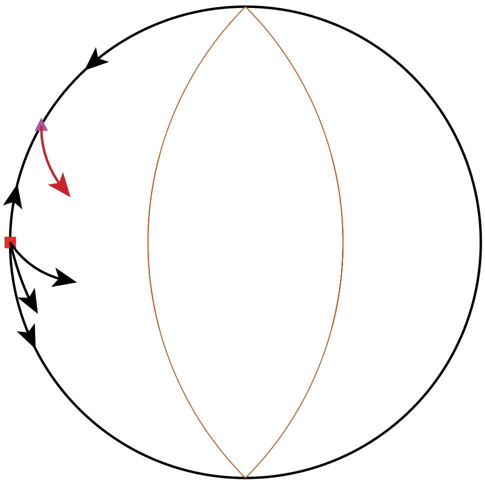
Set

$$\mathcal{Q} := \{(\alpha, d_l, d_c, d_r, t_l, t_c, t_r) \in R^7 : \\ d_l = 0, d_c = 0, t_r < 0, t_r^2 - d_r < 0\}.$$

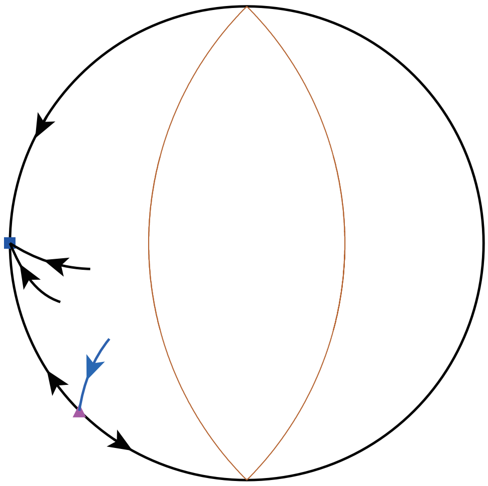
Fixing $\alpha, d_l, d_c, d_r, t_l, t_c$ in the parameter region $\mathcal{Q}$, let $t_r = 0$, system (1.3) is the DDC type, which is discussed in the next section. In the parameter region

$\mathcal{Q}$, system (1.3) has a focus $E_r(1+\frac{\alpha}{d_r}, \frac{\alpha t_r}{d_r}+t_c)$. In what follows, we research the limit cycle of system (1.3) in the parameter region $\mathcal{Q}$. First, we study the infinite equilibria of system (1.3). Similar to Lemma 2.2, we have the following lemma.

Lemma 4.1. *In the parameter region $\mathcal{Q}$, the infinite equilibria for $t_l < 0, \alpha > 0$ in the Poincaré disc of system (1.3) are showed in Figure 6.*



(a) $t_l < 0, \alpha > 0$



(b) $t_l > 0, \alpha > 0$

Figure 6. The infinite equilibria in the Poincaré disc of system (1.3).

In the following, we prove the uniqueness of limit cycles of system (1.3) as $t_l > 0$ in the parameter region $\mathcal{Q}$.

Lemma 4.2. *In the parameter region $\mathcal{Q}$, if $t_l > 0$, then system (1.3) admits at most a limit cycle.*

Proof. Define $w = w(x) = F(x)$. For $t_c > 0$, denote $\tilde{x}_1(w)$ $(\tilde{x}_2(w))$ by the branch of the inverse of $w(x)$ for $x \geq 1$ $(x < 1)$. With the transformation $(x, y) \to (w, y)$, system (1.3) can be transformed as

$$\frac{dy}{dw} = \frac{\tau_i(w)}{w-y}, \quad i = 1, 2, \tag{4.31}$$

where

$$\begin{aligned}\tau_1(w) = \tau(\tilde{x}_1(w)) &:= \frac{g(\tilde{x}_1(w)) - \alpha}{F'(\tilde{x}_1(w))} \\ &= \frac{d_r(w-t_c) - \alpha t_r}{t_r^2}, \quad w < t_c,\end{aligned}$$

and

$$\begin{aligned}\tau_2(w) = \tau(\tilde{x}_2(w)) &:= \frac{g(\tilde{x}_2(w)) - \alpha}{F'(\tilde{x}_2(w))} \\ &= \begin{cases} \frac{-\alpha}{t_l}, & w < -t_c, \\ -\frac{\alpha}{t_c}, & -t_c \leq w \leq t_c. \end{cases}\end{aligned}$$

By computation, we have

$$\tau_1(w) - \tau_2(w) = \begin{cases} \frac{d_r}{t_r^2}(w-t_c) + \alpha\left(\frac{1}{t_l} - \frac{1}{t_r}\right), & w < -t_c, \\ \frac{d_r(w-t_c)}{t_r^2} + \alpha\left(\frac{1}{t_c} - \frac{1}{t_r}\right), & -t_c \leq w \leq t_c. \end{cases}$$

Consider the equation

$$\tau_1(w) - \tau_2(w) = 0, \tag{4.32}$$

and its solutions may be $w_1 = t_c - \frac{t_r^2\alpha}{d_r}\left(\frac{1}{t_l} - \frac{1}{t_r}\right)$ and $w_1 = t_c - \frac{t_r^2\alpha}{d_r}\left(\frac{1}{t_c} - \frac{1}{t_r}\right)$. If $w = w_1$ is the solution of equation (4.32), then $w_1 < -t_c$, i.e, $\alpha > \frac{2t_cd_r}{t_r^2\left(\frac{1}{t_l} - \frac{1}{t_r}\right)} = \vartheta_1$. If $w = w_2$ is the solution of equation (4.32), then $-t_c \leq w_2 \leq t_c$, i.e, $\alpha \leq \frac{2t_cd_r}{t_r^2\left(\frac{1}{t_c} - \frac{1}{t_r}\right)} = \vartheta_2$. If $\vartheta_2 \leq \vartheta_1$, i.e, $t_c \leq t_l$, equation (4.32) has at most a solution, then we can prove that system (1.3) admits at most a limit cycle using similar way to the proof of Lemma 2.3. If $\vartheta_2 > \vartheta_1$, i.e, $t_c > t_l$, then equation (4.32) has two solutions. If system (1.3) admits at least two limit cycles as $(t_l, t_c, t_r, d_r, \alpha) = (t_l^0, t_c^0, t_r^0, d_r^0, \alpha^0)$, by Lemma 4.1 and stability, then we can assume that system (1.3) has at least three limit cycles as $(t_l, t_c, t_r, d_r, \alpha) = (t_l^0, t_c^0, t_r^0, d_r^0, \alpha^0)$, where Υ_1, Υ_2 and Υ_3 are the innermost three limit cycles, and Υ_1, Υ_3 are unstable and Υ_2 is stable. Otherwise, we can perturb such three stable or unstable limit cycles from semi-stable limit cycles. By Lemma 2.1, the amplitude of Υ_1, Υ_3 is decreasing(increasing) and Υ_2 is increasing(decreasing) as one of t_l, t_c increases.

We show that system (1.3) has at most a limit cycle by the induction method. In the first step, fixing $t_c, t_r, d_l, d_c, d_r, \alpha$, we look for $t_l = \mu_1 \in (t_c^0, t_l^0)$ such that Υ_1 and Υ_2 coincide. Otherwise, system (1.3) has at least two limit cycles as $t_l = t_c^0$. This is a contradiction. In the 2th step, fixing $t_l, t_r, d_l, d_c, d_r, \alpha$, we try to find $t_c = \nu_1 \in (t_c^0, \mu_1)$ such that Υ_2 and Υ_3 coincide. If ν_1 does not exist, then system (1.3) has at least two limit cycles as $t_c = \mu_1$, which contradict the uniqueness of the limit cycle of system (1.3) as $t_c = \mu_1$. In the 3th step, repeat the analysis of the 1th step, there exists a unique μ_2 such that Υ_1 and Υ_2 coincide. After finite steps, we obtain two sequences μ_n and ν_n. There exists t^* such that $\lim\limits_{n\to+\infty} \mu_n = \lim\limits_{n\to+\infty} \nu_n = t^*$. By the above analysis, system (1.3) has at least two limit cycles as $t_c = t_l = t^*$. This is a contradiction. Therefore, system (1.3) has at most a limit cycle for $t_c < t_l$.

For $t_c = 0$, denote $\check{x}_1(w)$ ($\check{x}_2(w)$) by the branch of the inverse of $w(x)$ for $x \geq 1$ ($x < -1$). By transformation $(x, y) \to (w, y)$, system (1.3) can be rewritten as

$$\frac{dy}{dw} = \frac{\check{\tau}_i(w)}{w - y}, \quad i = 1, 2, \tag{4.33}$$

where

$$\begin{aligned}\check{\tau}_1(w) = \tau(\check{x}_1(w)) &:= \frac{g(\check{x}_1(w)) - \alpha}{F'(\check{x}_1(w))} \\ &= \frac{d_r w - \alpha t_r}{t_r^2}, \quad w \leq 0,\end{aligned}$$

and

$$\check{\tau}_2(w) = \tau(\check{x}_1(w)) := \frac{g(\check{x}_2(w)) - \alpha}{F'(\check{x}_2(w))} = -\frac{\alpha}{t_l}, w \leq 0.$$

The equation $\check{\tau}_1(w) - \check{\tau}_2(w) = \frac{d_r w}{t_r^2} - \frac{\alpha}{t_r} + \frac{\alpha}{t_l} = 0$ has a solution $w = w_0 = \frac{t_r \alpha}{d_r} - \frac{t_r^2 \alpha}{t_l d_r}$. Similar to the proof of Lemma 2.5, we can prove that system (1.3) admits at most a limit cycle.

For $t_c < 0$, denote $x_1(w)$ ($x_2(w)$) by the branch of the inverse of $w(x)$ for $x \geq -1$ ($x < -1$). With the transformation $(x, y) \to (w, y)$, system (1.3) is transformed as

$$\frac{dy}{dw} = \frac{\tilde{\tau}_i(w)}{w - y}, \quad i = 1, 2, \tag{4.34}$$

where

$$\begin{aligned}\tilde{\tau}_1(w) = \tau(x_1(w)) &:= \frac{g(x_1(w)) - \alpha}{F'(x_1(w))} \\ &= \begin{cases} \frac{d_r}{t_r^2} w - \frac{d_r t_c}{t_r^2} - \frac{\alpha}{t_r}, & w < t_c, \\ -\frac{\alpha}{t_c}, & t_c \leq w \leq -t_c, \end{cases}\end{aligned}$$

and

$$\tilde{\tau}_2(w) = \tau(x_2(w)) := \frac{g(x_2(w)) - \alpha}{F'(x_2(w))} = -\frac{\alpha}{t_l}, \quad w < -t_c.$$

Then

$$\tilde{\tau}_1(w) - \tilde{\tau}_2(w) = \begin{cases} \frac{d_r}{t_r^2} w - \frac{d_r}{t_r^2} t_c - \alpha\left(\frac{1}{t_r} - \frac{1}{t_l}\right), & w < t_c, \\ -\alpha\left(\frac{1}{t_c} - \frac{1}{t_l}\right), & t_c \leq w \leq -t_c. \end{cases}$$

Therefore, the equation $\tilde{\tau}_1(w) - \tilde{\tau}_2(w) = 0$ has at most a solution. Similar to the proof of Lemma 2.5, we can prove that system (1.3) has at most a limit cycle. □

Lemma 4.3. *In the parameter region $\mathcal{Q}$, if $t_l \leq 0$, then system* (1.3) *has at most two limit cycle.*

Proof. For $t_c \leq 0$, system (1.3) has no limit cycles. In fact, if system (1.3) has a limit cycle $\Gamma_{\mathcal{Q}}^0$, then we have $\int_{\Gamma_{\mathcal{Q}}^0} F'(x)dt < 0$, i.e., $\Gamma_{\mathcal{Q}}^0$ is stable, which is contradiction to the stability of focus E_r. For $t_c > 0$, assume system (1.3) has at least two large limit cycles and $\Gamma_{\mathcal{Q}}^1$ and $\Gamma_{\mathcal{Q}}^2$ are the innermost two large limit cycles, where $A_{i+7}, B_{i+7}, C_{i+7}, D_{i+7} \in \Gamma_{\mathcal{Q}}^i$ for $i = 1, 2$, as shown in Figure 7.

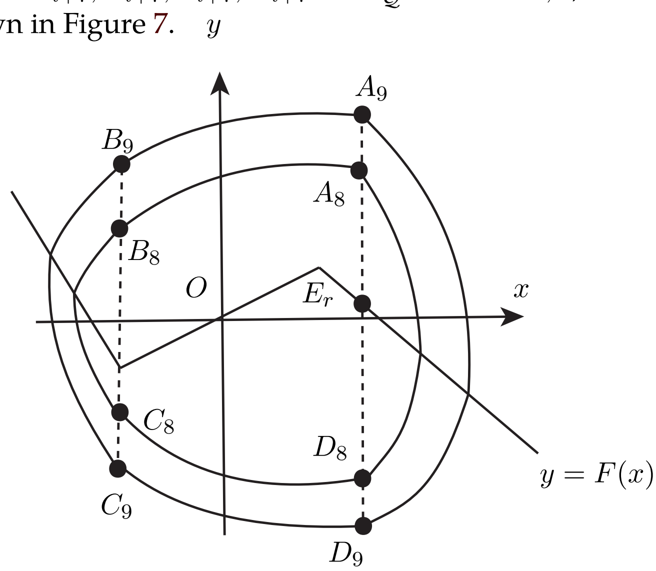


Figure 7. $\Gamma_{\mathcal{Q}}^1$ and $\Gamma_{\mathcal{Q}}^2$.

We call that $F(x_{E_r}) > F(-1)$. Otherwise, we have

$$(F(x) - F(x_{E_r}))(g(x) - \alpha) \leq 0,$$

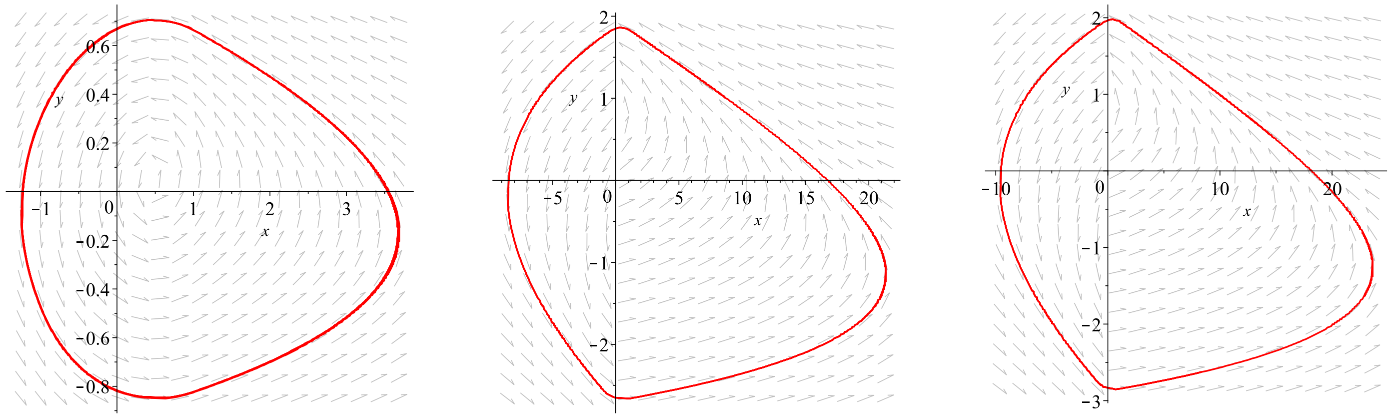
and

$$(F(x) - F(x_{E_r}))(g(x) - \alpha) \not\equiv 0.$$

Then

$$\begin{aligned} 0 &= \int_{\Gamma_{\mathcal{Q}}^i} dE_1 \\ &= \int_{\Gamma_{\mathcal{Q}}^i} (F(x) - F(x_{E_r}))(g(x) - \alpha)dt < 0 \quad (i = 1, 2), \end{aligned}$$

where $E_1(x, y) = \int_0^x g(s)ds + \frac{y^2}{2}$. This is a contradiction.



(a) $t_c = 0.1$ (b) $t_c = 0.8^{\frac{1}{2}}$ (c) $t_c = 1$

Figure 8. The limit cycle of system (1.3) for $(\alpha, d_l, d_c, d_r, t_l, t_r) = (0.1, 0, 0.2, 0, -0.1, -0.1)$.

By $t = -\sigma$, system (1.3) can be transformed into the form

$$\left\{\begin{array}{l} \frac{dx}{d\sigma} = y - F(x) \\ \frac{dy}{d\sigma} = \alpha - g(x) \end{array}\right. \tag{4.35}$$

Denote $y = y_8(x)$ and $y_9(x)$ by the orbit segments $\widetilde{A_8B_8}, \widetilde{A_9B_9}$ respectively, then $y_8(x) < y_9(x)$ for $x_{A_8} \leq x \leq x_{A_9}$. It follows that

$$\begin{aligned}
\int_{\widetilde{B_iA_i}} -F'(x)d\sigma &= \int_{x_{B_i}}^{x_{A_i}} \frac{F'(x)}{F(x) - y_i}dx \\
&= \int_{x_{B_i}}^{x_{A_i}} \frac{d\left(F(x) - y_i\right)}{F(x) - y_i(x)} \\
&\quad + \int_{x_{B_i}}^{x_{A_i}} \frac{y_i'(x)}{F(x) - y_i(x)}dx \\
&= \ln \frac{F\left(x_{A_i}\right) - y_i\left(x_{A_i}\right)}{F\left(x_{B_i}\right) - y_i\left(x_{B_i}\right)} \\
&\quad + \int_{x_{B_i}}^{x_{A_i}} \frac{g(x) - \alpha}{\left(F(x) - y_i(x)\right)^2}dx \\
&= \ln \frac{F\left(x_{A_i}\right) - y_i\left(x_{A_i}\right)}{F\left(x_{B_i}\right) - y_i\left(x_{A_i}\right)} \\
&\quad + \ln \frac{F\left(x_{B_i}\right) - y_i\left(x_{A_i}\right)}{F\left(x_{B_i}\right) - y_i\left(x_{B_i}\right)} \\
&\quad + \int_{x_{B_i}}^{x_{A_i}} \frac{g(x) - \alpha}{\left(F(x) - y_i(x)\right)^2}dx \\
&= \ln \frac{F\left(x_{A_i}\right) - y_i\left(x_{A_i}\right)}{F\left(x_{B_i}\right) - y_i\left(x_{A_i}\right)} \\
&\quad + \int_{x_{B_i}}^{x_{A_i}} \frac{y_i'(x)}{y_i(x) - F\left(x_{B_i}\right)}dx \\
&\quad + \int_{x_{B_i}}^{x_{A_i}} \frac{g(x) - \alpha}{\left(F(x) - y_i(x)\right)^2}dx \\
&= \ln \frac{F\left(x_{A_i}\right) - y_i\left(x_{A_i}\right)}{F\left(x_{B_i}\right) - y_i\left(x_{A_i}\right)} \\
&\quad + \int_{x_{B_i}}^{x_{A_i}} \frac{(g(x) - \alpha)(F(x) - F(x_{B_i}))}{(y_i(x) - F(x_{B_i})(F(x) - y_i(x))^2}dx,
\end{aligned}$$

where $i = 8, 9$.

From

$$\ln \frac{F\left(x_{A_8}\right) - y_8\left(x_{A_8}\right)}{F\left(x_{B_8}\right) - y_8\left(x_{A_8}\right)} < \ln \frac{F\left(x_{A_9}\right) - y_9\left(x_{A_9}\right)}{F\left(x_{B_9}\right) - y_9\left(x_{B_9}\right)},$$

and

$$\frac{1}{\left(y_8(x) - F\left(x_{A_8}\right)\right)\left(F(x) - y_8(x)\right)^2} > \frac{1}{\left(y_9(x) - F\left(x_{A_8}\right)\right)\left(F(x) - y_9(x)\right)^2}.$$

it yields that

$$\int_{\widetilde{B_8A_8}} -F'(x)d\sigma < \int_{\widetilde{B_9A_9}} -F'(x)d\sigma. \tag{4.36}$$

Similarly, we have

$$\int_{\widetilde{D_8C_8}} -F'(x)d\sigma < \int_{\widetilde{D_9C_9}} -F'(x)d\sigma. \tag{4.37}$$

Similar to the proof of Lemma 2.3, we have

$$\int_{\widetilde{C_8B_8}} -F'(x)d\sigma \leq \int_{\widetilde{C_9B_9}} -F'(x)d\sigma. \tag{4.38}$$

Moreover,

$$\int_{\widetilde{D_8A_8}} -F'(x)d\sigma = \int_{\widetilde{D_9A_9}} -F'(x)d\sigma = -\frac{t_r\pi}{\sqrt{t_r^2 - 4d_r}}. \tag{4.39}$$

From (4.36)-(4.39), for system (1.3), it follows that

$$\int_{\Gamma_Q^1} F'(x)dt > \int_{\Gamma_Q^2} F'(x)dt.$$

If system (1.3) has at least three limit cycles and Γ_Q^1, Γ_Q^2 and Γ_Q^3 are the innermost three limit cycles. From [9], system (1.3) has at most a small limit cycle. Thus, Γ_Q^2 and Γ_Q^3 are large limit cycles. We can assume that Γ_Q^1 is unstable, and $\Gamma_Q^2(\Gamma_Q^3)$ is stable(unstable). Otherwise, we can perturb such three stable or unstable limit cycles from the semi–stable limit cycle. Then we have $\int_{\Gamma_Q^2} F'(x)dt < \int_{\Gamma_Q^3} F'(x)dt$, which contradicts $\int_{\Gamma_Q^2} F'(x)dt > \int_{\Gamma_Q^3} F'(x)dt$. Therefore, system (1.3) has at most two limit cycles. □

In the following, we explain that there exists a parameter condition such that system (1.3) has two limit cycles. By Lemma 4.1, all orbits of system (1.3) are positively bounded for $t_l < 0, \alpha > 0$. From Theorem 4 in [33], there exist $t_c^{**} > 0$ and $\alpha^{**} > 0$ such that system (1.3) has an unstable small limit cycle for $t_l < 0, t_c = t_c^{**}, \alpha = \alpha^{**}$ in the parameter region $\mathcal{Q}_1$. Therefore, system (1.3) has two limit cycles by Poincaré–Bendixson Theorem. In Section 6, we provide a numerical example.

5 Proof of Theorem 1.4

In this section we deal with the case DDC. Set

$$\mathcal{M} := \{(\alpha, d_l, d_c, d_r, t_l, t_c, t_r) \in R^7 : \\ d_l = 0, d_c = 0, d_r > 0, t_r = 0\}.$$

Lemma 5.1. *In the parameter region $\mathcal{M}$, system (1.3) has at most a limit cycle.*

Proof. From Theorem 4 in [33], system (1.3) has no small limit cycles.

For $t_l t_c \geq 0$ and $t_l + t_c \neq 0$, assume that system (1.3) has a large limit cycle $\Gamma^0_{\mathcal{M}}$, then $\int_{\Gamma^0_{\mathcal{M}}} F'(x)dt \neq 0$, which implies that system (1.3) has at most a large limit cycle. Therefore, system (1.3) has at most a limit cycle.

For $t_l > 0, t_c < 0$, if system (1.3) has at least two large limit cycles, where $\Gamma^1_{\mathcal{M}}$ and $\Gamma^2_{\mathcal{M}}$ are the innermost two limit cycles, similar to the proof of Lemma 2.4, we have

$$\int_{\Gamma^1_{\mathcal{M}}} F'(x)dt < \int_{\Gamma^2_{\mathcal{M}}} F'(x)dt. \tag{5.40}$$

Thus, $\Gamma^1_{\mathcal{M}}$ is stable and $\Gamma^2_{\mathcal{M}}$ is unstable. Inequality (5.40) implies that system (1.3) has only two large limit cycles. By Lemma 2.1, system (1.3) has at least two limit cycles as $t_r = -\epsilon$, which contradicts the uniqueness of limit cycle of system (1.3) from the proof of Lemma 4.2. Therefore, system (1.3) has at most a limit cycle. Similarly, we can prove that system (1.3) has at most a limit cycle for $t_l < 0, t_c > 0$. □

For $t_l < 0, \alpha > 0$, the qualitative properties of infinite equilibria of system (1.3) in the parameter region $\mathcal{Q}$ is the same as the qualitative properties of infinite equilibria of system (1.3) in the parameter region $\mathcal{M}$. There exists $\alpha = \alpha^* > 0, t_l = t_l^* < 0, t_c = t_* > 0, d_r = d_r^* > 0, t_r = -\epsilon$ such that system (1.3) has two limit cycles in the parameter region $\mathcal{Q}$. Therefore, system (1.3) has a limit cycle for $\alpha = \alpha^* > 0, t_l = t_l^* < 0, t_c = t_* > 0, d_r = d_r^* > 0$ in the parameter region $\mathcal{M}$. In Section 6, we provide a numerical example.

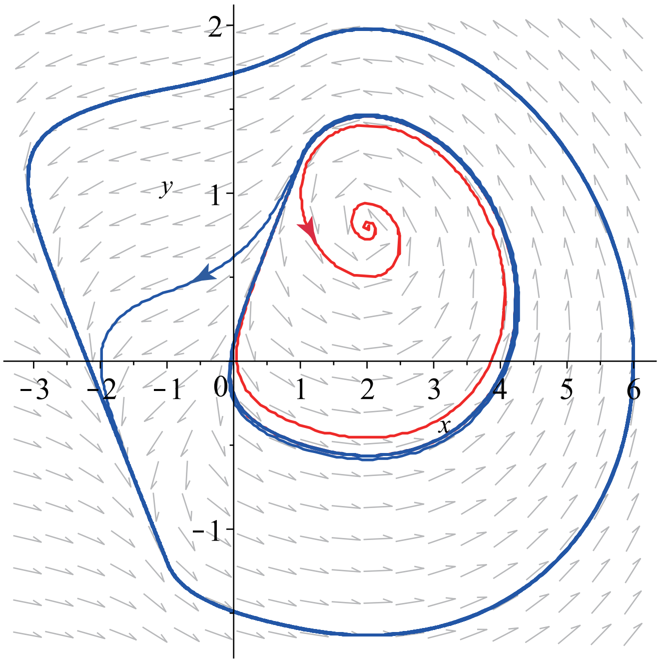


Figure 9. The limit cycle of system (1.3) of DDF type for $(\alpha, d_l, d_c, d_r, t_l, t_c, t_r) = (0.2, 0, 0, 0.2, -1, 1, -0.2)$.

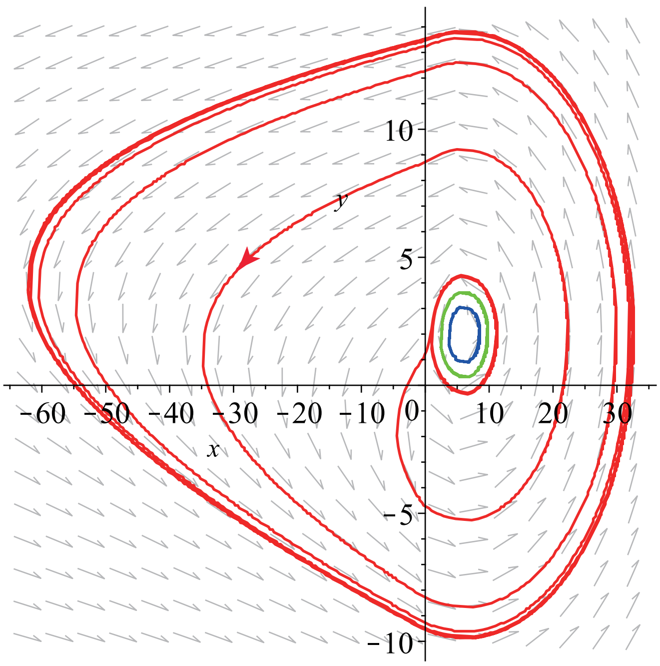


Figure 10. The limit cycle of system (1.3) of DDC type for $(\alpha, d_l, d_c, d_r, t_l, t_c, t_r) = (1, 0, 0, 0.2, -0.1, 2, 0)$.

6 Numerical examples

Fix $\alpha = 0.1, d_l = 0, d_c = 0.2, d_r = 0, t_l = -0.1, t_r = -0.1$, and let $t_c = 0.1$(resp. $t_c = 0.8^{\frac{1}{2}}$, $t_c = 1$), then system (1.3) is type DFD (resp. type $DN'D$, type DND) and has a limit cycle, as shown in Figure 8(a)(resp. Figure 8(b), Figure 8(c)). The left and right subsystems is degenerate, and the central subsystem has an unstable focus $E_c(\frac{1}{2}, \frac{1}{20})$(resp. an unstable node

$E_c(\frac{1}{2}, \sqrt{0.2})$ with equal eigenvalues, an unstable node $E_c(\frac{1}{2}, \frac{1}{2})$ with different eigenvalues).

Taking $\alpha = 0.2, d_l = 0, d_c = 0, d_r = 0.2, t_l = -1, t_c = 1, t_r = -0.2$, system (1.3) is type DDF. The phase portrait of system (1.3) is shown in Figure 9. System (1.3) has two limit cycles, and the left and central subsystem is degenerate, and the right subsystem has a stable focus $E_r(2, 0.6)$.

Set $\alpha = 1, d_l = 0, d_c = 0, d_r = 0.2, t_l = -0.1, t_c = 2, t_r = 0$, system (1.3) is DDC type. The phase portrait of system (1.3) is shown in Figure 10. System (1.3) has a limit cycle, and the left and central subsystem is degenerate, and the right subsystem has a center $E_r(2, 0.6)$.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

Ethical Approval and Consent to Participate

Not applicable.

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