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RESEARCH ARTICLE

Analysis of a Pest-Natural Enemy Model with Time Delay in Impulsive Releasing Natural Enemy

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Abstract

Releasing amount of natural enemy generally depends on its population data, while impulsive releasing natural enemy usually brings about a time delay after the data is observed in the practical pest management. Therefore, it is very important for pest managers to assess the impact of the time delay in pest management. In this paper, we construct a pest-natural enemy model with time delay in impulsive releasing natural enemy. We prove that the pest-free periodic solution of model (2.1) is globally attractive with $\eta\tau < \frac{\zeta y^*(1-e^{-\lambda\tau})}{\lambda}$. We also prove that model (2.1) is permanent with $\eta\tau > \frac{\zeta y^*(1-e^{-\lambda\tau})}{\lambda}$. Further influence of the time delay in impulsive releasing on dynamical behaviors of model (2.1) is investigated by numerical simulations. Our results provide reliable tactics for pest management.

Keywords: pest-natural enemy model, time delay, impulsive releasing, pest management, pest-free.

1 Introduction

Biological control refers to the method of using one organism against another. Biological control can be roughly divided into three categories: insect control, bird control and bacteria control. It is a way to reduce the population density of pests such as weeds and pests. It utilizes the interrelationship of biological species with one or one class of organisms. Its biggest advantage is that it does not pollute the environment, which wins out over pesticides and other non-biological pest control methods [1]. Lacey [2] demonstrated practical use of entomopathogenic microorganisms for pest control, including tables describing which pathogens are available commercially. Amarathunga et al. [3] proposed a comprehensive predator-prey population dynamic simulation model of the insect pest Western Flower Thrips (WFT) and its predator Orius in strawberries. Barclay [4] indicated that the release of sterile hosts alone was more efficient than release of parasitoids alone in controlling the hosts if population regulation was in the parasitoids. Liu et al. [5] investigated the complexity of a predator-prey dynamical model with impulsive releasing predator. Furthermore, time delays are usually introduced into modeling predator-prey systems, for example, Jiao et al. [6] investigated a delayed predator–prey model with impulsive perturbations on predator. Jiao et al. [7]

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also investigated a delayed predator–prey model with prey impulsively diffusing between two patches. Wu et al. [8] conducted two impulsive perturbations in a stage-structured ecological aquaculture management model. They just discussed the time delays in the differential equations of their models. While few researchers devoted to assess the impact of the time delay in impulsive items. For instance, Pei et al. [9] proposed impulsive selective harvesting in a logistic fishery model with time delay. Lawson et al. [10] presented a logistic population model with pulse delayed harvesting. But their studies limited to single population dynamical models.

2 Model formulation

With regards to time delay in impulsive releasing natural enemy on pest management, we construct a pest-natural enemy model with time delay in impulsive releasing natural enemy.

$$\begin{cases} \left.\begin{aligned} \frac{dx(t)}{dt} &= x(t)(\eta-\omega x(t)) - \frac{\zeta x(t)y(t)}{1+\gamma x(t)}, \\ \frac{dy(t)}{dt} &= \frac{c\zeta x(t)y(t)}{1+\gamma x(t)} - \lambda y(t), \end{aligned}\right\} t \in (n\tau,(n+1)\tau] \\ \left.\begin{aligned} \triangle x(t) &= 0, \\ \triangle y(t) &= \frac{\mu_{\max}}{1+\vartheta y(t-l\tau)}, \end{aligned}\right\} t=(n+1)\tau, 0<l<1, \end{cases} \tag{2.1}$$

where $x(t)$ is the number of the pest at time t. $y(t)$ is the number of natural enemy at time t. $\eta > 0$ is the growth coefficient of pest on $(n\tau,(n+1)\tau]$. $\omega > 0$ is the intraspecific competition coefficient of pest on $(n\tau,(n+1)\tau]$. $\frac{\zeta x(t)}{1+\gamma x(t)}$ is the Holling II type functional response on $(n\tau,(n+1)\tau]$. $c > 0$ is the conversion rate from pest into natural enemy on $(n\tau,(n+1)\tau]$. $\lambda > 0$ is the death coefficient of the natural enemy on $(n\tau,(n+1)\tau]$. Releasing amount of natural enemy generally depends on its population data, while impulsive releasing natural enemy usually brings about a time delay after the data is observed in the practical pest management. Therefore, It is very important for pest managers to assess the impact of the time delay in pest management. Therefore, the pest managers make decisions to releasing natural enemy at time $t=(n+1)\tau$, they firstly require to observe the number of the natural enemy at time $t=(n+1-l)\tau(0<l<1)$. Then, impulsive releasing amount of the natural enemy at time $t=(n+1)\tau$ is decided by $\frac{\mu_{max}}{1+\vartheta y(t-l\tau)}$. That is to say, after time interval $((n+1-l)\tau,(n+1)\tau]$, the successful releasing amount of natural enemy is $\frac{\mu_{max}}{1+\vartheta y(t-l\tau)}$, which depends on the observed number of natural enemies at time $t=(n+1-l)\tau$. $\mu_{max} \geq 0$ is the maximum releasing ability of the natural enemy at time $t=(n+1)\tau$, and $\vartheta > 0$ is the shape parameter. Obviously, the releasing amount of natural enemy is zero when the observed amounts of the natural enemy tend to the infinite, and the releasing amount of natural enemy is μ_{max} when observed amount of natural enemy tends to eradication. It is easy to see that this pattern of releasing natural enemy is more effective than constant releasing natural enemy for pest management. τ is the period of the impulsive releasing natural enemy.

Let $\phi(t)=(\phi_1(t),\phi_2(t))$ be a continuous function on $[-l\tau,0]$, we assume that all solutions of model (2.1) satisfy the initial conditions as follows

$$(x(t),y(t)) = (\phi_1(t),\phi_2(t)) > 0, \quad t\in[-l\tau,0],$$

$$(x(0^+),y(0^+)) = (x_0^+,y_0^+).$$

3 Main results

In this paper, we make notation as

$$R_0 = \frac{\lambda\eta\tau}{\zeta y^*(1-e^{-\lambda\tau})}.$$

Lemma 3.1. There exists a constant $K>0$ such that $x(t)\leq K, y(t)\leq K$ for each solution $(x(t),y(t))$ of model (2.1) with all t large enough.

Proof. Defining $V(t)=cx(t)+y(t)$, and when $t\in(n\tau,(n+1)\tau]$, we have

$$D^+V(t)+\lambda V(t) = -c\omega\left(x(t)-\frac{\lambda+\eta}{2\omega}\right)^2+\frac{c(\lambda+\eta)^2}{4\omega} \leq \frac{c(\lambda+\eta)^2}{4\omega}.$$

When $t=(n+1)\tau, V((n+1)\tau^+) = cx((n+1)\tau^+)+y((n+1)\tau^+) = cx((n+1)\tau)+y((n+1)\tau)+\frac{\mu_{\max}}{1+\gamma y((n+1-l)\tau}\leq V((n+1)\tau)+\mu_{\max}$.

By lemma 2.2 in [11] and for $t\in(n\tau,(n+1)\tau]$, we get

$$\begin{aligned} V(t) &\leq V(0)\exp(-\lambda t)+\frac{\frac{c(\lambda+\eta)^2}{4\omega}}{\lambda}(1-\exp(-\lambda t)) \\ &\quad + \frac{\mu_{\max}\exp(-\lambda(t-\tau))}{1-\exp(\lambda\tau)}+\frac{\mu_{\max}\exp(\lambda\tau)}{\exp(\lambda\tau)-1} \\ &\to \frac{c(\lambda+\eta)^2}{4\omega\lambda}+\mu_{\max}, \quad \text{as} \quad t\to\infty. \end{aligned}$$

So $V(t)$ is uniformly ultimately bounded. Hence, there exists a constant $K>0$ such that $x(t)\leq K, y(t)\leq K$ for t large enough. This completes the proof.

If $x(t) = 0$, we obtain the subsystem of model (2.1)

$$\begin{cases} \frac{dy(t)}{dt} = -\lambda y(t), t \in (n\tau, (n+1)\tau], \\ y(t^+) = y(t) + \frac{\mu_{\max}}{1+\vartheta y(t-l\tau)}, t = (n+1)\tau. \end{cases} \tag{3.2}$$

For $t \in (n\tau, (n+1)\tau]$, we can get the analysis solution of system (3.1) between impulsive moments

$$y(t) = y(n\tau^+)e^{-\lambda(t-n\tau)}. \tag{3.3}$$

So

$$y(t-l\tau) = y(n\tau^+)e^{-\lambda(t-l\tau-n\tau)}. \tag{3.4}$$

According to the impulsive releasing natural enemy at $t = (n+1)\tau$, we can easily derive the difference equation as

$$\left.\begin{aligned} y((n+1)\tau^+)) &= y((n+1)\tau) + \tfrac{\mu_{\max}}{1+\vartheta y((n+1-l)\tau)} \\ &= e^{-\lambda\tau}y(n\tau^+) + \tfrac{\mu_{\max}}{1+\vartheta e^{-\lambda(1-l)\tau}y(n\tau^+)}. \end{aligned}\right\} \tag{3.5}$$

Furthermore, the positive fixed point of difference equation (3.4) is computed as

$$y^* = \frac{-(1-e^{-\lambda\tau}) + \sqrt{(1-e^{-\lambda\tau})^2 + 4\mu_{\max}\vartheta e^{-\lambda(1-l)\tau}(1-e^{-\lambda\tau})}}{2\vartheta e^{-\lambda(1-l)\tau}(1-e^{-\lambda\tau})}. \tag{3.6}$$

Lemma 3.2. The positive fixed point y^* of difference equation (3.4) is globally asymptotically stable.

Proof. Making notation as

$$F(y(n\tau^+)) = e^{-\lambda\tau}y(n\tau^+) + \frac{\mu_{\max}}{1+\gamma e^{-\lambda(1-l)\tau}y(n\tau^+)},$$

we can easily obtain that

$$\begin{aligned} &\frac{\partial F(y(n\tau^+))}{\partial y(n\tau^+)} |_{y(n\tau^+)=y^*} \\ &= e^{-\lambda\tau} - \frac{\mu_{\max}\vartheta e^{-\lambda(1-l)\tau}}{(1+\vartheta e^{-\lambda(1-l)\tau}y^*)^2} < e^{-\lambda\tau} < 1. \end{aligned}$$

While

$$\begin{aligned} &\frac{\partial F(y(n\tau^+))}{\partial y(n\tau^+)} |_{y(n\tau^+)=y^*} \\ &= e^{-\lambda\tau} - \frac{\mu_{\max}\vartheta e^{-\lambda(1-l)\tau}}{(1+\vartheta e^{-\lambda(1-l)\tau}y^*)^2} \\ &> -1 + 2e^{-\lambda\tau} > -1. \end{aligned}$$

Obviously,

$$|\frac{\partial F(y(n\tau^+))}{\partial y(n\tau^+)} |_{y(n\tau^+)=y^*} | < 1,$$

Then, the positive fixed point y^* of difference equation (3.4) is locally stable. For the uniqueness of the positive fixed point y^* of difference equation (3.4), the positive fixed point y^* of difference equation (3.4) is globally asymptotically stable.

Similar to reference [7], we have the following lemma.

Lemma 3.3. System (3.1) has a positive periodic solution $\widehat{y(t)}$, which is globally asymptotically stable with

$$\widehat{y(t)} = y^* e^{-\lambda(t-n\tau)}, t \in (n\tau, (n+1)\tau], \tag{3.7}$$

and y^* is defined as (3.5).

Considering following auxiliary differential equation

$$\begin{cases} \frac{dy(t)}{dt} > ay(t), t \in (n\tau, (n+1)\tau], \\ y(t^+) = y(t) + \frac{\mu_{\max}}{1+\vartheta y(t-l\tau)}, t = (n+1)\tau. \\ y(0^+) = y_0^+, \end{cases} \tag{3.8}$$

and its comparative differential equation

$$\begin{cases} \frac{de(t)}{dt} = ae(t), t \in (n\tau, (n+1)\tau], \\ e(t^+) = e(t) + \frac{\mu_{\max}}{1+\vartheta e(t-l\tau)}, t = (n+1)\tau. \\ e(0^+) = y_0^+. \end{cases} \tag{3.9}$$

Similar to (3.4), we get

$$\left.\begin{aligned} e((n+1)\tau^+)) &= e((n+1)\tau) + \tfrac{\mu_{\max}}{1+\gamma e((n+1-l)\tau)} \\ &= e^{a\tau}e(n\tau^+) + \tfrac{\mu_{\max}}{1+\gamma e^{a(1-l)\tau}e(n\tau^+)} \end{aligned}\right\} \tag{3.10}$$

We may make notation as

$$\begin{aligned} e((n+1)\tau^+)) &\triangleq F(a) \\ &= e^{a\tau}e(n\tau^+) + \frac{\mu_{\max}}{1+\gamma e^{a(1-l)\tau}e(n\tau^+)}. \end{aligned} \tag{3.11}$$

Then,

$$\begin{aligned} \frac{\partial F(a)}{\partial a} &= \tau e^{a\tau}e(n\tau^+) \\ &- \frac{\vartheta\mu_{\max}(1-l)e^{-al\tau}}{(1+\gamma e^{a(1-l)\tau}e(n\tau^+)\vartheta\mu_{\max}(1-l)e^{-al\tau})^2} \\ &> e^{a\tau}(1-\vartheta\mu_{\max}(1-l)e^{-al\tau}). \end{aligned} \tag{3.12}$$

Therefore, we can easily obtain the following lemma.

Lemma 3.4. If $\vartheta\mu_{max}(1-l)e^{-al\tau} < 1$ and $y(0^+) > e(0^+)$, then $y(n\tau)^+) > e(n\tau)^+)$.

Proof. From (3.11) and $\vartheta\mu_{max}(1-l)e^{-al\tau} < 1$, we gain that $e((n+1)\tau)^+)$ is increasing function of a. For $y(0^+) > e(0^+)$, then we can easily have $y(n\tau)^+) > e(n\tau)^+)$.

Lemma 3.5. If $\vartheta\mu_{max}(1-l)e^{-al\tau} < 1$, then $y(t) \geq e(t), t \in (n\tau, (n+1)\tau]$, where $e(t)$ is the solution of (3.8).

Proof. From lemma 3.4., and the solutions of (3.7) (3.8) are respectively continuous and differentiable on $(n\tau, (n+1)\tau]$, this completes the proof.

Theorem 3.6. If $\vartheta\mu_{max}(1-l)e^{\lambda l\tau} < 1$, and

$$R_0 < 1, \tag{3.13}$$

holds, the pest-free periodic solution $(0, \widehat{y(t)})$ of model (2.1) is globally attractive.

Proof. The pest-free periodic solution $(0, \widehat{y(t)})$ of model (2.1) will be proved to be globally attractive in the next step. For any $\varepsilon > 0$ small enough, we have

$$\omega = e^{\int_0^\tau \eta - \zeta(\widehat{y(t)} - \varepsilon)dt} < 1.$$

From model (2.1), we know that $\frac{dy(t)}{dt} \geq -\lambda y(t)$. Then,

$$\begin{cases} \dfrac{dy_2(t)}{dt} = -\lambda y_2(t), t \in (n\tau, (n+1)\tau], \\ y_2(t^+) = y_2(t) + \dfrac{\mu_{\max}}{1 + \vartheta y_2(t - l\tau)}, t = (n+1)\tau, \\ y_2(0^+) = y(0^+). \end{cases} \tag{3.14}$$

From lemma 3.3, we can easily obtain that $\widehat{y(t)}$ is the globally asymptotically stable periodic solution of system (3.13). Referencing [5], we may get $y(t) \geq y_2(t)$ and $y_2(t) \to \widehat{y_2(t)} = \widehat{y(t)}$ as $t \to \infty$. Then

$$y(t) \geq y_2(t) \geq \widehat{y_2(t)} - \varepsilon = \widehat{y(t)} - \varepsilon, \tag{3.15}$$

for all t large enough. Substituting (3.14) into model (2.1), we have

$$\begin{cases} \dfrac{dx(t)}{dt} \leq x(t)[\eta - \zeta(\widehat{y(t)} - \varepsilon)], t \in (n\tau, (n+1)\tau], \\ x(t^+) = x(t), t = (n+1)\tau. \end{cases} \tag{3.16}$$

Integrating (3.15) on $(n\tau, (n+1)\tau]$, we can also have $x((n+1)\tau) \leq x(n\tau^+)e^{\int_{n\tau}^{(n+1)\tau}(\eta - \zeta(\widehat{y(t)} - \varepsilon))ds}$. Therefore, $x(n\tau) \leq x(0^+)\omega^n$ and $x(n\tau) \to 0$ as $n \to \infty$. Thereby $x(t) \to 0$ as $t \to \infty$.

For $0 < \varepsilon \leq \frac{\lambda}{c\zeta - \lambda\gamma}$ small enough, there must exist a $t_0 > 0$ such that $0 < x(t) < \varepsilon$ for all $t \geq t_0$. From model (2.1) and the increasing nature of the function $\frac{\zeta x}{1+\gamma x}$, we derive

$$-\lambda y(t) \leq \frac{dy(t)}{dt} \leq -(\lambda - \frac{c\zeta\varepsilon}{1+\gamma\varepsilon})y(t). \tag{3.17}$$

From lemma 3.5., it is well to know that $y_2(t) \leq y(t) \leq y_3(t)$ and $y_2(t) \to \widehat{y(t)}, y_3(t) \to \widehat{y_3(t)}$ as $t \to \infty$, where $y_2(t)$ and $y_3(t)$ are respectively the solutions of system (3.8) and

$$\begin{cases} \dfrac{dy_3(t)}{dt} = -(\lambda - \dfrac{c\zeta\varepsilon}{1+\gamma\varepsilon})y_3(t), t \in (n\tau, (n+1)\tau], \\ y_3(t^+) = y_3(t) + \dfrac{\mu_{\max}}{1 + \vartheta y_3(t - l\tau)}, t = (n+1)\tau, \\ y_3(0^+) = y(0^+), \end{cases} \tag{3.18}$$

According to lemma 3.3., we are aware of the periodic solution $\widehat{y_3(t)}$ of (3.17) being globally asymptotically stable, where

$$\widehat{y_3(t)} = y_3^* e^{-(\lambda - \frac{c\zeta\varepsilon}{1+\gamma\varepsilon})\tau}, t \in (n\tau, (n+1)\tau], \tag{3.19}$$

with

$$y_3^* = \frac{-\left(1 - e^{-\left(\lambda - \frac{c\zeta\varepsilon}{1+\gamma\varepsilon}\right)\tau}\right)}{2\vartheta e^{-\left(\lambda - \frac{c\zeta\varepsilon}{1+\gamma\varepsilon}\right)(1-l)\tau}\left(1 - e^{-\left(\lambda - \frac{c\zeta\varepsilon}{1+\gamma\varepsilon}\right)\tau}\right)} + \frac{\sqrt{\left(1 - e^{-\left(\lambda - \frac{c\zeta\varepsilon}{1+\gamma\varepsilon}\right)\tau}\right)^2 + 4\mu_{\max}\vartheta e^{-\left(\lambda - \frac{c\zeta\varepsilon}{1+\gamma\varepsilon}\right)(1-l)\tau}\left(1 - e^{-\left(\lambda - \frac{c\zeta\varepsilon}{1+\gamma\varepsilon}\right)\tau}\right)}}{2\vartheta e^{-\left(\lambda - \frac{c\zeta\varepsilon}{1+\gamma\varepsilon}\right)(1-l)\tau}\left(1 - e^{-\left(\lambda - \frac{c\zeta\varepsilon}{1+\gamma\varepsilon}\right)\tau}\right)}. \tag{3.20}$$

From lemma 3.5., and for $\varepsilon_1 > 0$ small enough, there exists a $t_1, t > t_1$ such that

$$\widehat{y_2(t)} - \varepsilon_1 < y(t) < \widehat{y_3(t)} + \varepsilon_1.$$

Let $\varepsilon \to 0$, we have

$$\widehat{y(t)} - \varepsilon_1 < y(t) < \widehat{y(t)} + \varepsilon_1,$$

for t large enough. This indicates $y(t) \to \widehat{y(t)}$ as $t \to \infty$.

Theorem 3.7. If $\vartheta\mu_{max}(1-l)e^{\lambda l\tau} < 1$, and

$$R_0 > 1, \tag{3.21}$$

holds, model (2.1) is permanent.

Proof. According to lemma 3.1, it is easy to know that there is a constant $K > 0$ such that $x(t) \le K, y(t) \le K$ for t large enough. From the proof procedure of theorem 3.5., and for any $\varepsilon_2 > 0$, we obtain that $y(t) > \widehat{y(t)} - \varepsilon_2$ for all t large enough. Then $y(t) \ge y^* e^{-\lambda(t-n\tau)} - \varepsilon_2 > y^* - \varepsilon_2 = m_2$ with t large enough. Thus, we only need to find $m_1 > 0$ such that $x(t) \ge m_1$ for t large enough. We will do it in the following.

By condition (3.20) of this theorem, we can select $0 < m_3 < \frac{\lambda}{c\zeta-\lambda\gamma}, \varepsilon_1 > 0$ small enough such that

$$\delta = (\eta-\omega m_3-\zeta\varepsilon_1)\tau - \frac{\zeta z^*}{\lambda - \frac{c\zeta m_3}{1+\gamma m_3}}(1-e^{-(\lambda-\frac{c\zeta m_3}{1+\gamma m_3})\tau}) > 0,$$

where z^* is defined as (3.5). We will prove $x(t) < m_3$ can not hold for $t \ge 0$. Otherwise,

$$\begin{cases} \frac{dy(t)}{dt} \le -(\lambda - \frac{c\zeta m_3}{1+\gamma m_3})y(t), t \in (n\tau, (n+1)\tau], \\ y(t^+) = y(t) + \frac{\mu_{\max}}{1+\vartheta y(t-l\tau)}, t = (n+1)\tau. \end{cases} \tag{3.22}$$

By lemmas 3.3, we have $y(t) \le z(t)$ and $z(t) \to \overline{z(t)}, t \to \infty$, where $z(t)$ is the solution of

$$\begin{cases} \frac{dz(t)}{dt} = -(\lambda - \frac{c\zeta m_3}{1+\gamma m_3})z(t), t \in (n\tau, (n+1)\tau], \\ z(t^+) = z(t) + \frac{\mu_{\max}}{1+\vartheta z(t-l\tau)}, t = (n+1)\tau. \end{cases} \tag{3.23}$$

and

$$\overline{z(t)} = z^* e^{-(\lambda-\frac{c\zeta m_3}{1+\gamma m_3})\tau}, t \in (n\tau, (n+1)\tau], \tag{3.24}$$

is the globally asymptotically stable periodic solution of (3.22) with

$$z^* = \frac{-\left(1-e^{-(\lambda-\frac{c\zeta m_3}{1+\gamma m_3})\tau}\right)}{2\vartheta e^{-(\lambda-\frac{c\zeta m_3}{1+\gamma m_3})(1-l)\tau}\left(1-e^{-(\lambda-\frac{c\zeta m_3}{1+\gamma m_3})\tau}\right)} + \frac{\sqrt{\left(1-e^{-(\lambda-\frac{c\zeta m_3}{1+\gamma m_3})\tau}\right)^2 + 4\mu_{\max}\vartheta e^{-(\lambda-\frac{c\zeta m_3}{1+\gamma m_3})(1-l)\tau}\left(1-e^{-(\lambda-\frac{c\zeta m_3}{1+\gamma m_3})\tau}\right)}}{2\vartheta e^{-(\lambda-\frac{c\zeta m_3}{1+\gamma m_3})(1-l)\tau}\left(1-e^{-(\lambda-\frac{c\zeta m_3}{1+\gamma m_3})\tau}\right)}. \tag{3.25}$$

From lemma 3.5. and for any $\varepsilon_1 > 0$, there exists a $T_1 > 0$, when $t \ge T_1$, we get

$$y(t) \le z(t) \le \overline{z(t)} + \varepsilon_1.$$

From model (2.1), we derive that

$$\begin{cases} \frac{dx(t)}{dt} \ge x(t)[\eta - \omega m_3 - \zeta(\overline{z(t)} + \varepsilon_1)], \\ \quad t \in (n\tau, (n+1)\tau], \\ x(t^+) = x(t), t = (n+1)\tau. \end{cases} \tag{3.26}$$

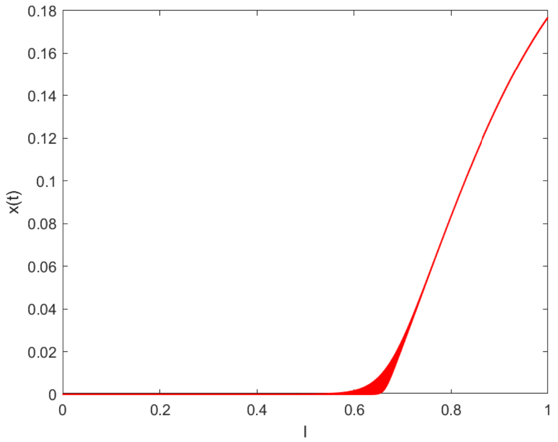
We can find $N_1 \in N$ and $N_1\tau > T_1$. For any $n \ge N_1$, (3.25) is integrated on $(n\tau, (n+1)\tau]$, thereby, we gain that

$$x((n+1)\tau) \ge x(n\tau^+)\exp(\int_{n\tau}^{(n+1)\tau}(\eta - \omega m_3 - \zeta(\overline{z(t)} + \varepsilon_2))dt) = x(n\tau^+)e^{\delta}.$$

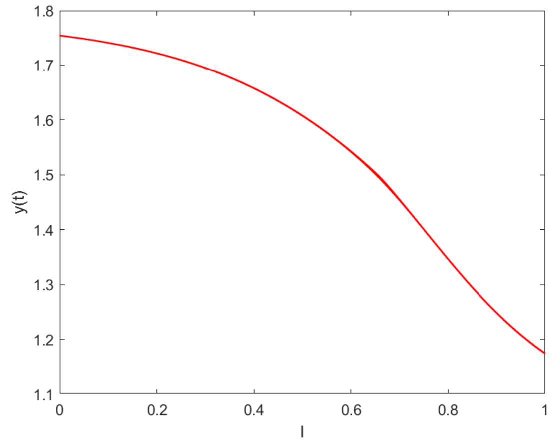
So $x((N_1+k)\tau) \ge x(N_1\tau^+)e^{k\sigma} \to \infty$, as $k \to \infty$, which is a contradiction to the boundedness of $x(t)$. Hence, there exists a $t_1 > 0$ $t > t_1$ such that $x(t) \ge m_3$.

4 Discussion

In this paper, we construct a pest-natural enemy model with impulsive delay releasing natural enemy. The pest-free periodic solution of model (2.1) is proved to be globally attractive with $R_0 < 1$. Model (2.1) is also proved to be permanent with $R_0 > 1$. If $x(0^+) = 0.5,\ y(0^+) = 0.5,\ \eta = 1.1,\ \omega = 0.8,\ \zeta = 3,\ \gamma = 1.5,\ \lambda = 2,\ \tau = 2,\ \vartheta = 0.5,\ c = 2,\ \mu_{max} = 1.75$, Figure 1 indicates that the number of natural enemy will decrease and the number of pest will increase with respect to parameter l. It shows that, after observing data of natural enemy, the managers should release natural enemies on interval $l \in [0, 0.65)$ to control pests in the practical pest management. Obviously, time delay in impulsive releasing natural enemy plays an important role on pest-free in model (2.1). This also show that model (2.1) provides an appropriate method to describe pest management. Further, if $x(0^+) = 0.5,\ y(0^+) = 0.5,\ \eta = 0.88,\ \omega = 0.54,\ c = 0.9,\ \zeta = 1.2,\ \gamma = 0.5,\ \lambda = 0.5,\ \tau = 2,\ \vartheta = 2,\ l = 0.7,\ \mu_{max} = 1.8$, we get $R_0 = 1.0650 > 1$, then the system is permanent as shown $(a)-(b)$ in Figure 2. If $\eta = 0.88,\ \omega = 0.54,\ c = 0.9,\ \zeta = 1.2,\ \gamma = 0.5,\ \lambda = 0.5,\ \tau = 2,\ \vartheta = 2,\ l = 0.7,\ \mu_{max} = 2$, we can also get $R_0 = 0.9981 < 1$, then the periodic solution $(0, \widetilde{y(t)})$ of model (2.1) is globally attractive as shown $(c-d)$

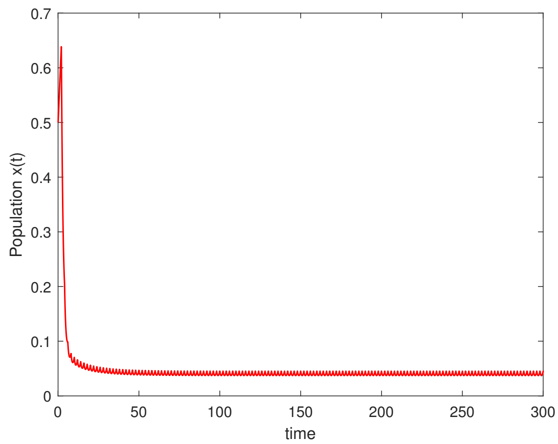


(a) Variation of $x(t)$ with respect to l

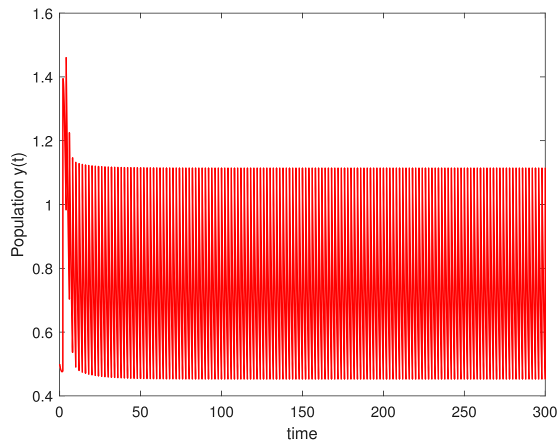


(b) Variation of $y(t)$ with respect to l

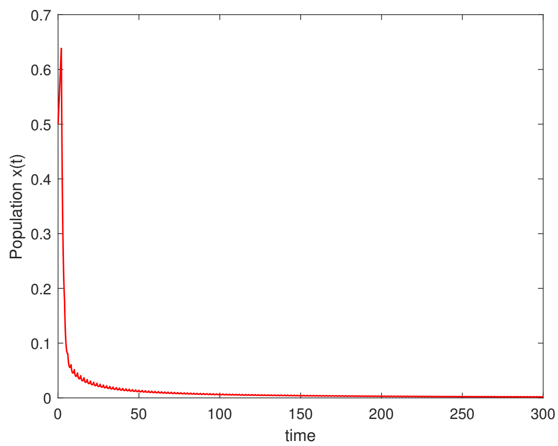
Figure 1. The variations of $x(t)$ and $y(t)$ with respect to parameter $0 < l < 1$ for $x(0^+) = 0.5$, $y(0^+) = 0.5$, $\eta = 1.1$, $\omega = 0.8$, $\zeta = 3$, $\gamma = 1.5$, $\lambda = 2$, $\tau = 2$, $\vartheta = 0.5$, $c = 2$, $\mu_{\max} = 1.75$.



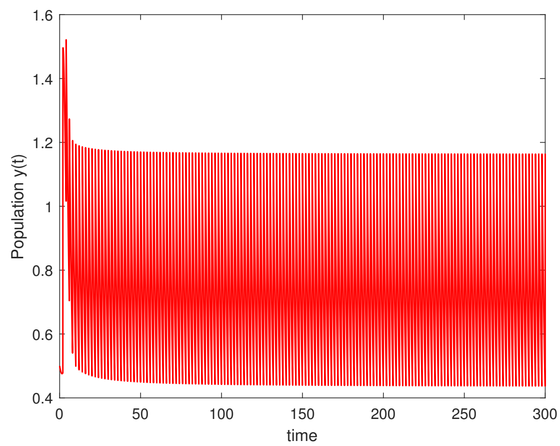
(a) Time-series of $x(t)$ (system permanence)



(b) Time-series of $y(t)$ (system permanence)

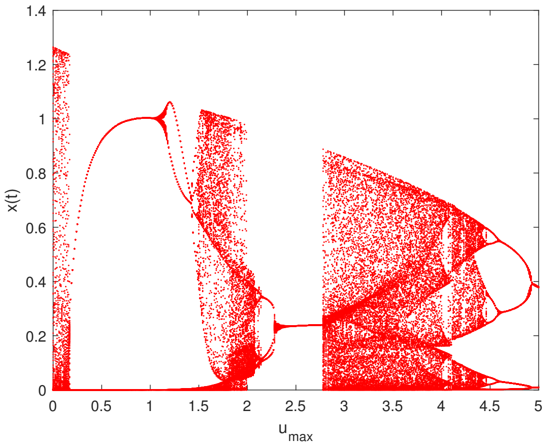


(c) Time-series of $x(t)$ (periodic solution)

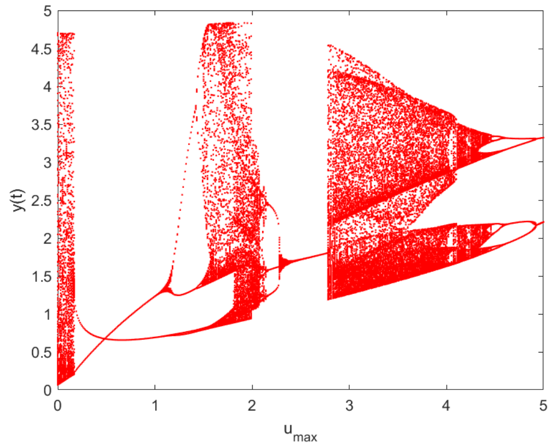


(d) Time-series of $y(t)$ (periodic solution)

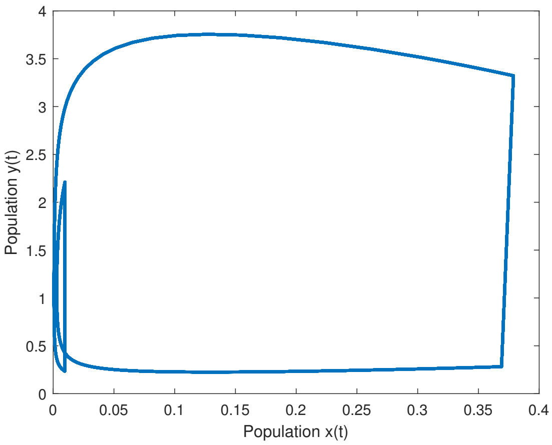
Figure 2. (a)-(b) Time-series of $x(t)$ and $y(t)$ of system permanence with $x(0^+) = 0.5$, $y(0^+) = 0.5$, $\eta = 0.88$, $\omega = 0.54$, $c = 0.9$, $\zeta = 1.2$, $\gamma = 0.5$, $\lambda = 0.5$, $\tau = 2$, $\vartheta = 2$, $l = 0.7$, $\mu_{\max} = 1.8$; (c)-(d) Time-series of $x(t)$ and $y(t)$ of periodic solution $(0, \widetilde{y(t)})$ of model (2.1) being globally attractive with $\eta = 0.88$, $\omega = 0.54$, $c = 0.9$, $\zeta = 1.2$, $\gamma = 0.5$, $\lambda = 0.5$, $\tau = 2$, $\vartheta = 2$, $l = 0.7$, $\mu_{\max} = 2$.



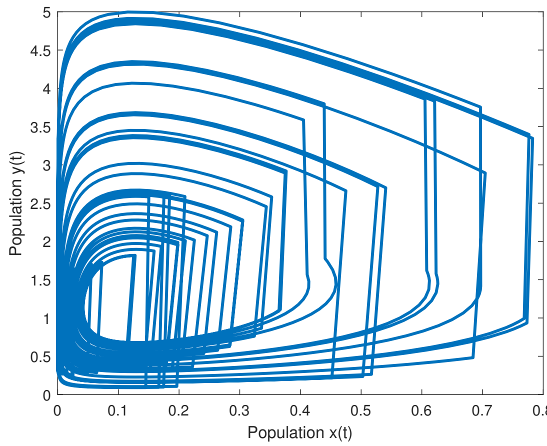
(a) Bifurcation graph of $x(t)$



(b) Bifurcation graph of $y(t)$

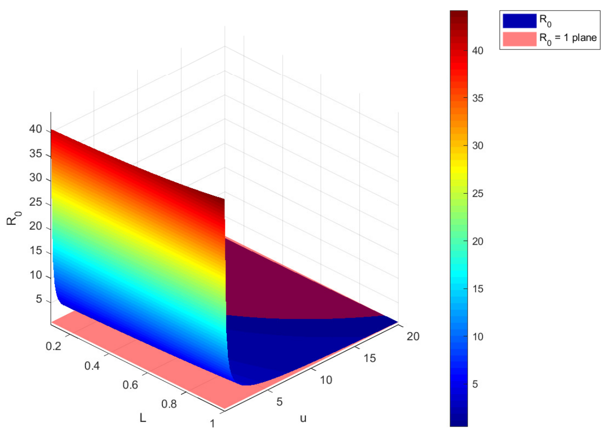


(c) $\mu_{max} = 5$, 2τ-period solution

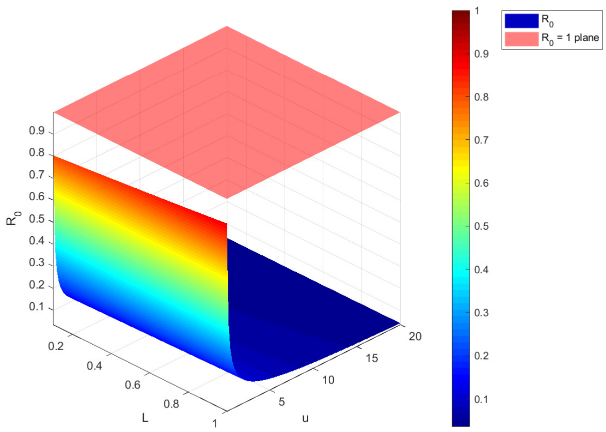


(d) $u_{max} = 3.5$, chaos

Figure 3. The bifurcation graphs of μ_{max} with $x(0^+) = 0.5,\ y(0^+) = 0.5, \eta = 1.4,\ \omega = 0.9,\ c = 3,\ \zeta = 1.3,\ \gamma = 2,\ \lambda = 0.4,\ \tau = 8,\ \vartheta = 1,\ l = 0.6.$



(a) The variation of R_0 with respect to l with $\eta = 1$, $\omega = 0.9$, $c = 0.55$, $\zeta = 0.5$, $\gamma = 2$, $\lambda = 1$, $\tau = 2$, $\vartheta = 1$



(b) The variation of R_0 with respect to $\mu_{\max}$ with $\eta = 0.26$, $\omega = 1.2$, $c = 2$, $\zeta = 2$, $\gamma = 2$, $\lambda = 0.4$, $\tau = 1$, $\vartheta = 4$

Figure 4. (a) The variation of R_0 references to l with $\eta = 1$, $\omega = 0.9$, $c = 0.55$, $\zeta = 0.5$, $\gamma = 2$, $\lambda = 1$, $\tau = 2$, $\vartheta = 1$. (b) The variation of R_0 reference to $\mu_{\max}$ with $\eta = 0.26$, $\omega = 1.2$, $c = 2$, $\zeta = 2$, $\gamma = 2$, $\lambda = 0.4$, $\tau = 1$, $\vartheta = 4$.

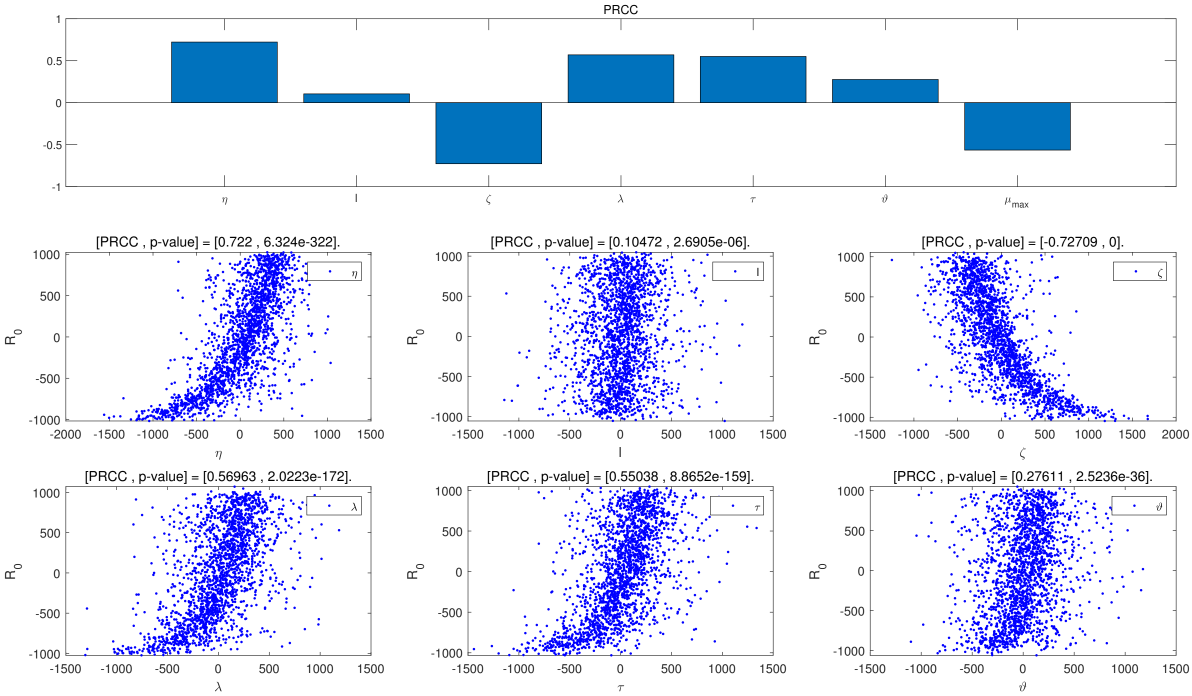


Figure 5. The scatter PRCC value of the R_0 to key parameters and the scatter plots of the parameters η, l, ζ, λ, τ, ϑ. The sample size is 2000 and all parameters are varied simultaneously.

in Figure 2. Then, simulation analysis in Figure 2. indicates that our results are verified to be correct.

In order to explore more rich dynamical properties, we choose some bifurcation parameters to plot bifurcation graphs. First, we let $x(0^+) = 0.5,\ y(0^+) = 0.5,\ \eta = 1.4,\ \omega = 0.9,\ c = 3,\ \zeta = 1.3,\ \gamma = 2,\ \lambda = 0.4,\ \tau = 8,\ \vartheta = 1,\ l = 0.6$ and vary μ_{max} form 0 to 5. They can be seen in Figure 3. We investigate how R_0 changes when l vary simultaneously with $x(0^+) = 0.5,\ y(0^+) = 0.5, \eta = 1,\ \omega = 0.9,\ c = 0.55,\ \zeta = 0.5,\ \gamma = 2,\ \lambda = 1,\ \tau = 2,\ \vartheta = 1$, we also investigate how R_0 changes when μ_{max} vary simultaneously with $\eta = 0.26,\ \omega = 1.2,\ c = 2,\ \zeta = 2,\ \gamma = 2,\ \lambda = 0.4,\ \tau = 1,\ \vartheta = 4$. Their varies can be seen in Figure 4.

Exploring the key parameters which affect the pest eradication, we conduct sensitivity analysis by calculating Partial Rank Correlation Coefficient (PRCC) using LHS method in this part. Takeing parameters as: $\eta = 0.8,\ \omega = 1,\ l = 0.5,\ \zeta = 0.75,\ \gamma = 2,\ \lambda = 0.4,\ \tau = 2,\ \vartheta = 2,\ c = 0.55,\ \mu_{max} = 2$, then we obtain Figure 5. From Figure 5 we get the PRCC values of $\eta,\ \lambda,\ \tau,\ \vartheta$ about R_0 are positive, which means the decreasing of these parameters can help eradicate the pest. The PRCC values of $\zeta,\ \mu_{max}$ about R_0 are negative, that is R_0 is decreasing with these parameters. Because the absolute value of PRCC values of $\eta,\ \zeta,\ \lambda\ ,\tau,\ \mu_{max}$ are greater than 0.4, the parameters lays an important role in pest management.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

Ethical Approval and Consent to Participate

Not applicable.

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