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RESEARCH ARTICLE

Dynamical Behavior of a Second-Order Exponential-Type Fuzzy Difference Equation with Quadratic Term

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Abstract

The paper discusses the dynamical characteristics of solutions to a model with quadratic term. More precisely, an exponential-type fuzzy difference equation is proposed as follows

$$a_{n+1} = \frac{D + Pe^{-a_n}}{T + a_{n-1}^2}, \quad n = 0, 1, \cdots,$$

here D, P, T and a_0, a_{-1} belong to positive fuzzy numbers. This model can be used to characterize the diffusion modeling of a class of infectious diseases with uncertainty, such as the transmission prediction of dengue fever, monkeypox, and other infectious diseases. In addition, by highlighting the advantages of using Stefanini's the generalization of division of fuzzy number (it is also known as g-division) and constructing a Lyapunov function, we primarily obtain the dynamical characteristics of the model discussed above, such as convergence of single positive equilibrium and persistence, global asymptotical stability and boundedness of positive solutions. Furthermore, some numerical examples are provided to confirm the theoretical findings.

Keywords: fuzzy difference equation, boundedness, global asymptotic behavior, g-Division.

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1 Introduction

Difference equations (DEs) are also called discrete dynamical systems. Difference equations are widely applied in various fields, including control engineering, biology, computer science, economics, ecology, and demography, etc. (see, for example, [1–6] and the references therein). Many differential equations need to be discretized into difference equations for analysis. Therefore, many researchers have shown considerable interest in the theory of DEs. In the past decades, the research on DEs has greatly advanced in both depth and breadth. In terms of depth, it involves not only the existence of solutions but also studies on stability, convergence and asymptotic behavior of positive equilibrium. Some researchers have also studied the oscillatory behavior, periodicity, the rate of convergence, bifurcation and chaos in certain models. In terms of breadth, the research reflects an expansion of the forms of difference equations and its system, including linear and nonlinear forms, exponential-type and logarithmic-type, maximum-type and various orders of equations (see [7–16, 38, 40], and the references therein).

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Particularly, as far as the exponential-type difference equations and their system are concerned, this is due to the fact that many population models are related to exponential-type difference equations, so many scholars are greatly interested in studying convergence, stability of positive solutions to these models, also discussing their uniqueness, boundedness and existence, etc. Give a few examples.

For example, EI-Metwally et al. [8] discussed the periodicity feature, asymptotic behaviour, existence and boundedness of solutions, along with stability of a single positive equilibrium for the ordinary DE below

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}, n \in N, \tag{1.1}$$

here x_{-i}, (i=0,1) α, β belong to nonnegative real numbers.

Papaschinopoulos et al. [9] discussed the asymptotic characteristics, the persistence, the boundedness of positive solutions of two exponential-type crisp DEs below

$$x_{n+1} = a + b x_{n-1} e^{-y_n}, \ y_{n+1} = c + d y_{n-1} e^{-x_n}, \tag{1.2}$$

here a, b, c, d, $x_{-i}, y_{-i} \in R^+$ (i=0,1) [37].

In 2006, Ozturk et al. [10] examined the boundedness, periodic nature and convergence of solutions to a second-order ordinary DE with an exponential term

$$x_{n+1} = \frac{\alpha_1 + \alpha_2 e^{-x_n}}{\alpha_3 + x_{n-1}}, n \in N, \tag{1.3}$$

here $x_i, i = -1, 0$ are nonnegative real numbers and $\alpha_1, \alpha_2, \alpha_3 \in R^+$.

In 2013, Bozkurt [11] has discussed an exponential-type crisp difference equation

$$x_{n+1} = \frac{\alpha e^{-x_n} + \beta e^{-x_{n-1}}}{\gamma + \alpha x_n + \beta x_{n-1}}, \tag{1.4}$$

here $x_i, i = -1, 0$ belong to any positive numbers and parameters $\alpha, \beta, \gamma \in R^+$. In their research, the author has obtained the local and global behaviour on the model's positive solutions discussed above.

However, as science and technology continue to advance, the relationships we face are becoming increasingly complex. Although difference equations can effectively describe numerous practical problems in real-life, they become quite challenging to study when dealing with issues related to fuzzy uncertainty or imprecision. In this context, fuzzy difference equations (FDEs) can address this shortcoming and effectively describe practical problems related to uncertainty or imprecision. Indeed, FDEs belong to one of the types of crisp DEs, they are currently applied in fields such as population prediction, strategic decision-making and control systems etc. In addition, in the analysis of fuzzy difference equations, one may generally regard the model initial conditions and parameters as fuzzy numbers and it goes without saying that the solutions are represented by fuzzy sequences. Especially in the recent two decades, fuzzy difference equations have received attention and discussion from an increasing number of scholars, leading to great interest in their theoretical research, and exponential-type fuzzy DEs in particular.

Wang et al. [14] examined the dynamical behaviour of a first-order fuzzy DE with exponential form

$$x_{n+1} = A + B x_n e^{-C x_n}, n \in N, \tag{1.5}$$

here x_0, A, B, C $\in \Re_F^+$.

In 2020, Zhang et al. [15] have discussed dynamics of a second-order FDE with form

$$x_{n+1} = \frac{A + B e^{-x_n}}{C + x_{n-1}}, n \in N, \tag{1.6}$$

here $A, B, C, x_{-i}, i \in \{0, 1\}$ belong to $\Re_F^+$.

It is noteworthy that the forms of difference equations can be linear or nonlinear. In the study of low-order nonlinear difference equations, some scholars are concerned with the form with quadratic terms. The formal expansion makes the research on difference equations deeper and more comprehensive, providing valuable references for the study of difference equations. Here are a few examples.

In 2020, Bešo et al. [17] have discussed the recursive sequence with a quadratic term as follows

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2}, n = 0, 1, \cdots, \tag{1.7}$$

here γ, δ and x_0, x_{-1} belong to R^+.

In [18], Khyat et al. investigated the following recursive sequence defined below with two quadratic terms

$$x_{n+1} = a + \frac{x_n^2}{x_{n-1}^2}, n \in N, \tag{1.8}$$

In fact, the authors obtained the single positive fixed point $\overline{x} = a + 1$, and they found it is globally stable if model (1.8) satisfies the relation $a > \sqrt{2}$. Furthermore, these authors determined the direction of the Neimark-Sacker bifurcation. Here the parameter a and initial conditions x_{-1}, x_0 belong to R^+.

In 2022, Zhang et al. [19] conducted research on the fuzzy DE below

$$x_{n+1} = A + \frac{Bx_n}{x_{n-1}^2}, n \in N, \tag{1.9}$$

where x_{-1}, x_0, A, B $\in \Re_F^+$.

It is also worth mentioning that the determination of initial conditions and parameters in difference equations depends on statistical methods. For dealing with the properties of fuzzy difference equation models, we usually employ common techniques and methods for handling crisp difference equations, such as iteration, inequality techniques, matrix theory, mathematical induction, and proof by contradiction, etc. It is helpful for addressing the qualitative behaviour of the most complex FDE models to use these approaches. Additionally, many scholars have explored various methods and techniques for studying fuzzy DEs, and the effectiveness of these methods has led to rapid development in the field such as finance, biologic models and population models, etc. Now let us make a historical flash back.

The concept of FDEs was first proposed by Lakshmikantham et al. [20] in 2002, who constructed the Lyapunov function to analyze the basic theory of fuzzy difference equation models and obtained comparison theorems for these models. Papaschinopoulos et al. [21] have studied global behaviors of FDE model $x_{n+1} = A + \frac{B}{x_n}$ by employing Zadeh extension principle, in which A and B and initial conditions x_0 belong to positive fuzzy numbers. At the same time, Mondal et al. [22] using Lagrange's multiplier method researched a linear fuzzy difference equation of order two. Stefanini [23] proposed a new method for studying some linear FDE models by employing a generalization of division of fuzzy numbers (it is also known as g-division). Khastan [24] has researched fuzzy logistic difference equations by utilizing the basic theory of Hukuhara Difference (H-Difference) of fuzzy numbers. In view of this, by employing these methods, many scholars have extended the study of various types of fuzzy difference equations and have derived many effective conclusions. For more details see ([25–33], and the references therein).

To the best of our knowledge, exponential-type fuzzy DEs are a special type of FDEs, as far as FDEs are concerned, it is a well-known fact that the parameters and initial conditions belong to positive fuzzy numbers, while it's solutions presents sequence of positive fuzzy numbers. However, due to the particular nature of the exponential form, we generally cannot obtain an explicit solution but can only express its implicit solution. Nevertheless, by using the Existence and Uniqueness Theorem of equation's solutions, it can still prove the existence and uniqueness of positive solutions for FDEs that correspond to a system of crisp DEs in our study.

Based on the points discussed above, this article aims to study the dynamics of solutions for second-order exponential-type FDE with quadratic terms using the g-division

$$a_{n+1} = \frac{D + Pe^{-a_n}}{T + a_{n-1}^2}, \quad n \in N, \tag{1.10}$$

here $D, P, T,\ a_{-i} \in \Re_F^+, i \in \{0, 1\}$. This model can be applied to the transmission prediction in the diffusion modeling of infectious diseases with uncertainty, such as dengue fever and monkeypox. In this context, a_n represents the number of infected individuals at a certain time, e^{-a_n} reflects the nonlinear infection probability of susceptible populations, and a_{n-1}^2 embodies the historical cumulative effect of infection transmission (e.g., the time lag of virus incubation period). The fuzzy parameters D, P, and T denote uncertain influencing factors such as transmission rate and the effect of isolation measures. For example, in the dengue fever transmission model, this equation can be used to analyze the fuzzy evolution trend of the epidemic under different prevention and control strategies, thereby assisting public health decision-making.

In a nutshell, based on previous research, this paper studies existence of positive solutions for quadratic FDE models by utilizing the g-division, Existence and Uniqueness Theorem for solutions of equations, the Lyapunov function and matrix theory etc. It also researches stability of unique equilibrium. Indeed, the core method of this paper is the generalized division of fuzzy numbers (g-division). Recently, numerous significant works published on the application of g-division (see [15, 19, 21, 27] and the references therein).

Researchers have found that g-division can overcome the disadvantages of the expansion of fuzzy intervals when applying the Zadeh Extension Principle, which can lead to increased fuzzy intervals. As a result, the finding of properties of FDEs has become more accurate and the results have become more representative. Consequently, scholars are increasingly interested in this method. In addition, the exponential-type difference equations date back to population dynamics, in a sense, due to the presence of certain fuzzy imprecise (or uncertain) phenomena in population dynamics, the role of exponential-type fuzzy difference equations becomes particularly important. In this context, our study offers new insights into population dynamics and paves the way for further investigation into exponential fuzzy difference equations. It offers valuable implications for the research on population dynamics models and provides new perspectives for the potential applications of fuzzy difference equation models.

Here, we provide a summary of the research approach and content of this paper, the detailed specifics are outlined below. In Section 2, we mainly introduce several important concepts related to this paper. In Section 3, we mainly investigated the dynamics of the exponential FDE model (1.10) with quadratic terms using g-division, inequality techniques, and matrix theory, etc. In Section 4, we analyzed the solutions of the fuzzy DE model (1.10) and showed that they are non-oscillatory under certain initial conditions and conditions related to equilibrium points. We provide a few examples demonstrating the effectiveness of theoretical findings we have obtained in Section 5. In Section 6, we summarize some important findings of this article.

2 Preliminary and definitions

To demonstrate the validity of the findings in this paper, we will first cite a few important basic concepts from previous research in this section.

Definition 2.1. [34] *If function $W : R \to [0,1]$ satisfies properties (i)-(iv):*
(i) W is normal, that is, $\exists a \in R$ with $W(a) = 1$;
(ii) W is fuzzy convex, that is, $\forall x \in [0,1]$ and $a_1, a_2 \in R$, one has

$$W(xa_1 + (1-x)a_2) \geq \min\{W(a_1), W(a_2)\};$$

(iii) W is upper semi-continuous;
(iv) The support of W, supp$W$ $= \overline{\bigcup_{\vartheta\in(0,1]}[W]_\vartheta} = \overline{\{a : W(a) > 0\}}$ is compact.

Then it is known as a fuzzy number.

For $\vartheta \in (0,1]$, the $\vartheta-$cuts of fuzzy number W is denoted by $[W]_\vartheta = \{a \in R : W(a) \geq \vartheta\}$, and for $\vartheta = 0$, the support of W is defined by supp$W$ $= [W]_0 = \overline{\{a \in R | W(a) > 0\}}$. One has the $[W]_\vartheta$ is a closed interval. If supp$W \subset (0,\infty)$, then the fuzzy number is known as positive. Indeed, W is a trivial fuzzy number (a positive real number), that is $[W]_\vartheta = [W, W], \vartheta \in (0,1]$.

Assume that $E, F \in \Re_F^+$ satisfy $[E]_\vartheta = [E_{L,\vartheta}, E_{R,\vartheta}], [F]_\vartheta = [F_{L,\vartheta}, F_{R,\vartheta}], \vartheta \in [0,1]$, and for $k > 0$, then the operations of addition E+F, scalar product kE, division $\frac{E}{F}$ and multiplication EF for fuzzy numbers E and F are defined as follows:

$$[E+F]_\vartheta = [E_{L,\vartheta} + F_{L,\vartheta}, E_{R,\vartheta} + F_{R,\vartheta}], \tag{2.1}$$

$$[kE]_\vartheta = [kE_{L,\vartheta}, kE_{R,\vartheta}], \tag{2.2}$$

$$\left[\frac{E}{F}\right]_\vartheta = \left(\min\left\{\tfrac{E_{L,\vartheta}}{F_{L,\vartheta}}, \tfrac{E_{L,\vartheta}}{F_{R,\vartheta}}, \tfrac{E_{R,\vartheta}}{F_{L,\vartheta}}, \tfrac{E_{R,\vartheta}}{F_{R,\vartheta}}\right\}, \max\left\{\tfrac{E_{L,\vartheta}}{F_{L,\vartheta}}, \tfrac{E_{L,\vartheta}}{F_{R,\vartheta}}, \tfrac{E_{R,\vartheta}}{F_{L,\vartheta}}, \tfrac{E_{R,\vartheta}}{F_{R,\vartheta}}\right\}\right), \quad [F]_\vartheta \neq 0, \tag{2.3}$$

$$[EF]_\vartheta = \begin{cases} (\min\{E_{L,\vartheta}F_{L,\vartheta}, E_{L,\vartheta}F_{R,\vartheta}, E_{R,\vartheta}F_{L,\vartheta}, E_{R,\vartheta}F_{R,\vartheta}\}, \\ \max\{E_{L,\vartheta}F_{L,\vartheta}, E_{L,\vartheta}F_{R,\vartheta}, E_{R,\vartheta}F_{L,\vartheta}, E_{R,\vartheta}F_{R,\vartheta}\}). \end{cases} \tag{2.4}$$

All fuzzy numbers, along with addition and scalar multiplication defined in (2.1) and (2.2), form a collection denoted by $\Re_F$ (where $\Re_F^+$ represents positive fuzzy numbers).

The definition of the metric space can be given as:

Definition 2.2. [34] *Let $E, F \in \Re_F$, the definition of distance is given as follows:*

$$D(E,F) = \sup_{\vartheta\in[0,1]} \max\{|E_{L,\vartheta} - F_{L,\vartheta}|, |E_{R,\vartheta} - F_{R,\vartheta}|\}. \tag{2.5}$$

Similarly, the norm of a set E in fuzzy space is defined as follows:

$$\|E\| = \sup_{\vartheta\in(0,1]} \max\{|E_{L,\vartheta}|, |E_{R,\vartheta}|\}.$$

It is evident that the metric space $(\Re_F, D)$ is complete.

Definition 2.3. [23] *Let $E, F \in \Re_F$ with ϑ-cuts $[E]_\vartheta = [E_{L,\vartheta}, E_{R,\vartheta}], [F]_\vartheta = [F_{L,\vartheta}, F_{R,\vartheta}]$, $0 \notin [F]_\vartheta, \forall\vartheta \in [0,1]$. Then the g-division $\div_g$ is an operational rule used for calculating fuzzy number $U = E \div_g F$ with $\vartheta-$cuts $[U]_\vartheta = [U_{L,\vartheta}, U_{R,\vartheta}]$(where $[U]_\vartheta^{-1} = [1/U_{R,\vartheta}, 1/U_{L,\vartheta}]$) defined by*

$$[U]_\vartheta = [E]_\vartheta \div_g [F]_\vartheta \Longleftrightarrow \begin{cases} (i) \quad [E]_\vartheta = [F]_\vartheta[E]_\vartheta, \\ \text{or} \\ (ii) \quad [F]_\vartheta = [E]_\vartheta[U]_\vartheta^{-1}, \end{cases} \tag{2.6}$$

if U represents a proper fuzzy number (where $U_{L,1} \leq U_{R,1}$*,* $U_{L,\vartheta}$ *is nondecreasing,* $U_{R,\vartheta}$ *is nonincreasing).*

Remark 2.1. *Based on the reference [23], let the fuzzy numbers E and F is positive, while* $U \in \Re_F^+$*, if* $E \div_g F = U \in \Re_F^+$ *exists, then case (i) and (ii) may occur*
Case (i). if $E_{L,\vartheta}F_{R,\vartheta} \leq E_{R,\vartheta}F_{L,\vartheta}, \forall\vartheta \in [0,1]$*, then* $U_{L,\vartheta} = \frac{E_{L,\vartheta}}{F_{L,\vartheta}}, U_{R,\vartheta} = \frac{E_{R,\vartheta}}{FR,\vartheta}$*,*
Case (ii). if $E_{L,\vartheta}F_{R,\vartheta} \geq E_{R,\vartheta}F_{L,\vartheta}, \forall\vartheta \in [0,1]$*, then* $U_{L,\vartheta} = \frac{E_{R,\vartheta}}{F_{R,\vartheta}}, U_{R,\vartheta} = \frac{E_{L,\vartheta}}{FL,\vartheta}$.

Definition 2.4.(see [21, 27]) *Assume there exists* $Q, H \in R^+$ *satisfy*

$$supp\ a_n \subset [Q, \infty)(resp.\ supp\ a_n \subset (0, H]), n \in N^+,$$

then sequence of positive fuzzy numbers $\{a_n\}$ *is persistent (resp. bounded).*

If $Q > 0, H > 0$ *satisfy*

$$supp\ a_n \subset [Q, H], n \in N^+.$$

Then sequence $\{a_n\}$ *is bounded and persistence.*

Furthermore, if there exists sequence $\|a_n\|, n \in N^+$*, is an unbounded norm, then sequence* $\{a_n\}, n \in N^+$ *is unbounded.*

Definition 2.5. [2] $\bar{a} \in \Re_F^+$ *is known as a positive equilibrium of the model (1.10) if it satisfies*

$$\bar{a} = \frac{D + Pe^{-\bar{a}}}{T + \bar{a}^2}$$

Let $a_n, a \in \Re_F^+, n \in N$, $a_n \to a$ as $n \to \infty$ if $\lim_{n\to\infty} D(a_n, a) = 0$.

Definition 2.6. (see [35]) *If* $u_-(\vartheta)$ *and* $u_+(\vartheta)$ *satisfy the condition* $[u]_\vartheta = [u_-(\vartheta), u_+(\vartheta)]$ *and the following three properties, where* $u \in \Re_F, \vartheta \in (0,1]$*, so we know* $u_-(\vartheta), u_+(\vartheta)$ *are functions defined on the interval (0,1].*

(i) $u_-(\vartheta)$ *is nondecreasing and left continuous;*

(ii) $u_+(\vartheta)$ *is nonincreasing and left continuous;*

(iii) $u_-(1) \leq u_+(1)$.

This definition implies that for any functions $f(\vartheta)$ *and* $g(\vartheta)$ *defined on the interval (0,1], if they satisfy the properties (i)-(iii) above, then* $\exists$ *u*$\in \Re_F$ *satisfies* $[u]_\vartheta = [f(\vartheta), g(\vartheta)]$ *for* $\vartheta \in (0,1]$.

Lemma 2.1. (see [2]) *Let differentiable functions f:* $I_b^2 \times I_c^2 \to I_b$ *and g:* $I_b^2 \times I_c^2 \to I_c$ *be continuous, the following discrete dynamical system*

$$\begin{cases} b_{n+1} = f(b_n, b_{n-1}, c_n, c_{n-1}), \\ c_{n+1} = g(b_n, b_{n-1}, c_n, c_{n-1}) \end{cases} \quad n = 0, 1, 2, \cdots, \tag{2.7}$$

has a unique solution $(b_i, c_i)_{i=-1}^{+\infty}$*, where the initial conditions* $(b_i, c_i) \in I_b \times I_c$ *for i= -1, 0.*

Consider the system (2.7), $(\bar{b}, \bar{c})$ *is called it's equilibrium point (or fixed point) if it satisfies*

$$\bar{b} = f(\bar{b}, \bar{b}, \bar{c}, \bar{c}),\ \bar{c} = g(\bar{b}, \bar{b}, \bar{c}, \bar{c}).$$

Lemma 2.2. (see [36]) *Let functions f:* $I_b \times I_c \to I_b$ *and g:* $I_b \times I_c \to I_c$ *be continuous, where* $I_b \times I_c$*= [s,t]×[u,v] be real intervals, for initial conditions* $(b_i, c_i) \in I_b \times I_c$*, i=-1, 0, consider the system (2.7), if the following propositions (i)-(iii) are correct.*

(i) f(b,c) is nonincreasing in both arguments b, c.

(ii) g(b,c) is nonincreasing in both arguments b, c.

(iii) Let us assume $(m_1, M_1, m_2, M_2) \in I_b \times I_c$ *is a solution of the following system*

$$\begin{cases} M_1 = f(m_1, m_2), \quad m_1 = f(M_1, M_2), \\ M_2 = g(m_1, m_2), \quad m_2 = g(M_1, M_2), \end{cases} \tag{2.8}$$

such that $m_1 = M_1$ *and* $m_2 = M_2$*. Thus, the system (2.7) has unique positive equilibrium* $(\bar{b}, \bar{c})$ *such that* $\lim_{n\to\infty}(b_n, c_n) = (\bar{b}, \bar{c})$*. The equilibrium* $(\bar{b}, \bar{c})$ *is also called global attractor if* $\lim_{n\to\infty}(b_n, c_n) = (\bar{b}, \bar{c})$.

Lemma 2.3. (see [16, 32]) *Suppose a recursive sequence* $a_{n+1} = f(a_n), n \in N$*, lead to* $\bar{a}$ *denotes a equilibrium of function f. Then we say that* $\bar{a}$ *is locally asymptotically stable if any root of the Jacobian matrix* J_f *about equilibrium* $\bar{a}$ *lie inside the open unit disk* $|\lambda| < 1$*. If at least one of these roots has a modulus greater than one, then the equilibrium* $\bar{a}$ *is unstable.*

Definition 2.7. [34] *A triangular fuzzy number is a triplet* $Z = (\xi, \zeta, \eta)$ *with the membership function*

$$Z(a) = \begin{cases} 0, & a \leq \xi; \\ \frac{a-\xi}{\zeta-\xi}, & \xi \leq a \leq \zeta; \\ 1, & a = \zeta; \\ \frac{\eta-a}{\eta-\zeta}, & \zeta \leq a \leq \eta; \\ 0, & a \geq \eta. \end{cases}$$

The ϑ-cuts of $Z = (\xi, \zeta, \eta)$ are defined by $[Z]_\vartheta = \{a \in R : Z(a) \geq \vartheta\} = [\xi + \vartheta(\zeta - \xi), \eta - \vartheta(\eta - \zeta)] = [Z_{L,\vartheta}, Z_{R,\vartheta}], \vartheta \in [0,1]$. From which we know $[Z]_\vartheta$ are closed intervals. Moreover, the function Z is known as a positive fuzzy number if the supp$Z \subset (0, \infty)$.

Theorem 2.1. STACKING THEOREM [34] *Let $\{Z_\vartheta : \vartheta \in [0,1]\}$ represents a not empty, convex and compact subset family of R^n satisfies the following properties:*

(i) $\overline{\cup Z_\vartheta} \subset Z_0$.

(ii) $Z_{\vartheta_2} \subset Z_{\vartheta_1},\ if\ \vartheta_1 \leq \vartheta_2$.

(iii) $Z_\vartheta = \cap_{t \geq 1} Z_{\vartheta_t}\ if\ \vartheta_t \uparrow \vartheta > 0$.

Then we have $u \in R_F^n$ such that $[u]_\vartheta = Z_\vartheta,\ for\ \forall \vartheta \in (0,1]$ and $[u]_0 = \overline{\cup_{0<\vartheta\leq 1} Z_\vartheta} \subset Z_0$.

3 Main results

3.1 Existence of positive solution of the model (1.10)

In this section, the existence of the solutions to an exponential-type fuzzy model (1.10) of order two is discussed. Firstly, the following lemma is presented, the lemma provided below is crucial for the derivation of Theorem 3.1.

Lemma 3.1. [27] Let $f : R^+ \times R^+ \times R^+ \times R^+ \to R^+$ be continuous, $A, A_0, A_1, A_2 \in \Re_F$. Then

$$\vartheta = f([A]_\vartheta, [A_0]_\vartheta, [A_1]_\vartheta, [A_2]_\vartheta),\quad \vartheta \in (0,1]. \tag{3.1}$$

Theorem 3.1. *Consider model (1.10), in which $D, P \in \Re_F^+$. Then for $a_{-1}, a_0 \in \Re_F^+$, then the fuzzy model (1.10) exists unique positive solution a_n.*

Proof. Let $\{a_n\}$ denote a sequence of positive fuzzy numbers such that the fuzzy model (1.10) holds true with initial conditions a_{-1}, a_0. To take into account the $\vartheta-$cuts, for $\vartheta \in (0,1]$, one has

$$\begin{cases} [D]_\vartheta = [D_{L,\vartheta}, D_{R,\vartheta}], \quad [P]_\vartheta = [P_{L,\vartheta}, P_{R,\vartheta}], \\ \qquad n = 0, 1, 2, \cdots \\ [T]_\vartheta = [T_{L,\vartheta}, T_{R,\vartheta}], \quad [a_n]_\vartheta = [a_{n,L,\vartheta}, a_{n,R,\vartheta}]. \end{cases} \tag{3.2}$$

From (1.10), (3.2) and applying Lemma 3.1, one has

$$\begin{aligned} \vartheta = [a_{n+1,L,\vartheta}, a_{n+1,R,\vartheta}] &= \left[\frac{D + Pe^{-a_n}}{T + a_{n-1}^2}\right]_\vartheta \\ &= \frac{[D]_\vartheta + [P]_\vartheta \times [e^{-a_n}]_\vartheta}{[T]_\vartheta + [a_{n-1}^2]_\vartheta} \\ &= \frac{[D_{L,\vartheta} + P_{L,\vartheta}e^{-a_{n,R,\vartheta}}, D_{R,\vartheta} + P_{R,\vartheta}e^{-a_{n,L,\vartheta}}]}{[T_{L,\vartheta} + a_{n-1,L,\vartheta}^2, T_{R,\vartheta} + a_{n-1,R,\vartheta}^2]}. \end{aligned} \tag{3.3}$$

Based on the g-division of fuzzy numbers, and noting Remark 2.1, we can easily deduce that either case (i) or case (ii) occurs.
Case (i)

$$\begin{aligned} \vartheta =& [a_{n+1,L,\vartheta}, a_{n+1,R,\vartheta}] \\ =& \left[\frac{D_{L,\vartheta} + P_{L,\vartheta}e^{-a_{n,R,\vartheta}}}{T_{L,\vartheta} + a_{n-1,L,\vartheta}^2},\right. \\ & \left.\frac{D_{R,\vartheta} + P_{R,\vartheta}e^{-a_{n,L,\vartheta}}}{T_{R,\vartheta} + a_{n-1,R,\vartheta}^2}\right]. \end{aligned} \tag{3.4}$$

Case (ii)

$$\begin{aligned} \vartheta =& [a_{n+1,L,\vartheta}, a_{n+1,R,\vartheta}] \\ =& \left[\frac{D_{R,\vartheta} + P_{R,\vartheta}e^{-a_{n,L,\vartheta}}}{T_{R,\vartheta} + a_{n-1,R,\vartheta}^2},\right. \\ & \left.\frac{D_{L,\vartheta} + P_{L,\vartheta}e^{-a_{n,R,\vartheta}}}{T_{L,\vartheta} + a_{n-1,L,\vartheta}^2}\right]. \end{aligned} \tag{3.5}$$

Suppose Case (i) occurs, that is, $\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-a_{n,R,\vartheta}}}{D_{R,\vartheta}+P_{R,\vartheta}e^{-a_{n,L,\vartheta}}} \leq \frac{T_{L,\vartheta}+a_{n-1,L,\vartheta}^2}{T_{R,\vartheta}+a_{n-1,R,\vartheta}^2}$, for $n = 0, 1, \cdots$, from (3.3), one gets, for $\beta \in (0,1]$

$$\begin{aligned} a_{n+1,L,\vartheta} &= \frac{D_{L,\vartheta} + P_{L,\vartheta}e^{-a_{n,R,\vartheta}}}{T_{L,\vartheta} + a_{n-1,L,\vartheta}^2}, \\ a_{n+1,R,\vartheta} &= \frac{D_{R,\vartheta} + P_{R,\vartheta}e^{-a_{n,L,\vartheta}}}{T_{R,\vartheta} + a_{n-1,R,\vartheta}^2}. \end{aligned} \tag{3.6}$$

Thus, we can suppose that for arbitrary initial conditions $(a_{j,L,\vartheta}, a_{j,R,\vartheta}), j = -1, 0, \vartheta \in (0,1]$, from which it has a unique solution $(a_{n,L,\vartheta}, a_{n,R,\vartheta})$. Next, we will demonstrate that $[a_{n,L,\vartheta}, a_{n,R,\vartheta}], \vartheta \in (0,1]$, where $(a_{n,L,\vartheta}, a_{n,R,\vartheta})$ is the solution of system (3.5) with initial conditions $(a_{j,L,\vartheta}, a_{j,R,\vartheta}), j = -1, 0$, determines the solution a_n of (1.10) with initial conditions a_{-1}, a_0 such that

$$[a_n]_\vartheta = [a_{n,L,\vartheta}, a_{n,R,\vartheta}],\ \vartheta \in (0,1],\ n = 0, 1, 2, \cdots. \tag{3.7}$$

From [19], and since $a_j \in \Re_F^+, j = -1, 0$, for $\vartheta_1, \vartheta_2 \in (0,1]$ *with* $\vartheta_1 \leq \vartheta_2$, we have

$$0 < a_{j,L,\vartheta_1} \leq a_{j,L,\vartheta_2} \leq a_{j,R,\vartheta_2} \leq a_{j,R,\vartheta_1}, j = -1, 0. \tag{3.8}$$

We claim that

$$a_{n,L,\vartheta_1} \leq a_{n,L,\vartheta_2} \leq a_{n,R,\vartheta_2} \leq a_{n,R,\vartheta_1}, n = 0, 1, 2, \cdots. \tag{3.9}$$

Working inductively. From (3.7), we can deduce that (3.8) holds true for $n = 0, 1, \cdots$. Assume (3.8) holds true for $n \leq k, k \in N^+$. According to (3.5) and (3.7), for n=k+1, then we can get

$$\begin{aligned} a_{k+1,L,\vartheta_1} &= \frac{D_{L,\vartheta_1} + P_{L,\vartheta_1} e^{-a_{k,R,\vartheta_1}}}{T_{L,\vartheta_1} + a^2_{k-1,L,\vartheta_1}} \\ &\leq \frac{D_{L,\vartheta_2} + P_{L,\vartheta_2} e^{-a_{k,R,\vartheta_2}}}{T_{L,\vartheta_2} + a^2_{k-1,L,\vartheta_2}} \\ &= a_{k+1,L,\vartheta_2} = \frac{D_{L,\vartheta_2} + P_{L,\vartheta_2} e^{-a_{k,R,\vartheta_2}}}{T_{L,\vartheta_2} + a^2_{k-1,L,\vartheta_2}} \\ &\leq \frac{D_{R,\vartheta_2} + P_{R,\vartheta_2} e^{-a_{k,L,\vartheta_2}}}{T_{R,\vartheta_2} + a^2_{k-1,R,\vartheta_2}} = a_{k+1,R,\vartheta_2} \\ &= \frac{D_{R,\vartheta_2} + P_{R,\vartheta_2} e^{-a_{k,L,\vartheta_2}}}{T_{R,\vartheta_2} + a^2_{k-1,R,\vartheta_2}} \\ &\leq \frac{D_{R,\vartheta_1} + P_{R,\vartheta_1} e^{-a_{k,L,\vartheta_1}}}{T_{R,\vartheta_1} + a^2_{k-1,R,\vartheta_1}} \\ &= a_{k+1,R,\vartheta_1}. \end{aligned}$$

Therefore, (3.8) satisfies. Moreover, from (3.5) we have

$$\begin{aligned} a_{1,L,\vartheta} &= \frac{D_{L,\vartheta} + P_{L,\vartheta} e^{-a_{0,R,\vartheta}}}{T_{L,\vartheta} + a^2_{-1,L,\vartheta}}, \\ a_{1,R,\vartheta} &= \frac{D_{R,\vartheta} + P_{R,\vartheta} e^{-a_{0,L,\vartheta}}}{T_{R,\vartheta} + a^2_{-1,R,\vartheta}}, \quad \vartheta \in (0,1]. \end{aligned} \tag{3.10}$$

Since the initial conditions $a_j \in \Re_F^+, j \in \{-1, 0\}$, and parameter D, P, T $\in \Re_F^+$, then $a_{0,L,\vartheta}, a_{0,R,\vartheta}$, $a_{-1,L,\vartheta}$ and $a_{-1,R,\vartheta}$ are left continuous. Then from (3.9), one gets that $a_{1,L,\vartheta}, a_{1,R,\vartheta}$ are also left continuous, in the same way as the proof of (3.8), one gets $a_{n,L,\vartheta}, a_{n,R,\vartheta}, n \in N^+$ are left continuous.

Now, we show supp $a_n = \overline{\bigcup_{\vartheta \in (0,1]} [a_{n,L,\vartheta}, a_{n,R,\vartheta}]}$ is compact. Noting $\bigcup_{\vartheta \in (0,1]} [a_{n,L,\vartheta}, a_{n,R,\vartheta}]$ is bounded. Additionally, since $a_j, j = -1, 0$ and $D, P, T \in \Re_F^+$, let positive integers M_D, N_D, N_P, M_P, $M_T, N_T, M_j, N_j, j = -1, 0$ satisfy, for $\forall \vartheta \in (0,1]$,

$$\begin{cases} [D_{L,\vartheta}, D_{R,\vartheta}] \subset [M_D, N_D], \\ [P_{L,\vartheta}, P_{R,\vartheta}] \subset [M_P, N_P], \\ [T_{L,\vartheta}, T_{R,\vartheta}] \subset [M_T, N_T], \\ [a_{j,L,\vartheta}, a_{j,R,\vartheta}] \subset [M_j, N_j], \quad j = -1, 0. \end{cases} \tag{3.11}$$

Hence, it follows from (3.9) and (3.10) that

$$\subset \left[\frac{M_D + M_P e^{-N_0}}{N_T + N^2_{-1}}, \frac{N_D + N_P e^{-M_0}}{M_T + M^2_{-1}} \right], \quad \vartheta \in (0,1]. \tag{3.12}$$

From which, it follows that

$$\bigcup_{\vartheta \in (0,1]} [a_{1,L,\vartheta}, a_{1,R,\vartheta}] \subset \left[\frac{M_D + M_P e^{-N_0}}{N_T + N^2_{-1}}, \frac{N_D + N_P e^{-M_0}}{M_T + M^2_{-1}} \right], \quad \vartheta \in (0,1]. \tag{3.13}$$

Therefore, it follows from (3.12) that $\overline{\bigcup_{\vartheta \in (0,1]} [a_{1,L,\vartheta}, a_{1,R,\vartheta}]}$ is compact and $\overline{\bigcup_{\vartheta \in (0,1]} [a_{1,L,\vartheta}, a_{1,R,\vartheta}]} \subset (0, \infty)$. Deducing inductively it can easily get that $\overline{\bigcup_{\vartheta \in (0,1]} [a_{n,L,\vartheta}, a_{n,R,\vartheta}]}$ is compact, and

$$\overline{\bigcup_{\vartheta \in (0,1]} [a_{n,L,\vartheta}, a_{n,R,\vartheta}]} \subset (0, \infty), \ n = 1, 2, \cdots. \tag{3.14}$$

Hence, from (3.8), (3.13), and since $[a_{n,L,\vartheta}, a_{n,R,\vartheta}]$ are left continuous, one has $[a_{n,L,\vartheta}, a_{n,R,\vartheta}]$ determines a sequence of positive fuzzy numbers $\{a_n\}$ lead to (3.6) is valid.

We shall now show that for arbitrary initial conditions a_{-1}, a_0, the sequence a_n determines the solution of the model (1.10). Since for $\forall \vartheta \in (0,1]$, one has

$$\begin{aligned} \vartheta &= [a_{n,L,\vartheta}, a_{n,R,\vartheta}] \\ &= \left[\frac{D_{L,\vartheta} + P_{L,\vartheta} e^{-a_{n,R,\vartheta}}}{T_{L,\vartheta} + a^2_{n-1,L,\vartheta}}, \frac{D_{R,\vartheta} + P_{R,\vartheta} e^{-a_{n,L,\vartheta}}}{T_{R,\vartheta} + a^2_{n-1,R,\vartheta}} \right] \\ &= \frac{[D_{L,\vartheta} + P_{L,\vartheta} e^{-a_{n,R,\vartheta}}, D_{R,\vartheta} + P_{R,\vartheta} e^{-a_{n,L,\vartheta}}]}{[T_{L,\vartheta} + a^2_{n-1,L,\vartheta}, T_{R,\vartheta} + a^2_{n-1,R,\vartheta}]} \\ &= \frac{[D]_\vartheta + [P]_\vartheta \times [e^{-a_n}]_\vartheta}{[T]_\vartheta + [a^2_{n-1}]_\vartheta} = \left[\frac{D + P e^{-a_n}}{T + a^2_{n-1}} \right]_\vartheta, \end{aligned} \tag{3.15}$$

from which we can conclude for arbitrary initial conditions a_{-1}, a_0, the sequence a_n makes certain the solution of the model (1.10).

Additionally, for arbitrary initial conditions a_{-1}, a_0, if fuzzy equation (1.10) also has another positive fuzzy solution $\overline{a_n}$, from which we obtain for $n \in N^+$

$$[\overline{a_n}]_\vartheta = [a_{n,L,\vartheta}, a_{n,R,\vartheta}], \quad \vartheta \in (0,1]. \tag{3.16}$$

Then from (3.6) and (3.14), one has $[a_n]_\vartheta = [\overline{a_n}]_\vartheta, n = 0, 1, 2, \cdots, \vartheta \in (0,1]$, so for $n = 0, 1, \cdots$ we can deduce that $a_n = \overline{a_n}$.

Assuming Case (ii) holds, we omit it's proof since it can be proven similarly based on Case (i). Therefore, Theorem 3.1 is proved. This completes the proof.

3.2 Dynamics of fuzzy difference equation (1.10)

To discuss the dynamical properties of the model (1.10), it is necessary to consider the corresponding ordinary difference equation system. We will examine the two previously mentioned cases, i.e., Case (i) and Case (ii) by using the generalized division of fuzzy numbers.

In the following, we need to present the following several Lemmas, which are essential for our subsequent discussion if Case (i) holds.

Lemma 3.2 *Consider the following system with constant parameters*

$$b_{n+1}=\frac{\alpha+\beta e^{-c_n}}{\gamma+b_{n-1}^2},\quad c_{n+1}=\frac{\alpha_1+\beta_1 e^{-b_n}}{\gamma_1+c_{n-1}^2},\quad n\in N, \tag{3.17}$$

here $\alpha,\alpha_1,\beta,\beta_1,\gamma,\gamma_1,b_{-1},b_0,c_{-1},c_0\in(0,+\infty)$*. If* $h=\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+(\frac{\alpha+\beta}{\gamma})^2}$, $q=\frac{\alpha_1+\beta_1 e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma_1+(\frac{\alpha_1+\beta_1}{\gamma_1})^2}$, $H=\frac{\alpha+\beta}{\gamma}$, $Q=\frac{\alpha_1+\beta_1}{\gamma_1}$*. Then we can obtain the following correct conclusions.*

(i) All positive solutions (b_n,c_n) *of system (3.15) are bounded and persistent.*

(ii) System (3.15) has a unique positive equilibrium $(\overline{b},\overline{c})\in[h,H]\times[q,Q]$,

if

$$\frac{\gamma+3\left(\frac{\alpha+\beta}{\gamma}\right)^2}{\left[\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+\left(\frac{\alpha+\beta}{\gamma}\right)^2}\left(\gamma+\left(\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+\left(\frac{\alpha+\beta}{\gamma}\right)^2}\right)^2\right)-\alpha\right]\ln\left[\frac{\beta}{\frac{\alpha+\beta}{\gamma}\left(\gamma+\left(\frac{\alpha+\beta}{\gamma}\right)^2\right)-\alpha}\right]}$$

$$<\frac{\gamma_1+\left(\ln\left[\frac{\beta}{\frac{\alpha+\beta}{\gamma}\left(\gamma+\left(\frac{\alpha+\beta}{\gamma}\right)^2\right)-\alpha}\right]\right)^2}{\gamma_1+3\left(\ln\left[\frac{\beta}{\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+\left(\frac{\alpha+\beta}{\gamma}\right)^2}\left(\gamma+\left(\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+\left(\frac{\alpha+\beta}{\gamma}\right)^2}\right)^2\right)-\alpha}\right]\right)^2}, \tag{3.18}$$

and

$$\frac{\gamma_1+3\left(\frac{\alpha_1+\beta_1}{\gamma_1}\right)^2}{\left[\frac{\alpha_1+\beta_1 e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma_1+\left(\frac{\alpha_1+\beta_1}{\gamma_1}\right)^2}\left(\gamma_1+\left(\frac{\alpha_1+\beta_1 e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma_1+\left(\frac{\alpha_1+\beta_1}{\gamma_1}\right)^2}\right)^2\right)-\alpha_1\right]\ln\left[\frac{\beta_1}{\frac{\alpha_1+\beta_1}{\gamma_1}\left(\gamma_1+\left(\frac{\alpha_1+\beta_1}{\gamma_1}\right)^2\right)-\alpha_1}\right]}$$

$$<\frac{\gamma+\left(\ln\left[\frac{\beta_1}{\frac{\alpha_1+\beta_1}{\gamma_1}\left(\gamma_1+\left(\frac{\alpha_1+\beta_1}{\gamma_1}\right)^2\right)-\alpha_1}\right]\right)^2}{\gamma+3\left(\ln\left[\frac{\beta_1}{\frac{\alpha_1+\beta_1 e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma_1+\left(\frac{\alpha_1+\beta_1}{\gamma_1}\right)^2}\left(\gamma_1+\left(\frac{\alpha_1+\beta_1 e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma_1+\left(\frac{\alpha_1+\beta_1}{\gamma_1}\right)^2}\right)^2\right)-\alpha_1}\right]\right)^2}. \tag{3.19}$$

Proof. (i) Suppose system (3.15) has arbitrary positive solution (b_n,c_n), according to (3.15), we can deduce that

$$b_n\le\frac{\alpha+\beta}{\gamma}=H,\quad c_n\le\frac{\alpha_1+\beta_1}{\gamma_1}=Q, \tag{3.20}$$

From (3.15) and (3.18), we conclude for $n=0,1,2,\cdots$ that

$$\begin{aligned} b_n&\ge\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+(\frac{\alpha+\beta}{\gamma})^2}=h,\\ c_n&\ge\frac{\alpha_1+\beta_1 e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma_1+(\frac{\alpha_1+\beta_1}{\gamma_1})^2}=q. \end{aligned} \tag{3.21}$$

Then, by combining (3.18) and (3.19), one has

$$h\le b_n\le H,\ q\le c_n\le Q.$$

From the above arguments, it follows that the assertion is true.

(ii) Consider algebraic system

$$b=\frac{\alpha+\beta e^{-c}}{\gamma+b^2},\quad c=\frac{\alpha_1+\beta_1 e^{-b}}{\gamma_1+c^2}. \tag{3.22}$$

Through transformation and simplification, by virtue of (3.20), one has

$$e^{-c}=\frac{b(\gamma+b^2)-\alpha}{\beta},\ e^{-b}=\frac{c(\gamma_1+c^2)-\alpha_1}{\beta_1}. \tag{3.23}$$

Assume that $(b,c)\in(h,H]\times(q,Q]$, based on (3.21), then we can get that

$$c=\ln\left[\frac{\beta}{b(\gamma+b^2)-\alpha}\right],\ b=\ln\left[\frac{\beta_1}{c(\gamma_1+c^2)-\alpha_1}\right]. \tag{3.24}$$

Let c=f(b)=ln $\left[\frac{\beta}{b(\gamma+b^2)-\alpha}\right]$, b $\in(h,H]$, and denoting

$$F(b)=\ln\left[\frac{\beta_1}{f(b)(\gamma_1+f^2(b))-\alpha_1}\right]-b. \tag{3.25}$$

Then

$$f'(b)=\frac{df(b)}{db}=-\frac{\gamma+3b^2}{b(\gamma+b^2)-\alpha}, \tag{3.26}$$

from (3.16), (3.23) and (3.24), we have

$$\begin{aligned}F'(b)&=\frac{dF(b)}{db}=-\frac{f'(b)(\gamma_1+3f^2(b))}{f(b)(\gamma_1+f^2(b))-\alpha_1}-1\\&=\frac{\frac{\gamma+3b^2}{b(\gamma+b^2)-\alpha}\left(\gamma_1+3\left(\ln[\frac{\beta}{b(\gamma+b^2)-\alpha}]\right)^2\right)}{\ln\left[\frac{\beta}{b(\gamma+b^2)-\alpha}\right]\left(\gamma_1+\left(\ln\left[\frac{\beta}{b(\gamma+b^2)-\alpha}\right]\right)^2\right)}-1\\&\le\frac{\frac{\gamma+3H^2}{h(\gamma+h^2)-\alpha}\left(\gamma_1+3\left(\ln[\frac{\beta}{h(\gamma+h^2)-\alpha}]\right)^2\right)}{\ln\left[\frac{\beta}{H(\gamma+H^2)-\alpha}\right]\left(\gamma_1+\left(\ln\left[\frac{\beta}{H(\gamma+H^2)-\alpha}\right]\right)^2\right)}-1\\&=\frac{\frac{\gamma+3\left(\frac{\alpha+\beta}{\gamma}\right)^2}{\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+\left(\frac{\alpha+\beta}{\gamma}\right)^2}\left(\gamma+\left(\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+\left(\frac{\alpha+\beta}{\gamma}\right)^2}\right)^2\right)-\alpha}\left(\gamma_1+3\left(\ln\left[\frac{\beta}{\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+\left(\frac{\alpha+\beta}{\gamma}\right)^2}\left(\gamma+\left(\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+\left(\frac{\alpha+\beta}{\gamma}\right)^2}\right)^2\right)-\alpha}\right]\right)^2\right)}{\ln\left[\frac{\beta}{\frac{\alpha+\beta}{\gamma}\left(\gamma+\left(\frac{\alpha+\beta}{\gamma}\right)^2\right)-\alpha}\right]\left(\gamma_1+\left(\ln\left[\frac{\beta}{\frac{\alpha+\beta}{\gamma}(\gamma+\left(\frac{\alpha+\beta}{\gamma}\right)^2)-\alpha}\right]\right)^2\right)}-1\\&<0.\end{aligned} \tag{3.27}$$

Therefore, the function F(b)=0 is monotonically decreasing in the interval [h,H]. Additionally, from (3.18), (3.19) and (3.23), we can obtain

$$F(h)=\ln\left[\frac{\beta_1}{f(h)(\gamma_1+f^2(h))-\alpha_1}\right]-h>0, \tag{3.28}$$

if and only if

$$\ln\left[\beta_1\left(\ln\left[\frac{\beta}{h(\gamma+h^2)-\alpha}\right]\left(\gamma_1+\left(\ln\left[\frac{\beta}{h(\gamma+h^2)-\alpha}\right]\right)^2\right)-\alpha_1\right)^{-1}\right]>h,$$

$$F(H)=\ln\left[\frac{\beta_1}{f(H)(\gamma_1+f^2(H))-\alpha_1}\right]-H<0, \tag{3.29}$$

if and only if

$$\ln\left[\beta_1\left(\ln\left[\frac{\beta}{H(\gamma+H^2)-\alpha}\right]\left(\gamma_1+\left(\ln\left[\frac{\beta}{H(\gamma+H^2)-\alpha}\right]\right)^2\right)-\alpha_1\right)^{-1}\right]<H.$$

Therefore, $F(b)$ has at least a positive solution $b\in[h,H]$. From which we know F(b)=0 has a unique positive equilibrium $\bar{b}\in[h,H]$. In the same way, we can get that $F(c)=0$ has a unique positive equilibrium $\bar{c}\in[q,Q]$ if inequality (3.17) holds true.

Lemma 3.3 *Consider the constant parameters system (3.15), if the positive equilibrium*

$$(\bar{b},\bar{c})\in\left[h=\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+(\frac{\alpha+\beta}{\gamma})^2},H=\frac{\alpha+\beta}{\gamma}\right]\times\left[q=\frac{\alpha_1+\beta_1 e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma_1+(\frac{\alpha_1+\beta_1}{\gamma_1})^2},Q=\frac{\alpha_1+\beta_1}{\gamma_1}\right].$$

Then we can obtain the following correct conclusions.

(i) If

$$\frac{2(\alpha+\beta)^2}{\gamma^3}+\frac{2(\alpha_1+\beta_1)^2}{\gamma_1^3}+\frac{4(\alpha+\beta)^2(\alpha_1+\beta_1)^2}{\gamma^3\gamma_1^3}+\frac{\beta\beta_1}{\gamma_1\gamma}<1. \tag{3.30}$$

Then the positive equilibrium $(\bar{b},\bar{c})$ of the system (3.15) is locally asymptotically stable.

(ii) The unique positive equilibrium $(\bar{b},\bar{c})$ of the system (3.15) is a global attractor [39, 41]. If

$$\begin{aligned}&\frac{2(\alpha+\beta)}{\gamma}\left(\alpha+\beta e^{-\frac{\alpha_1+\beta_1 e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma_1+(\frac{\alpha_1+\beta_1}{\gamma_1})^2}}\right)\\&\ge\left(\gamma+\left(\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+(\frac{\alpha+\beta}{\gamma})^2}\right)^2\right)^2,\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}&\frac{2(\alpha_1+\beta_1)}{\gamma_1}\left(\alpha_1+\beta_1 e^{-\frac{\alpha+\beta e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma+(\frac{\alpha+\beta}{\gamma})^2}}\right)\\&\ge\left(\gamma_1+\left(\frac{\alpha_1+\beta_1 e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma_1+(\frac{\alpha_1+\beta_1}{\gamma_1})^2}\right)^2\right)^2.\end{aligned} \tag{3.32}$$

(iii) If (i) and (ii) hold true, then $(\bar{b},\bar{c})$ is global asymptotically stable.

Proof. (i) Using statement (ii) of Lemma 3.2, the linearized equation of the system (3.15) around the fixed points $(\bar{b},\bar{c})$ can be expressed below

$$Z_{n+1}=DZ_n, \tag{3.33}$$

where $Z_n=(b_n,b_{n-1},c_n,c_{n-1})^T$, and the Jacobian matrix $D_{(\bar{b},\bar{c})}$ of the system (3.15) around the equilibrium $(\bar{b},\bar{c})$ can be expressed below

$$D_{(\bar{b},\bar{c})}=\begin{bmatrix}0 & -\frac{2\bar{b}(\alpha+\beta e^{-\bar{c}})}{(\gamma+\bar{b}^2)^2} & -\frac{\beta e^{-\bar{c}}}{\gamma+\bar{b}^2} & 0\\ 1 & 0 & 0 & 0\\ -\frac{\beta_1 e^{-\bar{b}}}{\gamma_1+\bar{c}^2} & 0 & 0 & -\frac{2\bar{c}(\alpha_1+\beta_1 e^{-\bar{b}})}{(\gamma_1+\bar{c}^2)^2}\\ 0 & 0 & 1 & 0\end{bmatrix}.$$

Then the characteristic polynomial of (3.15) around

$(\bar{b},\bar{c})$ is given as follows

$$\begin{aligned} f(\lambda) =&\lambda^4 - \left(-\frac{2\bar{b}(\alpha+\beta e^{-\bar{c}})}{(\gamma+\bar{b}^2)^2} - \frac{2\bar{c}(\alpha_1+\beta_1 e^{-\bar{b}})}{(\gamma_1+\bar{c}^2)^2}\right. \\ &\left.+\frac{\beta e^{-\bar{c}}}{\gamma+\bar{b}^2}\frac{\beta_1 e^{-\bar{b}}}{\gamma_1+\bar{c}^2}\right)\lambda^2 \\ &+\frac{2\bar{b}(\alpha+\beta e^{-\bar{c}})}{(\gamma+\bar{b}^2)^2}\frac{2\bar{c}(\alpha_1+\beta_1 e^{-\bar{b}})}{(\gamma_1+\bar{c}^2)^2} \\ &=0. \end{aligned} \tag{3.34}$$

Assuming condition (3.28) holds true, we obtain

$$\begin{aligned} &\left|-\frac{2\bar{b}(\alpha+\beta e^{-\bar{c}})}{(\gamma+\bar{b}^2)^2} - \frac{2\bar{c}(\alpha_1+\beta_1 e^{-\bar{b}})}{(\gamma_1+\bar{c}^2)^2} + \frac{\beta e^{-\bar{c}}}{\gamma+\bar{b}^2}\frac{\beta_1 e^{-\bar{b}}}{\gamma_1+\bar{c}^2}\right| \\ &+\frac{2\bar{b}(\alpha+\beta e^{-\bar{c}})}{(\gamma+\bar{b}^2)^2}\frac{2\bar{c}(\alpha_1+\beta_1 e^{-\bar{b}})}{(\gamma_1+\bar{c}^2)^2} \\ \leq&\frac{2\bar{b}(\alpha+\beta e^{-\bar{c}})}{(\gamma+\bar{b}^2)^2} + \frac{2\bar{c}(\alpha_1+\beta_1 e^{-\bar{b}})}{(\gamma_1+\bar{c}^2)^2} \\ &+\frac{\beta e^{-\bar{c}}}{\gamma+\bar{b}^2}\frac{\beta_1 e^{-\bar{b}}}{\gamma_1+\bar{c}^2} + \frac{2\bar{b}(\alpha+\beta e^{-\bar{c}})}{(\gamma+\bar{b}^2)^2}\frac{2\bar{c}(\alpha_1+\beta_1 e^{-\bar{b}})}{(\gamma_1+\bar{c}^2)^2} \\ \leq&\frac{2\bar{b}(\alpha+\beta)}{\gamma^2} + \frac{2\bar{c}(\alpha_1+\beta_1)}{\gamma_1^2} + \frac{\beta\beta_1}{\gamma_1\gamma} + \frac{2\bar{b}(\alpha+\beta)}{\gamma^2}\frac{2\bar{c}(\alpha_1+\beta_1)}{\gamma_1^2} \\ \leq&\frac{2(\alpha+\beta)^2}{\gamma^3} + \frac{2(\alpha_1+\beta_1)^2}{\gamma_1^3} + \frac{4(\alpha+\beta)^2(\alpha_1+\beta_1)^2}{\gamma^3\gamma_1^3} + \frac{\beta\beta_1}{\gamma_1\gamma} \\ <&1. \end{aligned} \tag{3.35}$$

Based on Theorem 1.2.1 of book [1], every root of (3.32) lie inside the unit disk $|\lambda| < 1$. Thus (i) has proven.

(ii) Let

$$f(b,c) = \frac{\alpha+\beta e^{-c}}{\gamma+b^2},\ g(b,c) = \frac{\alpha_1+\beta_1 e^{-b}}{\gamma_1+c^2}. \tag{3.36}$$

From (3.33), we can deduce that $f'(b)$ and $g'(b)$ are both less than zero. Similarly, $f'(c)$ and $g'(c)$ are also less than zero. Thus we know functions f(b,c) and g(b,c) are all nonincreasing in both b and c.

According to Lemma 2.2. and (3.33), we have

$$\begin{cases} M_1 = \frac{\alpha+\beta e^{-m_2}}{\gamma+m_1^2},\ m_1 = \frac{\alpha+\beta e^{-M_2}}{\gamma+M_1^2} \\ M_2 = \frac{\alpha_1+\beta_1 e^{-m_1}}{\gamma_1+m_2^2},\ m_2 = \frac{\alpha_1+\beta_1 e^{-M_1}}{\gamma_1+M_2^2} \end{cases}. \tag{3.37}$$

Additionally, we can present arguments analogous to those employed in the proof of Theorem 1.16 from [36], let us assume that

$$H \geq M_1 \geq m_1 \geq h,\quad Q \geq M_2 \geq m_2 \geq q. \tag{3.38}$$

From (3.34), we obtain

$$\begin{aligned} m_1 - M_1 =& \frac{\alpha(m_1^2 - M_1^2) + \beta\gamma(e^{-M_2} - e^{-m_2})}{(\gamma+M_1^2)(\gamma+m_1^2)} \\ &+\frac{\beta(m_1^2 e^{-M_2} - M_1^2 e^{-m_2})}{(\gamma+M_1^2)(\gamma+m_1^2)}. \end{aligned}$$

Since $M_2 \geq m_2$, then we can conclude that $e^{-m_2} \geq e^{-M_2}$, so it follows that

$$m_1 - M_1 \leq \frac{2H(\alpha+\beta e^{-q})}{(\gamma+h^2)^2}(m_1 - M_1). \tag{3.39}$$

From (3.36), if (3.29) holds true, we can deduce that $m_1 \geq M_1$. Since from (3.35), i.e., $m_1 \leq M_1$, so we can get $m_1 = M_1$. Of course, in the same way, we can easily show that $m_2 = M_2$ if (3.30) holds true. Thus, based on Lemma 2.2., the unique positive equilibrium $(\bar{b},\bar{c})$ is a global attractor.

(iii) From (i) and (ii), the conclusion is clearly true. Thus, the Lemma 3.3 has been proven.

Theorem 3.2 *Consider FDE (1.10), in which $D, P \in \Re_F^+$, and $a_{-1}, a_0 \in \Re_F^+$. If*

$$\begin{aligned} &\frac{D_{L,\vartheta} + P_{L,\vartheta}e^{-a_{n,R,\vartheta}}}{D_{R,\vartheta} + P_{R,\vartheta}e^{-a_{n,L,\vartheta}}} \leq \frac{T_{L,\vartheta} + a_{n-1,L,\vartheta}^2}{T_{R,\vartheta} + a_{n-1,R,\vartheta}^2}, \\ &\vartheta \in (0,1],\ n = 0,1,2,\cdots. \end{aligned} \tag{3.40}$$

Thus we have the following three correct propositions.

(i) All positive solution a_n of the model (1.10) are bounded and persistent.

(ii) The exponential-type fuzzy model (1.10) has a single positive equilibrium $\overline{a}$, for $\vartheta \in (0,1]$, if

$$\begin{aligned} &\frac{T_{L,\vartheta}+3\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\right)^2}{\left[\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}}}{T_{L,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\right)^2}\left(T_{L,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}}}{T_{L,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\right)^2}\right)^2\right)-D_{L,\vartheta}\right]} \\ &\times\frac{1}{\ln\left[\frac{P_{L,\vartheta}}{\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\left(T_{L,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\right)^2\right)-D_{L,\vartheta}}\right]} \\ &<\frac{T_{R,\vartheta}+\left(\ln\left[\frac{P_{L,\vartheta}}{\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\left(T_{L,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\right)^2\right)-D_{L,\vartheta}}\right]\right)^2}{T_{R,\vartheta}+3\left(\ln\left[\frac{P_{L,\vartheta}}{\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}}}{T_{L,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\right)^2}\left(T_{L,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}}}{T_{L,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\right)^2}\right)^2\right)-D_{L,\vartheta}}\right]\right)^2}, \end{aligned} \tag{3.41}$$

and

$$\frac{T_{R,\vartheta}+3\left(\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}\right)^2}{\left[\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}}}{T_{R,\vartheta}+\left(\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}\right)^2}\left(T_{R,\vartheta}+\left(\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}}}{T_{R,\vartheta}+\left(\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}\right)^2}\right)^2\right)-D_{R,\vartheta}\right]}$$

$$\times\frac{1}{\ln\left[\frac{P_{R,\vartheta}}{\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}\left(T_{R,\vartheta}+\left(\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}\right)^2\right)-D_{R,\vartheta}}\right]}$$

$$<\frac{T_{L,\vartheta}+\left(\ln\left[\frac{P_{R,\vartheta}}{\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}\left(T_{R,\vartheta}+\left(\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}\right)^2\right)-D_{R,\vartheta}}\right]\right)^2}{T_{L,\vartheta}+3\left(\ln\left[\frac{P_{R,\vartheta}}{\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}}}{T_{R,\vartheta}+\left(\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}\right)^2}\left(T_{R,\vartheta}+\left(\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}}}{T_{R,\vartheta}+\left(\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}\right)^2}\right)^2\right)-D_{R,\vartheta}}\right]\right)^2}. \tag{3.42}$$

(iii) All positive solution a_n of the model (1.10) converge to the single equilibrium $\overline{a}$ when n is sufficiently large, for $\vartheta \in (0,1]$, if

$$\frac{2(D_{L,\vartheta}+P_{L,\vartheta})}{T_{L,\vartheta}}\left(D_{L,\vartheta}+P_{L,\vartheta}e^{-\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}}}{T_{R,\vartheta}+\left(\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}\right)^2}}\right)$$

$$\geq\left(T_{L,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}}}{T_{L,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\right)^2}\right)^2\right)^2, \tag{3.43}$$

and

$$\frac{2(D_{R,\vartheta}+P_{R,\vartheta})}{T_{R,\vartheta}}\left(D_{R,\vartheta}+P_{R,\vartheta}e^{-\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}}}{T_{L,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\right)^2}}\right)$$

$$\geq\left(T_{R,\vartheta}+\left(\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}}}{T_{R,\vartheta}+\left(\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}\right)^2}\right)^2\right)^2. \tag{3.44}$$

Proof. (i) Let's assume a_n denotes a positive solution of the model (1.10), if (3.10) and (3.37) holds true, by virtue of Lemma 3.2. From (3.5) it follows that

$$\begin{aligned} a_{n,L,\vartheta} &\geq \frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}}}{T_{L,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\right)^2} \\ &\geq \frac{M_D+M_Pe^{-\frac{N_D+N_P}{M_T}}}{N_T+\left(\frac{N_D+N_P}{M_T}\right)^2} =: \Phi, \\ a_{n,R,\vartheta} &\leq \frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}} \leq \frac{N_D+N_P}{M_T} =: \Phi. \end{aligned} \tag{3.45}$$

Then from (3.42), we get

$$[a_n]_\vartheta \subset [a_{n,L,\vartheta}, a_{n,R,\vartheta}] \subset [\phi, \Phi]. \tag{3.46}$$

From (3.42) and (3.43), we have $\bigcup_{\vartheta\in(0,1]}[a_{n,L,\vartheta}, a_{n,R,\vartheta}] \subset [\phi,\Phi]$, and so $\overline{\bigcup_{\vartheta\in(0,1]}[a_{n,L,\vartheta}, a_{n,R,\vartheta}]} \subset [\phi,\Phi]$. Thus a_n is bounded and persistent. It is clear that (i) has been proven.

(ii) If (3.38) and (3.39) are valid, consider the following system

$$\begin{aligned} a_{L,\vartheta} &= \frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-a_{R,\vartheta}}}{T_{L,\vartheta}+a_{L,\vartheta}^2}, \\ a_{R,\vartheta} &= \frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-a_{L,\vartheta}}}{T_{R,\vartheta}+a_{R,\vartheta}^2}, \quad \vartheta\in(0,1], \end{aligned} \tag{3.47}$$

from (3.10), (3.44) and Lemma 3.2 , we can get that

$$\begin{aligned} &\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}}}{T_{L,\vartheta}+(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}})^2} \leq a_{L,\vartheta} \leq \frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}, \\ &\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}}}{T_{R,\vartheta}+(\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}})^2} \leq a_{R,\vartheta} \leq \frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}. \end{aligned} \tag{3.48}$$

Assume that the model (1.10) exists a positive solution a_n such that $[a_n]_\vartheta = [a_{n,L,\vartheta}, a_{n,R,\vartheta}]$, for $n \in N, \vartheta \in (0,1]$. If (3.37) holds true, we obtain

$$\begin{aligned} a_{n+1,L,\vartheta} &= \frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-a_{n,R,\vartheta}}}{T_{L,\vartheta}+a_{n-1,L,\vartheta}^2}, \\ a_{n+1,R,\vartheta} &= \frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-a_{n,L,\vartheta}}}{T_{R,\vartheta}+a_{n-1,R,\vartheta}^2}, \quad \vartheta\in(0,1]. \end{aligned} \tag{3.49}$$

From (3.38), (3.39) and (3.40), by employing lemma 3.2 and lemma 3.3, we deduce for $n = 0,1,2,\cdots$ that (3.46) exists a unique positive equilibrium $(a_{L,\vartheta}, a_{R,\vartheta}), \vartheta\in(0,1]$, such that

$$\lim_{n\to\infty} a_{n,L,\vartheta} = a_{L,\vartheta}, \ \lim_{n\to\infty} a_{n,R,\vartheta} = a_{R,\vartheta}. \tag{3.50}$$

From (3.42) and (3.45), one has for $0 < \vartheta_1 \leq \vartheta_2 < 1$

$$0 < a_{L,\vartheta_1} \leq a_{L,\vartheta_2} \leq a_{R,\vartheta_2} \leq a_{R,\vartheta_1}. \quad (3.51)$$

Since $D_{L,\vartheta}, P_{L,\vartheta}, T_{L,\vartheta}, D_{R,\vartheta}, P_{R,\vartheta}, T_{R,\vartheta}$, are left continuous, based on (3.44), we know $a_{L,\vartheta}, a_{R,\vartheta}$ are also left continuous.

Therefore, from (3.10) and (3.45), we obtain

$$a_{R,\vartheta} \leq \frac{N_D + N_P}{N_T} =: \eta. \quad (3.52)$$

Thus from (3.45) and (3.49), it follows that

$$a_{L,\vartheta} \geq \frac{M_D + M_P e^{-\eta}}{N_T + \left(\frac{N_D+N_P}{M_T}\right)^2} =: \mu. \quad (3.53)$$

Then (3.49) and (3.50) imply that $[a_{L,\vartheta}, a_{R,\vartheta}] \subset [\mu, \eta]$, and so $\bigcup_{\vartheta\in(0,1]}[a_{L,\vartheta}, a_{R,\vartheta}] \subset [\mu, \eta]$, from which it is sufficient to demonstrate that $\bigcup_{\vartheta\in(0,1]}[a_{L,\vartheta}, a_{R,\vartheta}]$ is compact, and

$$\bigcup_{\vartheta\in(0,1]} [a_{L,\vartheta}, a_{R,\vartheta}] \subset (0, \infty). \quad (3.54)$$

Then according to Definition 2.6., and from (3.44), (3.48), (3.51), taking into account $a_{L,\vartheta}, a_{R,\vartheta}, \vartheta \in (0,1]$ such that a $\in \Re_F^+$ satisfies

$$a = \frac{D + Pe^{-a}}{T + a^2}, \ [a]_\vartheta = [a_{L,\vartheta}, a_{R,\vartheta}], \ \vartheta \in (0,1]. \quad (3.55)$$

Therefore, it implies that $\overline{a}$ is a positive equilibrium of (1.10).

Let's assume that (1.10) has other positive equilibrium $\widetilde{a}$, then it is clear that the functions $\overline{a}_{L,\vartheta} : (0,1] \to (0,\infty), \overline{a}_{R,\vartheta} : (0,1] \to (0,\infty)$, such that

$$\widetilde{a} = \frac{D + Pe^{-\widetilde{a}}}{T + \widetilde{a}^2}, \ [\widetilde{a}]_\vartheta = [\widetilde{a}_{L,\vartheta}, \widetilde{a}_{R,\vartheta}], \ \vartheta \in (0,1]. \quad (3.56)$$

From (3.44) and (3.53), we can deduce that

$$\widetilde{a}_{L,\vartheta} = \frac{D_{L,\vartheta} + P_{L,\vartheta} e^{-\widetilde{a}_{R,\vartheta}}}{T_{L,\vartheta} + \widetilde{a}_{L,\vartheta}^2}, \qquad \widetilde{a}_{R,\vartheta} = \frac{D_{R,\vartheta} + P_{R,\vartheta} e^{-\widetilde{a}_{L,\vartheta}}}{T_{R,\vartheta} + \widetilde{a}_{R,\vartheta}^2}, \quad \vartheta \in (0,1].$$

It implies that $a_{L,\vartheta} = \widetilde{a}_{L,\vartheta}, a_{R,\vartheta} = \widetilde{a}_{R,\vartheta}, \vartheta \in (0,1]$. Hence $\overline{a} = \widetilde{a}$. So it is sufficient to demonstrate that the unique positive equilibrium of fuzzy difference equation (1.10) is $\overline{a}$.

(iii) From (3.47), if relation (3.40) and (3.41) hold true, we have

$$\lim_{n\to\infty} D(a_n, a) = \lim_{n\to\infty} \sup_{\vartheta\in(0,1]} \{\max\{|a_{n,L,\vartheta} - a_{L,\vartheta}|, |a_{n,R,\vartheta} - a_{R,\vartheta}|\}\} = 0. \quad (3.57)$$

As n approach infinity, this implies from (3.54) that we have shown the convergence of positive solution for the model (1.10). This concludes the proof of part (iii). Concluding the proof.

Suppose Case (ii) occurs, that is, $\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-a_{n,L,\vartheta}}}{D_{L,\vartheta}+P_{L,\vartheta}e^{-a_{n,R,\vartheta}}} \leq \frac{T_{R,\vartheta}+a_{n-1,R,\vartheta}^2}{T_{L,\vartheta}+a_{n-1,L,\vartheta}^2}$, for $n = 0, 1, \cdots$, from (3.4), one gets, for $\vartheta \in (0,1]$

$$a_{n+1,L,\vartheta} = \frac{D_{R,\vartheta} + P_{R,\vartheta} e^{-a_{n,L,\vartheta}}}{T_{R,\vartheta} + a_{n-1,R,\vartheta}^2}, \qquad a_{n+1,R,\vartheta} = \frac{D_{L,\vartheta} + P_{L,\vartheta} e^{-a_{n,R,\vartheta}}}{T_{L,\vartheta} + a_{n-1,L,\vartheta}^2}. \quad (3.58)$$

In the following, we need to present two Lemmas, which are essential for our subsequent discussion if Case (ii) holds.

Lemma 3.4 *Consider difference equations*

$$b_{n+1} = \frac{\alpha_1 + \beta_1 e^{-b_n}}{\gamma_1 + c_{n-1}^2}, \qquad c_{n+1} = \frac{\alpha + \beta e^{-c_n}}{\gamma + b_{n-1}^2}, \quad n \in N, \quad (3.59)$$

here $\alpha, \alpha_1, \beta, \beta_1, \gamma, \gamma_1, b_{-1}, b_0, c_{-1}, c_0 \in (0, +\infty)$. *If it adheres to the subsequent relations*

$$H = \frac{\alpha + \beta}{\gamma}, \quad Q = \frac{\alpha_1 + \beta_1}{\gamma_1}, \qquad H_1 = \frac{\alpha + \beta e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma + (\frac{\alpha_1+\beta_1}{\gamma_1})^2}, \quad Q_1 = \frac{\alpha_1 + \beta_1 e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma_1 + (\frac{\alpha+\beta}{\gamma})^2}. \quad (3.60)$$

Thus, the propositions below are valid.

(i) All positive solutions (b_n, c_n) *of system (3.56) are bounded and persistent.*

(ii) System (3.56) has a unique positive equilibrium $(\overline{b}, \overline{c}) \in [Q_1, Q] \times [H_1, H]$, *if*

$$\left(\beta e^{-H_1}(H+1) + \alpha\right)\left(\beta_1 e^{-\Upsilon}(\Psi + 1) + \alpha_1\right) < 4H_1^2 \Upsilon^3 \sqrt{\frac{\alpha_1 + \beta_1 e^{-\Upsilon}}{\Upsilon}} - \gamma_1, \quad (3.61)$$

and

$$\begin{aligned}&\left(\beta_1 e^{-Q_1}(Q+1)+\alpha_1\right)\left(\beta e^{-\Upsilon_1}(\Psi_1+1)+\alpha\right)\\&< 4Q_1^2\Upsilon_1^3\sqrt{\frac{\alpha+\beta e^{-\Upsilon_1}}{\Upsilon_1}}-\gamma,\end{aligned} \tag{3.62}$$

where

$$\Upsilon=\sqrt{\frac{\alpha+\beta e^{-H}}{H}}-\gamma,\ \Psi=\sqrt{\frac{\alpha+\beta e^{-H_1}}{H_1}}-\gamma,$$

$$\Upsilon_1=\sqrt{\frac{\alpha_1+\beta_1 e^{-Q}}{Q}}-\gamma_1,\ \Psi_1=\sqrt{\frac{\alpha_1+\beta_1 e^{-Q_1}}{Q_1}}-\gamma_1. \tag{3.63}$$

Proof. (i) Let (b_n, c_n) denote an arbitrary positive solution of (3.56), then we can easily deduce follows from (3.56) and (3.57) that

$$b_n\le\frac{\alpha_1+\beta_1}{\gamma_1}=Q,\quad c_n\le\frac{\alpha+\beta}{\gamma}=H, \tag{3.64}$$

From (3.56), (3.57) and (3.61), we obtain

$$b_n\ge\frac{\alpha_1+\beta_1 e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma_1+(\frac{\alpha+\beta}{\gamma})^2}=Q_1, \tag{3.65}$$

$$c_n\ge\frac{\alpha+\beta e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma+(\frac{\alpha_1+\beta_1}{\gamma_1})^2}=H_1. \tag{3.66}$$

Then, combining (3.61) with (3.62), one obtains

$$Q_1\le b_n\le Q,\ H_1\le c_n\le H.$$

It is clear from arguments above that the positive solutions b_n, c_n are bounded and persists. Thus the assertion is true.

(ii) Consider the following algebraic equations

$$b=\frac{\alpha_1+\beta_1 e^{-b}}{\gamma_1+c^2},\quad c=\frac{\alpha+\beta e^{-c}}{\gamma+b^2}. \tag{3.67}$$

Transform and simplify the system (3.63), we can deduce that

$$c=\sqrt{\frac{\alpha_1+\beta_1 e^{-b}}{b}}-\gamma_1,\ b=\sqrt{\frac{\alpha+\beta e^{-c}}{c}}-\gamma. \tag{3.68}$$

From (3.64), and setting

$$F(c)=\sqrt{\frac{\alpha_1+\beta_1 e^{-f(c)}}{f(c)}}-\gamma_1-c, \tag{3.69}$$

where

$$\begin{aligned}&f(c)=b=\sqrt{\frac{\alpha+\beta e^{-c}}{c}}-\gamma,\\&c\in\left[H_1=\frac{\alpha+\beta e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma+(\frac{\alpha_1+\beta_1}{\gamma_1})^2},\ \frac{\alpha+\beta}{\gamma}=H\right].\end{aligned} \tag{3.70}$$

From (3.65) and (3.66), we have

$$\begin{aligned}F(c):&\left[H_1=\frac{\alpha+\beta e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma+(\frac{\alpha_1+\beta_1}{\gamma_1})^2},\ \frac{\alpha+\beta}{\gamma}=H\right]\\&\mapsto\left[H_1=\frac{\alpha+\beta e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma+(\frac{\alpha_1+\beta_1}{\gamma_1})^2},\ \frac{\alpha+\beta}{\gamma}=H\right].\end{aligned}$$

By virtue of (3.66), we have

$$f'(c)=-\frac{\beta e^{-c}(c+1)+\alpha}{2c^2}\left(\frac{\alpha+\beta e^{-c}}{c}-\gamma\right)^{-\frac{1}{2}}. \tag{3.71}$$

From (3.65), (3.66) and (3.67), one has

$$F'(c)=\tfrac{\beta e^{-c}(c+1)+\alpha}{2c^2}\left(\tfrac{\alpha+\beta e^{-c}}{c}-\gamma\right)^{-\frac{1}{2}}\times$$

$$\begin{aligned}&\frac{\beta_1 e^{-\sqrt{\frac{\alpha+\beta e^{-c}}{c}}-\gamma}\left(\sqrt{\frac{\alpha+\beta e^{-c}}{c}}-\gamma+1\right)+\alpha_1}{2\left(\frac{\alpha+\beta e^{-c}}{c}-\gamma\right)}\\&\times\left(\frac{\alpha_1+\beta_1 e^{-\sqrt{\frac{\alpha+\beta e^{-c}}{c}}-\gamma}}{\sqrt{\frac{\alpha+\beta e^{-c}}{c}}-\gamma}-\gamma_1\right)^{-\frac{1}{2}}-1.\end{aligned} \tag{3.72}$$

According to (3.68), (3.60) and since $c\in\left[H_1=\frac{\alpha+\beta e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma+(\frac{\alpha_1+\beta_1}{\gamma_1})^2},\ \frac{\alpha+\beta}{\gamma}=H\right]$, and (3.58) holds true, so we can conclude that

$$F'(c) \leq \frac{\beta e^{-H_1}(H+1)+\alpha}{2H_1^2}\left(\frac{\alpha+\beta e^{-H}}{H}-\gamma\right)^{-\frac{1}{2}} \times$$

$$\frac{\beta_1 e^{-\sqrt{\frac{\alpha+\beta e^{-H}}{H}-\gamma}}\left(\sqrt{\frac{\alpha+\beta e^{-H_1}}{H_1}-\gamma}+1\right)+\alpha_1}{2\left(\frac{\alpha+\beta e^{-H}}{H}-\gamma\right)}$$

$$\times\left(\frac{\alpha_1+\beta_1 e^{-\sqrt{\frac{\alpha+\beta e^{-H}}{H}-\gamma}}}{\sqrt{\frac{\alpha+\beta e^{-H}}{H}-\gamma}}-\gamma_1\right)^{-\frac{1}{2}}-1$$

$$=\frac{\beta e^{-H_1}(H+1)+\alpha}{2H_1^2}\left(\Upsilon^2\right)^{-\frac{1}{2}} \times \frac{\beta_1 e^{-\Upsilon}(\Psi+1)+\alpha_1}{2\Upsilon^2}$$

$$\times\left(\frac{\alpha_1+\beta_1 e^{-\Upsilon}}{\Upsilon}-\gamma_1\right)^{-\frac{1}{2}}-1$$

$$=\frac{\left(\beta e^{-H_1}(H+1)+\alpha\right)\left(\beta_1 e^{-\Upsilon}(\Psi+1)+\alpha_1\right)}{4H_1^2\Upsilon^3}$$

$$\times\left(\frac{\alpha_1+\beta_1 e^{-\Upsilon}}{\Upsilon}-\gamma_1\right)^{-\frac{1}{2}}-1<0. \tag{3.73}$$

Therefore, the function F(c)=0 is monotonically decreasing in the interval $[H_1, H]$. Moreover, from (3.65), we have

$$F(H_1)=\sqrt{\frac{\alpha_1+\beta_1 e^{-f(H_1)}}{f(H_1)}-\gamma_1}-H_1>0, \tag{3.74}$$

if and only if

$$\left(\left(\alpha_1+\beta_1 e^{-\sqrt{\frac{\alpha+\beta e^{-H_1}}{H_1}-\gamma}}\right)\left(\frac{\alpha+\beta e^{-H_1}}{H_1}-\gamma\right)\right)^{-\frac{1}{2}}-\gamma_1\Bigg)^{-\frac{1}{2}}>H_1,$$

$$F(H)=\sqrt{\frac{\alpha_1+\beta_1 e^{-f(H)}}{f(H)}-\gamma_1}-H<0, \tag{3.75}$$

if and only if

$$\left(\left(\alpha_1+\beta_1 e^{-\sqrt{\frac{\alpha+\beta e^{-H}}{H}-\gamma}}\right)\left(\frac{\alpha+\beta e^{-H}}{H}-\gamma\right)\right)^{-\frac{1}{2}}-\gamma_1\Bigg)^{-\frac{1}{2}}<H.$$

Therefore, F(c)=0 exists at least one positive solution in the interval $\left[H_1=\frac{\alpha+\beta e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma+(\frac{\alpha_1+\beta_1}{\gamma_1})^2}, \frac{\alpha+\beta}{\gamma}=H\right]$. From which we know F(c)=0 has a single positive equilibrium $\bar{c} \in [H_1, H]$. In the same way, we can get that F(b)=0 exists a single positive equilibrium $\bar{b} \in [Q_1, Q]$ if inequality (3.59) holds true.

Lemma 3.5 *Let's consider the constant parameters system (3.56), let the equilibrium*

$$(\bar{b},\bar{c}) \in \left[H_1=\frac{\alpha+\beta e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma+(\frac{\alpha_1+\beta_1}{\gamma_1})^2}, \frac{\alpha+\beta}{\gamma}=H\right] \times\left[Q_1=\frac{\alpha_1+\beta_1 e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma_1+(\frac{\alpha+\beta}{\gamma})^2}, \frac{\alpha_1+\beta_1}{\gamma_1}=Q\right].$$

Thus the following propositions are true.

(i) If

$$\frac{\beta_1 e^{-H_1}}{\gamma_1+Q_1^2}+\frac{\beta e^{-Q_1}}{\gamma+H_1^2}+\frac{\beta\beta_1 e^{-H_1-Q_1}}{\left(\gamma+H_1^2\right)\left(\gamma_1+Q_1^2\right)} +\frac{4HQ\left(\alpha+\beta e^{-Q_1}\right)\left(\alpha_1+\beta_1 e^{-H_1}\right)}{\left(\gamma+H_1^2\right)^2\left(\gamma_1+Q_1^2\right)^2}<1. \tag{3.76}$$

Then the equilibrium of (3.56) is locally asymptotically stable.

(ii) The equilibrium $(\bar{b},\bar{c})$ is global asymptotically stable if it satisfies the following conditions

$$\alpha_1+\beta_1 e^{-Q_1}<\bar{b}\left(\gamma_1+H_1^2\right),\ \alpha+\beta e^{-H_1}<\bar{c}\left(\gamma+Q_1^2\right). \tag{3.77}$$

Proof. (i) Based on (3.56), we assume $(\bar{b},\bar{c})$ to be the unique positive equilibrium, so the linearized equation of system (3.56) around the equilibrium $(\bar{b},\bar{c})$ is given by

$$Z_{n+1}=G_{(\bar{b},\bar{c})}Z_n, \tag{3.78}$$

where $Z_n=(b_n,b_{n-1},c_n,c_{n-1})^T$, and the Jacobian matrix $G_{(\bar{b},\bar{c})}$ of (3.56) be given below

$$G_{(\bar{b},\bar{c})}=\begin{pmatrix} A_1 & 0 & 0 & A_2\\ 1 & 0 & 0 & 0\\ 0 & B_1 & B_2 & 0\\ 0 & 0 & 1 & 0\end{pmatrix},$$

where $A_1=-\frac{\beta_1 e^{-\bar{b}}}{\gamma_1+\bar{c}^2}$, $A_2=-\frac{2\bar{c}\left(\alpha_1+\beta_1 e^{-\bar{b}}\right)}{\left(\gamma_1+\bar{c}^2\right)^2}$, $B_1=-\frac{2\bar{b}\left(\alpha+\beta e^{-\bar{c}}\right)}{\left(\gamma+\bar{b}^2\right)^2}$, $B_2=-\frac{\beta e^{-\bar{c}}}{\gamma+\bar{b}^2}$.

The characteristic polynomial of $G_{(\bar{b},\bar{c})}$ about the fixed point $(\bar{b},\bar{c})$ is given by

$$\lambda^4+\eta_1\lambda^3+\eta_2\lambda^2-\eta_3=0, \tag{3.79}$$

where $\eta_1 = -(A_1 + B_2),\ \eta_2 = A_1B_2,\ \eta_3 = A_2B_1$. If relation (3.71) holds true, we have

$$\begin{aligned}\sum_{i=1}^{3}|\eta_i| &= \frac{\beta_1 e^{-\bar{b}}}{\gamma_1+\bar{c}^2} + \frac{\beta e^{-\bar{c}}}{\gamma+\bar{b}^2} + \frac{\beta_1 e^{-\bar{b}}}{\gamma_1+\bar{c}^2} \times \frac{\beta e^{-\bar{c}}}{\gamma+\bar{b}^2} \\ &+ \frac{2\bar{c}\left(\alpha_1+\beta_1 e^{-\bar{b}}\right)}{\left(\gamma_1+\bar{c}^2\right)^2} \times \frac{2\bar{b}\left(\alpha+\beta e^{-\bar{c}}\right)}{\left(\gamma+\bar{b}^2\right)^2} \\ &\le \frac{\beta_1 e^{-H_1}}{\gamma_1+Q_1^2} + \frac{\beta e^{-Q_1}}{\gamma+H_1^2} + \frac{\beta\beta_1 e^{-H_1-Q_1}}{\left(\gamma+H_1^2\right)\left(\gamma_1+Q_1^2\right)} \\ &+ \frac{4HQ\left(\alpha+\beta e^{-Q_1}\right)\left(\alpha_1+\beta_1 e^{-H_1}\right)}{\left(\gamma+H_1^2\right)^2\left(\gamma_1+Q_1^2\right)^2} \\ &< 1.\end{aligned} \tag{3.80}$$

Hence, from Remark 1.3.1 of book [8] and inequality (3.75), we can deduce that the modules $|\lambda_i| < 1$ of the characteristic equation (3.74). we can conclude that the equilibrium $(\bar{b}, \bar{c})$ is locally asymptotically stable.

(ii) Let Lyapunov function

$$\Gamma_n = \bar{b}\left(\frac{b_n}{\bar{b}} - \ln\frac{b_n}{\bar{b}} - 1\right) + \bar{c}\left(\frac{c_n}{\bar{c}} - \ln\frac{c_n}{\bar{c}} - 1\right). \tag{3.81}$$

Since $b - \ln b - 1 \ge 0, \forall b > 0$, then Γ_n is a nonnegative real number. Furthermore, we have

$$\begin{aligned}-\ln\frac{b_{n+1}}{b_n} &= \ln\frac{b_n}{b_{n+1}} = \ln\left(1-\left(1-\frac{b_n}{b_{n+1}}\right)\right) \\ &\le -\left(1-\frac{b_n}{b_{n+1}}\right) \le -\frac{b_{n+1}-b_n}{b_{n+1}},\end{aligned} \tag{3.82}$$

$$\begin{aligned}-\ln\frac{c_{n+1}}{c_n} &= \ln\frac{c_n}{c_{n+1}} = \ln\left(1-\left(1-\frac{c_n}{c_{n+1}}\right)\right) \\ &\le -\left(1-\frac{c_n}{c_{n+1}}\right) \le -\frac{c_{n+1}-c_n}{c_{n+1}}.\end{aligned} \tag{3.83}$$

Suppose that (3.72) holds true, from (3.76), we obtain

$$\begin{aligned}&\Gamma_{n+1} - \Gamma_n = \bar{b}\left(\frac{b_{n+1}}{\bar{b}} - \ln\frac{b_{n+1}}{\bar{b}} - 1\right) \\ &+ \bar{c}\left(\frac{c_{n+1}}{\bar{c}} - \ln\frac{c_{n+1}}{\bar{c}} - 1\right) \\ &- \bar{b}\left(\frac{b_n}{\bar{b}} - \ln\frac{b_n}{\bar{b}} - 1\right) - \bar{c}\left(\frac{c_n}{\bar{c}} - \ln\frac{c_n}{\bar{c}} - 1\right) \\ =&(b_{n+1}-b_n) + (c_{n+1}-c_n) - \bar{b}\ln\frac{b_{n+1}}{b_n} - \bar{c}\ln\frac{c_{n+1}}{c_n} \\ \le&(b_{n+1}-b_n) + (c_{n+1}-c_n) - \bar{b}\frac{b_{n+1}-b_n}{b_{n+1}} - \bar{c}\frac{c_{n+1}-c_n}{c_{n+1}} \\ =&(b_{n+1}-b_n)\left(1-\frac{\bar{b}}{b_{n+1}}\right) + (c_{n+1}-c_n)\left(1-\frac{\bar{c}}{c_{n+1}}\right) \\ =&(b_{n+1}-b_n)\left(1-\frac{\bar{b}\left(\gamma_1+c_{n-1}^2\right)}{\alpha_1+\beta_1 e^{-b_n}}\right) \\ &+ (c_{n+1}-c_n)\left(1-\frac{\bar{c}\left(\gamma+b_{n-1}^2\right)}{\alpha+\beta e^{-c_n}}\right) \\ \le&(H-H_1)\left(\frac{\alpha_1+\beta_1 e^{-Q_1} - \bar{b}\left(\gamma_1+H_1^2\right)}{\alpha_1+\beta_1 e^{-Q_1}}\right) \\ &+ (Q-Q_1)\left(\frac{\alpha+\beta e^{-H_1} - \bar{c}\left(\gamma+Q_1^2\right)}{\alpha+\beta e^{-H_1}}\right) \\ \le&0.\end{aligned} \tag{3.84}$$

This implies that Γ_n is monotonically decreasing sequence of nonnegative real numbers, namely, $\lim_{n\to\infty}\Gamma_n \ge 0$, it is clear that $\lim_{n\to\infty}(\Gamma_{n+1}-\Gamma_n) = 0$, so we can deduce that $\lim_{n\to\infty} b_n = \bar{b}$ and $\lim_{n\to\infty} c_n = \bar{c}$, by virtue of the statements of (i), the unique positive equilibrium $(\bar{b}, \bar{c}) \in [H_1, H] \times [Q_1, Q]$ is globally asymptotically stable. Thus, the Lemma 3.5 has been proven.

Theorem 3.3 *Consider FDE (1.10), in which $D, P \in \Re_F^+$, and $x_1, x_0 \in \Re_F^+$. If*

$$\begin{aligned}&\frac{D_{R,\vartheta} + P_{R,\vartheta}e^{-a_{n,L,\vartheta}}}{D_{L,\vartheta} + P_{L,\vartheta}e^{-a_{n,R,\vartheta}}} \le \frac{T_{R,\vartheta} + a_{n-1,R,\vartheta}^2}{T_{L,\vartheta} + a_{n-1,L,\vartheta}^2}, \\ &\vartheta \in (0,1],\ n = 0,1,2,\cdots.\end{aligned} \tag{3.85}$$

Thus we have the following several correct propositions.

(i) All positive solution a_n of the fuzzy model (1.10) are bounded and persistent.

(ii) the fuzzy model (1.10) exists a unique positive

equilibrium $\bar{a}$. If

$$\left(P_{L,\vartheta}e^{-H_{1,\vartheta}}(H_\vartheta+1)+D_{L,\vartheta}\right)\left(P_{R,\vartheta}e^{-\Upsilon_\vartheta}(\Psi_\vartheta+1)+D_{R,\vartheta}\right) < 4H_{1,\vartheta}^2\Upsilon_\vartheta^3\sqrt{\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-\Upsilon_\vartheta}}{\Upsilon_\vartheta}}-T_{R,\vartheta}, \quad (3.86)$$

and

$$\left(P_{R,\vartheta}e^{-Q_{1,\vartheta}}(Q_\vartheta+1)+D_{R,\vartheta}\right)\left(P_{L,\vartheta}e^{-\Upsilon_{1,\vartheta}}(\Psi_{1,\vartheta}+1)+D_{L,\vartheta}\right) < 4Q_{1,\vartheta}^2\Upsilon_{1,\vartheta}^3\sqrt{\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-\Upsilon_{1,\vartheta}}}{\Upsilon_{1,\vartheta}}}-T_{L,\vartheta}, \quad (3.87)$$

where

$$\Upsilon_\vartheta=\sqrt{\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-H_\vartheta}}{H_\vartheta}}-T_{L,\vartheta},$$

$$\Psi_\vartheta=\sqrt{\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-H_{1,\vartheta}}}{H_{1,\vartheta}}}-T_{L,\vartheta}.$$

$$\Upsilon_{1,\vartheta}=\sqrt{\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-Q_\vartheta}}{Q_\vartheta}}-T_{R,\vartheta},$$

$$\Psi_{1,\vartheta}=\sqrt{\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-Q_{1,\vartheta}}}{Q_{1,\vartheta}}}-T_{R,\vartheta}.$$

$$H_\vartheta=\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}},\ H_{1,\vartheta}=\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}}}{T_{L,\vartheta}+(\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}})^2},$$

$$Q_\vartheta=\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}},\ Q_{1,\vartheta}=\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}}}{T_{R,\vartheta}+(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}})^2}.$$

(iii) All positive solution a_n of the fuzzy model (1.10) tend to unique equilibrium $\bar{a}$ when n is sufficiently large. If

$$\begin{aligned} D_{R,\vartheta}+P_{R,\vartheta}e^{-Q_{1,\vartheta}} &< \bar{b}\left(T_{R,\vartheta}+H_{1,\vartheta}^2\right), \\ D_{L,\vartheta}+P_{L,\vartheta}e^{-H_{1,\vartheta}} &< \bar{c}\left(T_{L,\vartheta}+Q_{1,\vartheta}^2\right). \end{aligned} \quad (3.88)$$

Here

$$H_{1,\vartheta}=\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}}}{T_{L,\vartheta}+(\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}})^2},$$

$$Q_{1,\vartheta}=\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}}}{T_{R,\vartheta}+(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}})^2}.$$

Proof. (i) Suppose that (3.80) and Lemma 3.4 are valid. Let a_n denotes a positive fuzzy solution of (1.10), and from (3.55) and (3.10), we have

$$\begin{aligned} a_{n,L,\vartheta} &\geq \frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-\frac{D_{R,\vartheta}+P_{R,\vartheta}}{T_{R,\vartheta}}}}{T_{R,\vartheta}+\left(\frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\right)^2} \\ &\geq \frac{M_D+M_Pe^{-\frac{N_D+N_P}{M_T}}}{N_T+\left(\frac{N_D+N_P}{M_T}\right)^2}=:\Phi, \qquad (3.89) \\ a_{n,R,\vartheta} &\leq \frac{D_{L,\vartheta}+P_{L,\vartheta}}{T_{L,\vartheta}}\leq\frac{N_D+N_P}{M_T}=:\Phi. \end{aligned}$$

Then from (3.84), we get

$$[a_n]_\vartheta\subset[a_{n,L,\vartheta},a_{n,R,\vartheta}]\subset[\phi,\Phi]. \quad (3.90)$$

From (3.84) and (3.85), we have $\bigcup_{\vartheta\in(0,1]}[a_{n,L,\vartheta},a_{n,R,\vartheta}]\subset[\phi,\Phi]$, and so $\overline{\bigcup_{\vartheta\in(0,1]}[a_{n,L,\vartheta},a_{n,R,\vartheta}]}\subset[\phi,\Phi]$. Thus a_n is bounded and persistent.

(ii) Since the proof of Theorem 3.3. is similar to that of (ii) of Theorem 3.2., we have omitted the proof here.

(iii) By virtue of (iii) of Theorem 3.2., (iii) is proved. Concluding the proof.

In the next section, the non-oscillatory properties of positive solutions for the model (1.10) are discussed. Additionally, since the proof of case (i) is the same as that of case (ii), we will only consider the proof of case (i) around equilibrium $(\bar{b},\bar{c})$. For the proof of case (ii), we can rely on case (i) to proof it in a similar way, in view of this, so we omit the proof of case (ii).

4 Nonoscillatory behavior analysis of the FDE model (1.10)

For this part, our objective is to analyze the non-oscillatory properties of case (i) about equilibrium $(\bar{b},\bar{c})$, in order to study whether it is non-oscillatory around equilibrium $(\bar{b},\bar{c})$ of case (i). The following definition and lemma are needed.

Let $\{u_n\},\{v_n\}$ are sequences of positive numbers, then the sequence (u_n,v_n) is non-oscillatory about (u,v), here u,v belong to R^+, if there exists $t_0\in N$ and $p,q\in N^+$, for $p,q\geq\tau_0$ such that

$$(u_p-u)(u_q-u)\geq 0,(v_p-v)(v_q-v)\geq 0. \quad (4.1)$$

Next, let's recall the fuzzy analog of non-oscillatory define (see [31] and the reference therein).

Suppose that $\{y_n\}$ represents sequence of positive fuzzy numbers and y denotes $\Re^+_F$. Thus the fuzzy sequence $\{y_n\}$ is non-oscillatory at y, if $\exists\ \tau_0, p, q \in N^+$, for $p, q \geq \tau_0$, satisfying

(a) $Min\{y_p, y\} = y_p$ *and* $Min\{y_q, y\} = y_q$,

(b) $Min\{y_p, y\} = y$ *and* $Min\{y_q, y\} = y$.

Lemma 4.1. *Let's consider the system of crisp DEs (3.15), in which initial conditions b_{-1}, c_{-1}, b_0, c_0 and parameters $\alpha, \alpha_1, \beta, \beta_1, \gamma, \gamma_1$ are all positive real numbers. Then solution (b_n, c_n) of (3.15) is not oscillatory around equilibrium $(\bar{b}, \bar{c})$ of (3.15) if and only if one of the relations are valid below.*

$$\begin{aligned}&(i)\ \ c_0 > \bar{c},\ b_0 < \bar{b},\ b_{-1} > \bar{b},\ c_{-1} < \bar{c},\\ &or\ \ (ii)\ \ c_0 < \bar{c},\ b_0 > \bar{b},\ b_{-1} < \bar{b},\ c_{-1} > \bar{c}.\end{aligned} \tag{4.2}$$

Proof. Let us assume the system (3.15) has positive solution (b_n, c_n), if case (i) of (4.1) holds true. It follows from system (3.15) that

$$\begin{aligned} b_1 &= \frac{\alpha + \beta e^{-c_0}}{\gamma + b_{-1}^2} < \frac{\alpha + \beta e^{-\bar{c}}}{\gamma + \bar{b}^2} = \bar{b},\\ c_1 &= \frac{\alpha_1 + \beta_1 e^{-b_0}}{\gamma_1 + c_{-1}^2} > \frac{\alpha_1 + \beta_1 e^{-\bar{b}}}{\gamma_1 + \bar{c}^2} = \bar{c}.\end{aligned} \tag{4.3}$$

by using mathematical induction, one has for $n = 0, 1, \cdots$,

$$b_n < \bar{b},\ \ c_n > \bar{c}. \tag{4.4}$$

Likewise, suppose that case (ii) of (4.2) is valid, it becomes evident that for $n \in N$,

$$b_n > \bar{b},\ \ c_n < \bar{c}. \tag{4.5}$$

So from (4.4) and (4.5), it is sufficient to demonstrate that positive solution (b_n, c_n) isn't oscillatory around equilibrium $(\bar{b}, \bar{c})$ of (3.15). The lemma is proved in this way.

Theorem 4.1. *Let's consider the fuzzy model (1.10), where initial conditions a_{-1}, a_0 and parameters D, P, T are all positive fuzzy numbers. If*

$$\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-a_{n,R,\vartheta}}}{D_{R,\vartheta}+P_{R,\vartheta}e^{-a_{n,L,\vartheta}}} \leq \frac{T_{L,\vartheta}+a^2_{n-1,L,\vartheta}}{T_{R,\vartheta}+a^2_{n-1,R,\vartheta}},\ \vartheta \in (0,1],\ n \in N, \tag{4.6}$$

is satisfied. Then solution a_n of fuzzy system (1.10) exhibits non-oscillation at equilibrium $\bar{a}$ if and only if the relation (i) or relation (ii) is satisfied. Suppose that the relations is

$$\begin{cases} (i)\ \ a_{0,R,\vartheta} > a_{R,\vartheta},\ a_{0,L,\vartheta} < a_{L,\vartheta},\\ \quad a_{-1,L,\vartheta} > a_{L,\vartheta},\ a_{-1,R,\vartheta} < a_{R,\vartheta},\\ or\\ (ii)\ \ a_{0,R,\vartheta} < a_{R,\vartheta},\ a_{0,L,\vartheta} > a_{L,\vartheta},\\ \quad a_{-1,L,\vartheta} < a_{L,\vartheta},\ a_{-1,R,\vartheta} > a_{R,\vartheta}.\end{cases} \tag{4.7}$$

Proof. Let us assume conditions (4.7) hold true, then from (3.5), (3.46) and Lemma 4.1, for $n_0 = 0$, any u, v ≥ 0, $\vartheta \in (0, 1]$ such that

$$\begin{aligned}(a_{u,L,\vartheta} - a_{L,\vartheta})(a_{v,L,\vartheta} - a_{L,\vartheta}) > 0 \quad \text{and}\\ (a_{u,R,\vartheta} - a_{R,\vartheta})(a_{v,R,\vartheta} - a_{R,\vartheta}) > 0.\end{aligned} \tag{4.8}$$

Hence, inequalities (4.8) are equivalent to

$$\begin{cases}[\min\{a_{u,L,\vartheta}, a_{L,\vartheta}\}, \min\{a_{u,R,\vartheta}, a_{R,\vartheta}\}] = [a_{u,L,\vartheta}, a_{u,R,\vartheta}],\\ [\min\{a_{v,L,\vartheta}, a_{L,\vartheta}\}, \min\{a_{v,R,\vartheta}, a_{R,\vartheta}\}] = [a_{v,L,\vartheta}, a_{v,R,\vartheta}].\end{cases} \tag{4.9}$$

or

$$\begin{cases}[\min\{a_{u,L,\vartheta}, a_{L,\vartheta}\}, \min\{a_{u,R,\vartheta}, a_{R,\vartheta}\}] = [a_{L,\vartheta}, a_{R,\vartheta}],\\ [\min\{a_{v,L,\vartheta}, a_{L,\vartheta}\}, \min\{a_{v,R,\vartheta}, a_{R,\vartheta}\}] = [a_{L,\vartheta}, a_{R,\vartheta}].\end{cases} \tag{4.10}$$

According to (4.9) and (4.10), one has

$$\begin{cases}\min\{a_u, \bar{a}\} = a_u,\ \ \min\{a_v, \bar{a}\} = a_v,\\ \qquad\qquad \text{or}\\ \min\{a_u, \bar{a}\} = \bar{a},\ \ \min\{a_v, \bar{a}\} = \bar{a}.\end{cases} \tag{4.11}$$

From which, it is clear that the solution a_n isn't oscillatory at equilibrium $\bar{a}$. In addition, if case (ii) holds true, we can prove it in a similar manner to case (i).

5 Numerical examples

In this part, to validate the effectiveness of our results, we also give several numerical examples to support our theoretical findings.

Example 5.1 To take into account the following exponential-type FDE

$$a_{n+1} = \frac{D + Pe^{-a_n}}{T + a^2_{n-1}},\ \ n \in N, \tag{5.1}$$

we take D, P, T and a_{-1}, a_0 denote triangular fuzzy numbers, such that

$$D(t)=\begin{cases}2t-6, & 3\le t\le 3.5\\ -2t+8, & 3.5\le t\le 4\end{cases},\qquad a_{-1}(t)=\begin{cases}4t-2, & 0.5\le t\le 0.75\\ -4t+4, & 0.75\le t\le 1\end{cases}\tag{5.2}$$

$$P(t)=\begin{cases}t-12, & 12\le t\le 13\\ -t+14, & 13\le t\le 14\end{cases},\qquad a_0(t)=\begin{cases}t-5, & 5\le t\le 6\\ -t+7, & 6\le t\le 7\end{cases}\tag{5.3}$$

$$T(t)=\begin{cases}\frac{2}{3}t-6, & 9\le t\le 10.5\\ -\frac{2}{3}t+8, & 10.5\le t\le 12\end{cases}.\tag{5.4}$$

From (5.2), we get

$$\begin{aligned}{}_{\vartheta} &= [3+0.5\vartheta, 4-0.5\vartheta],\\ [a_{-1}]_{\vartheta} &= [0.5+0.25\vartheta, 1-0.25\vartheta],\\ &\vartheta\in(0,1].\end{aligned}\tag{5.5}$$

From (5.3) and (5.4), we have

$$\begin{aligned}{}_{\vartheta} &= [12+\vartheta, 14-\vartheta],\\ [a_0]_{\vartheta} &= [5+\vartheta, 7-\vartheta],\\ [T]_{\vartheta} &= [9+1.5\vartheta, 12-1.5\vartheta],\\ &\vartheta\in(0,1].\end{aligned}\tag{5.6}$$

Therefore, it follows that

$$\overline{\bigcup_{\vartheta\in(0,1]}[D]_{\vartheta}}=[3,4],\quad \overline{\bigcup_{\vartheta\in(0,1]}[P]_{\vartheta}}=[12,14],\tag{5.7}$$

$$\overline{\bigcup_{\vartheta\in(0,1]}[T]_{\vartheta}}=[9,12],\ \overline{\bigcup_{\vartheta\in(0,1]}[a_{-1}]_{\vartheta}}=[0.5,1],\ \overline{\bigcup_{\vartheta\in(0,1]}[a_0]_{\vartheta}}=[5,7].\tag{5.8}$$

From (5.1), it results in two second-order exponential-type difference equations with parameter $\vartheta\in(0,1]$,

$$\begin{aligned}a_{n+1,L,\vartheta} &= \frac{3+0.5\vartheta+(12+\vartheta)e^{-a_{n,R,\vartheta}}}{9+1.5\vartheta+a^2_{n-1,L,\vartheta}},\\ a_{n+1,R,\vartheta} &= \frac{4-0.5\vartheta+(14-\vartheta)e^{-a_{n,L,\vartheta}}}{12-1.5\vartheta+a^2_{n-1,R,\vartheta}},\ \vartheta\in(0,1].\end{aligned}\tag{5.9}$$

Therefore, if Theorem 3.2 holds true, and satisfying the positive fuzzy initial conditions a_{-1}=(0.5, 0.75, 1), a_0=(5, 6, 7) and fuzzy parameters D, P, T, then each positive solution of the model (1.10) is both bounded and persistent. Additionally, through the statements of Theorem 3.2, the unique positive equilibrium of the model (1.10) $\overline{a}$ =(0.3255, 0.3192, 0.3149). Moreover, all positive solution a_n of the model (1.10) converge to the unique equilibrium $\overline{a}$ as $n\to\infty$. (see Figures 1, 2 and 3)

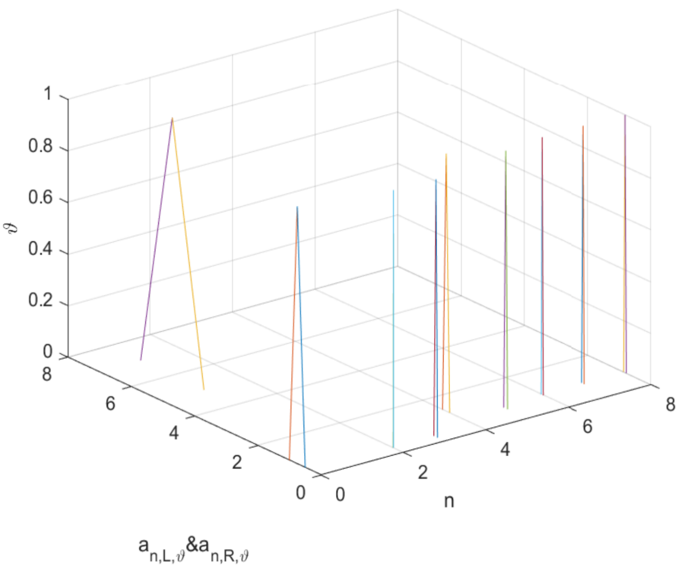

Figure 1. Behavior of the system (5.8).

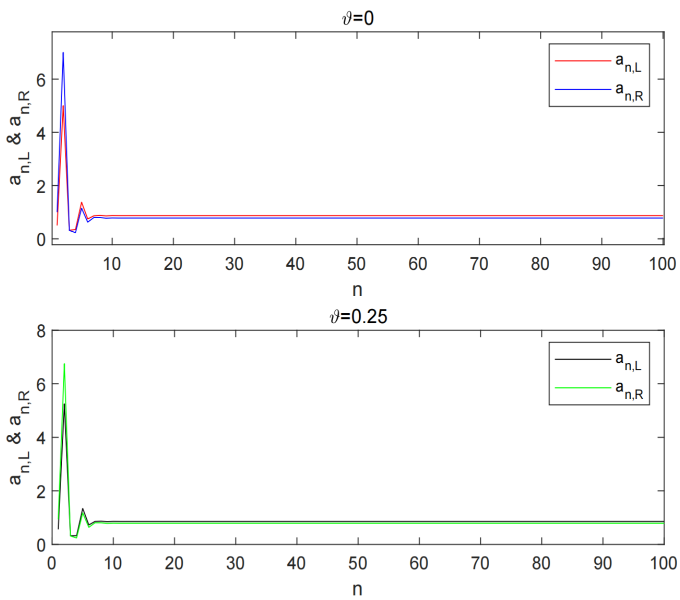

Figure 2. The positive solution of the system (5.8) at $\vartheta=0$ and $\vartheta=0.25$.

Example 4.2 Let us to take into account the following

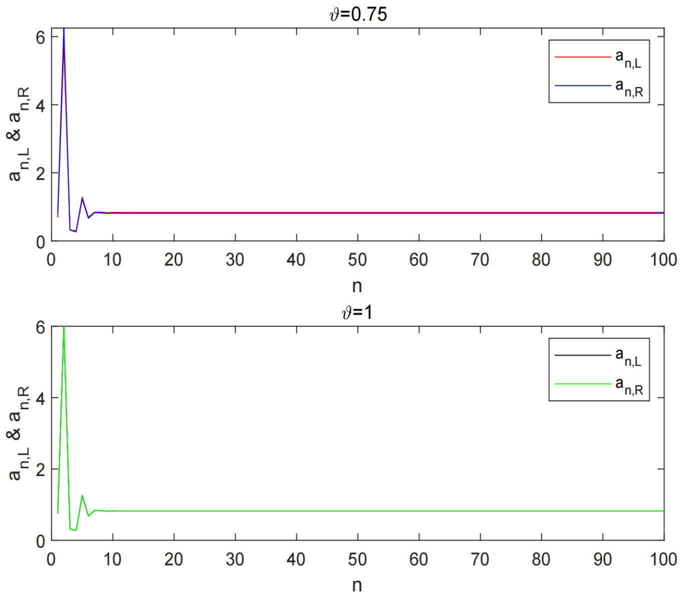

Figure 3. The positive solution of the system (5.8) at $\vartheta = 0.75$ and $\vartheta = 1$.

fuzzy difference equation

$$a_{n+1} = \frac{D + Pe^{-a_n}}{T + a_{n-1}^2}, \quad n \in N, \tag{5.10}$$

here D, P, T and a_{-1}, a_0 are satisfied

$$D(t) = \begin{cases} 5t - 10, & 2 \le t \le 2.2 \\ -5t + 12, & 2.2 \le t \le 2.4 \end{cases}, \qquad a_{-1}(t) = \begin{cases} 5t - 4, & 0.8 \le t \le 1 \\ -5t + 6, & 1 \le t \le 1.2 \end{cases} \tag{5.11}$$

$$P(t) = \begin{cases} 5t - 2, & 0.4 \le t \le 0.6 \\ -5t + 4, & 0.6 \le t \le 0.8 \end{cases}, \qquad a_0(t) = \begin{cases} t - 0.5, & 0.5 \le t \le 1.5 \\ -t + 2.5, & 1.5 \le t \le 2.5 \end{cases} \tag{5.12}$$

$$T(t) = \begin{cases} \frac{1}{2}t - \frac{3}{4}, & 1.5 \le t \le 3.5 \\ -\frac{1}{2}t + \frac{11}{4}, & 3.5 \le t \le 5.5 \end{cases}. \tag{5.13}$$

From (5.2), we get

$$\begin{aligned} {}_{\vartheta} &= [2 + 0.2\vartheta, 2.4 - 0.2\vartheta], \\ [a_{-1}]_{\vartheta} &= [0.8 + 0.2\vartheta, 1.2 - 0.2\vartheta], \\ \vartheta &\in (0, 1]. \end{aligned} \tag{5.14}$$

From (5.3) and (5.4), we have

$$\begin{aligned} {}_{\vartheta} &= [0.4 + 0.2\vartheta, 0.8 - 0.2\vartheta], \\ [a_0]_{\vartheta} &= [0.5 + \vartheta, 2.5 - \vartheta], \\ [T]_{\vartheta} &= [1.5 + 2\vartheta, 5.5 - 2\vartheta], \\ \vartheta &\in (0, 1]. \end{aligned} \tag{5.15}$$

Therefore, it follows that

$$\begin{aligned} \overline{\bigcup_{\vartheta \in (0,1]} [D]_{\vartheta}} &= [2, 2.4], \\ \overline{\bigcup_{\vartheta \in (0,1]} [P]_{\vartheta}} &= [0.4, 0.8], \\ \overline{\bigcup_{\vartheta \in (0,1]} [T]_{\vartheta}} &= [1.5, 5.5], \\ \overline{\bigcup_{\vartheta \in (0,1]} [a_{-1}]_{\vartheta}} &= [0.8, 1.2], \\ \overline{\bigcup_{\vartheta \in (0,1]} [a_0]_{\vartheta}} &= [0.5, 2.5]. \end{aligned} \tag{5.16}$$

From (4.8), it results in two second-order exponential-type difference equation with parameter $\vartheta \in (0, 1]$,

$$\begin{aligned} a_{n+1,L,\vartheta} &= \frac{2.4 - 0.2\vartheta + (0.8 - 0.2\vartheta)e^{-a_{n,L,\vartheta}}}{5.5 - 2\vartheta + a_{n-1,R,\vartheta}^2}, \\ a_{n+1,R,\vartheta} &= \frac{2 + 0.2\vartheta + (0.4 + 0.2\vartheta)e^{-a_{n,R,\vartheta}}}{1.5 + 2\vartheta + a_{n-1,L,\vartheta}^2}. \end{aligned} \tag{5.17}$$

Obviously, if Theorem 3.3. holds true, and such that the positive fuzzy initial conditions a_{-1}=(0.8, 1, 1.2), a_0=(0.5, 1.5, 2.5) and positive fuzzy parameters D, P, T are satisfied. so by employing Lemma 3.5, we can get that the model (1.10) exists unique positive equilibrium $\overline{a} = (0.4157, 0.5186, 0.9499)$. Furthermore, all positive solution a_n of the model (1.10) converge to the unique equilibrium $\overline{a}$ as $n \to \infty$. (see Figures 4, 5 and 6)

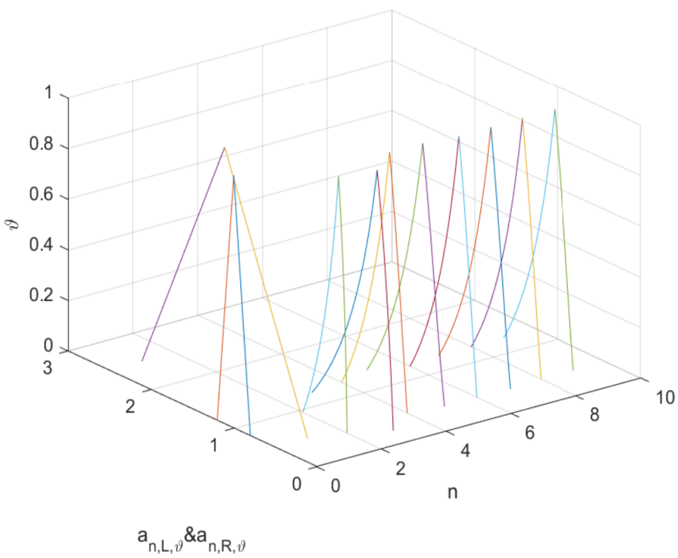

Figure 4. Behavior of system (5.16).

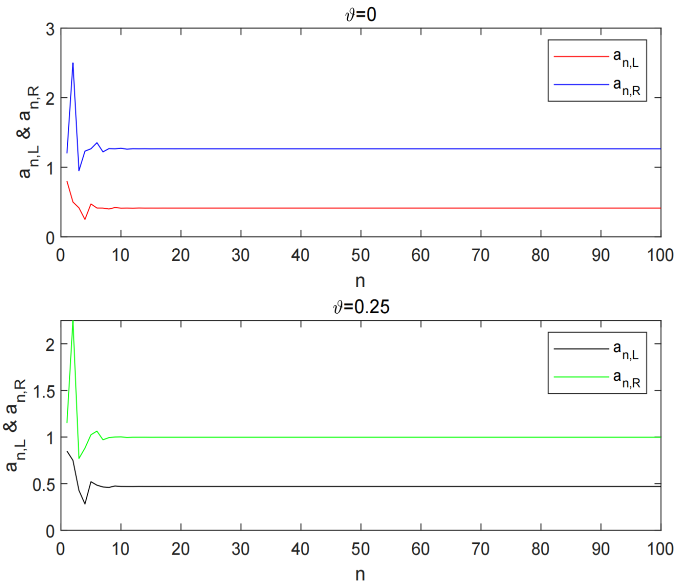

Figure 5. The positive solution of the system (5.16) at $\vartheta = 0$ and $\vartheta = 0.25$.

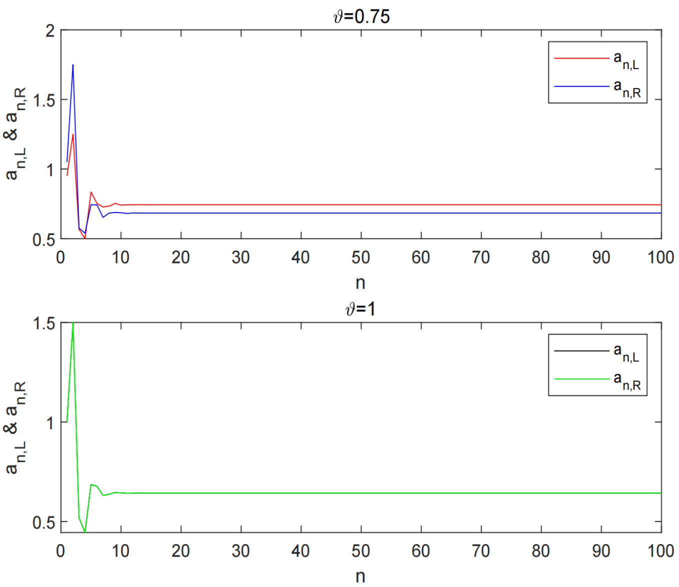

Figure 6. The positive solution of the system (5.16) at $\vartheta = 0.75$ and $\vartheta = 1$.

6 Conclusion

To sum up, in this work, we discussed a second-order exponential form of FDEs with quadratic term, i.e., $a_{n+1} = \frac{D+Pe^{-a_n}}{T+a_{n-1}^2}$, according to g-division, Lyapunov function, etc., we study the qualitative behaviors related to model (1.10), including the existence, non-oscillatory nature, boundedness and persistence of positive solutions and the global stability of unique positive equilibrium $\overline{a}$ of (1.10). More precisely, we find the conclusions below:

(i) Suppose Case (i) is valid, i.e., $\frac{D_{L,\vartheta}+P_{L,\vartheta}e^{-a_{n,R,\vartheta}}}{D_{R,\vartheta}+P_{R,\vartheta}e^{-a_{n,L,\vartheta}}} \leq \frac{T_{L,\vartheta}+a_{n-1,L,\vartheta}^2}{T_{R,\vartheta}+a_{n-1,R,\vartheta}^2}$, $\vartheta \in (0,1]$, $n = 0,1,2,\cdots$. Then each positive solutions of the model (1.10) is both bounded and persistent. Moreover, if the Theorem 3.2 is satisfied, so the model (1.10) has single positive equilibrium $\overline{a}$, and all positive solutions a_n of the model (1.10) tend to the unique equilibrium $\overline{a}$ as $n \to \infty$.

(ii) If Case (ii) is valid, i.e., $\frac{D_{R,\vartheta}+P_{R,\vartheta}e^{-a_{n,L,\vartheta}}}{D_{L,\vartheta}+P_{L,\vartheta}e^{-a_{n,R,\vartheta}}} \leq \frac{T_{R,\vartheta}+a_{n-1,R,\vartheta}^2}{T_{L,\vartheta}+a_{n-1,L,\vartheta}^2}$, $\vartheta \in (0,1]$, $n \in N$. Thus each positive solution of the fuzzy model (1.10) is both bounded and persistent. Furthermore, if the Theorem 3.3 is satisfied, then the fuzzy model (1.10) exists single positive equilibrium $\overline{a}$, and any solution x_n of the model (1.10) converges to the single equilibrium $\overline{a}$ as $n \to \infty$.

(iii) If (4.5) and (4.6) are valid, we have proven the non-oscillation of the model (1.10) under the Case (i). Additionally, using the same method as in case 1, we can similarly prove the non-oscillation in case 2. Therefore, we omit the proof.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

Ethical Approval and Consent to Participate

Not applicable.

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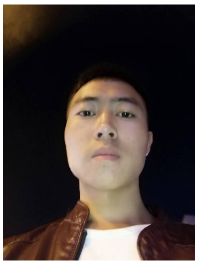

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