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RESEARCH ARTICLE

Analysis of Trajectory Structure and GAS for a High-Order Nonlinear Difference Equation

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Abstract

This article delves into the trajectory structure rules of a specific fifth-order rational difference equation:

$$s_{m+1} = \frac{s_m s_{m-2} s_{m-3} s_{m-4} + s_m s_{m-2} + s_m s_{m-3} + s_{m-2} s_{m-3} + s_{m-4} + a}{s_m s_{m-2} s_{m-3} + s_m s_{m-2} s_{m-4} + s_m s_{m-3} s_{m-4} + s_{m-2} s_{m-3} s_{m-4} + 1 + a}$$

where the initial conditions satisfy $s_i \in (0, \infty)$, $i = -4, -3, -2, -1, 0$, **and the parameters** $a \in [0, \infty)$. **As the initial values vary, the lengths of consecutive positive and negative semi-cycles for non-trivial solutions exhibit a periodic pattern with a prime period of 31. The rule within one period is** $1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-, 4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+$. **Through the application of this rule,the global asymptotic stability(GAS) of the positive fixed point of the equation is proven. In the end, three instances are utilized to demonstrate the accuracy of the theoretical conclusions.**

Keywords: global asymptotic stability, semi-cycle analysis, trajectory structure, nonlinear difference equation.

1 Introduction

Difference equations, as a fundamental and crucial cornerstone in the field of discrete mathematics, demonstrate remarkable flexibility and exert significant influence in their wide applications across numerous diverse fields. These fields include physics, where they help simulate discrete physical phenomena, biology, which they assist in understanding population dynamics and gene sequences [1, 2], economics, where they play a key role in predicting economic trends and analyzing market behaviors, and engineering, where they are important in designing digital filters and signal processing algorithms. Over the past few years, the field of difference equations has witnessed a significant surge in research activities, resulting in a large number of remarkable achievements [3, 4]. One of the focuses has been on delving deeply into nonlinear difference equations, which exhibit complex and fascinating behaviors that are difficult to capture through linear models. In addition, rational difference equations have received extensive attention due to their prevalence in various mathematical and scientific contexts, characterized by rational functions. Moreover, significant progress has also been made in the research regarding systems of difference equations [5–7], providing insights into the dynamics of coupled systems that involve multiple interconnected equations. Researchers in this field have adopted a series of sophisticated theories and models to explore the properties and behaviors of these equations in depth. Through

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rigorous mathematical analysis, computational simulations, and empirical verification, they have uncovered a wealth of theoretical knowledge, laying a solid foundation for practical applications. This research has not only enriched our understanding of fundamental mathematical principles but also opened up innovative ways to solve real-world problems.

In recent years, the field of mathematical research has shown significant interest in and exploration of the global characteristics of nonlinear difference equations. This emerging research area has attracted the attention of a large number of scholars, who have each contributed unique insights and methods, thereby deepening the collective understanding of these complex systems. Thanks to collaborative efforts, the research on nonlinear difference equations has produced fruitful results, revealing their complex workings and impacts across disciplines [8–10]. Among the studied aspects, trajectory structure rules and global asymptotic stability (GAS) are particularly crucial. A deep understanding of these properties is vital for uncovering the inherent dynamic behaviors of systems modeled by these equations. By analyzing trajectory structures, researchers can unravel patterns determining how systems evolve over time, providing insights into their long-term behaviors and potential outcomes. Moreover, research on GAS offers a theoretical foundation for practical applications of nonlinear difference equations. In real-world scenarios, systems are subject to various perturbations, and understanding convergence to a stable equilibrium is essential. This knowledge aids in designing control strategies, predicting, and mitigating instabilities, enhancing system robustness and reliability.

In the research on nonlinear difference equations, the analysis of trajectory structure is a core issue. The structure of the trajectory doesn't merely uncover the variation patterns of the equation solutions, but is also intricately linked to the system's stability. The semi-cycle analysis method is an effective means for studying the trajectory structure of difference equations [11, 12]. Through this method, the periodic change laws of the solutions can be revealed, and then the stability of the system can be judged. Previous studies have shown that the semi-cycle analysis method has achieved remarkable results in the research on the trajectory structure of nonlinear difference equations.

In [13], Shen studied the structural rules of the trajectory and GAS of the positive fixed point of a difference equation of third order with nonlinearity:

$$z_{m+1} = \frac{z_{m-1}^{\alpha} z_{m-2} + 1 + c}{z_{m-1}^{\alpha} + z_{m-2} + c}, \quad m = 0, 1, 2, \cdots,$$

where the initial conditions satisfy $z_0, z_{-1}, z_{-2} \in (0, \infty)$, and the parameters $\alpha \in [0, 1], c \in [0, \infty)$. The structural rules of the trajectory with prime period 7 were examined, and the GAS of the positive fixed point was verified.

In [14], Chen et al. explored the cycle-length bifurcation and GAS of a high-order difference equation:

$$x_{m+1} = \frac{x_{m-1}x_{m-2} + x_{m-1}x_{m-4} + x_{m-2}x_{m-4} + 1 + b}{x_{m-1}x_{m-2}x_{m-4} + x_{m-1} + x_{m-2} + x_{m-4} + b}, \quad m = 0, 1, 2, \cdots,$$

where the initial values $x_i \in (0, \infty)$, $i = -4, -3, -2, -1, 0$, and the parameters $b \in [0, \infty)$.

In [15], Zhang et al. delved into the GAS and structural rules of the trajectory of the difference equations:

$$s_{i+1} = \frac{s_{i-1}s_{i-2}s_{i-4} + s_{i-1} + s_{i-2} + s_{i-4} + c}{s_{i-1}s_{i-2} + s_{i-1}s_{i-4} + s_{i-2}s_{i-4} + 1 + c}, \quad i = 0, 1, 2, \cdots,$$

where the initial values $s_i \in (0, \infty)$, $i = -4, -3, -2, -1, 0$, and the parameters $c \in [0,\infty)$.Through the application of the semi-cycle analysis approach, during a prime period, the consecutive lengths of positive and negative semi-cycles for any non-trivial solution display periodicity in the sequence: 2, 3, 4, 6, 12.

In [16], Sun et al. conducted an investigation into the GAS of a high-order rational difference equation:

$$y_{i+1} = \frac{g(y_{i-r_1}, \cdots, y_{i-r_k})h(y_{i-m_1}, \cdots, y_{i-m_l}) + 1}{g(y_{i-r_1}, \cdots, y_{i-r_k}) + h(y_{i-m_1}, \cdots, y_{i-m_l})}, \quad i = 0, 1, 2, \cdots,$$

where $g \in C((0, +\infty)^k, (0, +\infty))$, $h \in C((0, +\infty)^l, (0, +\infty))$ and $k, l \in \{1, 2, \ldots\}$, $r_k > \cdots > r_1 \geq 0$ and $m_l > \cdots > m_1 \geq 0$.

In [17], Li et al. explored the pattern of semi-cycle length in a certain type of fifth-order difference equation:

$$u_{i+1} = \frac{M(u_i, u_{i-2}, u_{i-3}, u_{i-4})}{N(u_i, u_{i-2}, u_{i-3}, u_{i-4})}, \quad i \in N,$$

where

$$M(a,b,c,d) = a^x b^y + a^x c^z + a^x d^w + b^y c^z + b^y d^w + c^z d^w + a^x b^y c^z d^w + 1 + r,$$

$$N(a,b,c,d) = a^x + b^y + c^z + d^w + a^x b^y c^z + a^x b^y d^w + a^x c^z d^w + b^y c^z d^w + r,$$

the parameters satisfy $r \in [0,\infty), x \in (0,1], y,z,w \in (0,\infty)$, and the initial values $u_i \in (0,\infty)$, $i = -4,-3,-2,-1,0$.

Drawing inspiration from these investigations, this article will conduct a more in-depth exploration of the structural rules of the trajectory and GAS of a new type of fifth-order nonlinear difference equation, the form of this equation is as follows:

$$s_{m+1} = \frac{s_m s_{m-2} s_{m-3} s_{m-4} + s_m s_{m-2} + s_m s_{m-3} + s_{m-2} s_{m-3} + s_{m-4} + a}{s_m s_{m-2} s_{m-3} + s_m s_{m-2} s_{m-4} + s_m s_{m-3} s_{m-4} + s_{m-2} s_{m-3} s_{m-4} + 1 + a} \quad (1)$$

where the initial values $s_i \in (0,\infty)$, $i = -4,-3,-2,-1,0$ and the parameters $a \in [0,\infty)$, m=0,1,2,$\cdots$.

Unlike previous studies, the equation considered in this paper has a more complex nonlinear form, which increases the difficulty of analysis but also presents new challenges and opportunities for the research. The core objective of this paper is to employ the semi-cycle analysis method to ascertain the structural rules of the trajectory of the positive and negative semi-cycles of the equation's solutions, and to examine the GAS of its positive fixed point.

Through detailed theoretical analysis, this paper is anticipated to offer novel perspectives for the research on the trajectory structure of nonlinear difference equations and offer theoretical support for practical applications in related fields.

2 Preliminary and definitions

Clearly, the positive fixed point $\bar{s}$ of Eq.(1) satisfies

$$\bar{s} = \frac{\bar{s}^4 + 3\bar{s}^2 + \bar{s} + a}{4\bar{s}^3 + 1 + a}, \quad (2)$$

from this equation, it's clear that Eq. (1) has a unique positive fixed point $\bar{s}$=1.

Definition 1. [18] *A positive semicycle of a solution $\{s_m\}_{m=-4}^{\infty}$ of Eq.(1) consists of a string of terms $\{s_k, s_{k+1}, \cdots, s_t\}$, all greater than or equal to the fixed point $\bar{s}$, with $k \geq -4$ and $t \leq \infty$ such that*

$$\textit{either } k = -4 \textit{ or } k > -4 \textit{ and } s_{k-1} < \bar{s},$$

and

$$\textit{either } t = \infty \textit{ or } t < \infty \textit{ and } s_{t+1} < \bar{s}.$$

A negative semicycle of a solution $\{s_m\}_{m=-4}^{\infty}$ of Eq.(1) consists of a string of terms $\{s_k, s_{k+1}, \cdots, s_t\}$, all less than $\bar{s}$, with $k \geq -4$ and $t \leq \infty$ provided that

$$\textit{either } k = -4 \textit{ or } k > -4 \textit{ and } s_{k-1} \geq \bar{s},$$

and

$$\textit{either } t = \infty \textit{ or } t < \infty \textit{ and } s_{t+1} \geq \bar{s}.$$

The length of a semicycle sequence represents the number of terms included in it.

Definition 2. *A solution $\{s_m\}_{m=-4}^{\infty}$ of Eq.(1) is deemed eventually trivial if it ultimately becomes equal to $\bar{s} = 1$,otherwise it is considered nontrivial.*

Theorem 1. *A positive solution $\{s_m\}_{m=-4}^{\infty}$ of Eq.(1) is eventually trivial iff*

$$(-1 + s_{-4})(s_{-3} - 1)(-1 + s_{-2})(-1 + s_{-1})(-1 + s_0) = 0. \quad (3)$$

Proof. Let (3) holds. Thus it follows from Eq.(1).

(i) If $s_{-4} = 1$,then $s_{5m-4} = 1$,for $m \geq 1$.

(ii) If $s_{-3} = 1$,then $s_{5m-3} = 1$,for $m \geq 1$.

(iii) If $s_{-2} = 1$,then $s_{5m-2} = 1$,for $m \geq 1$.

(iv) If $s_{-1} = 1$,then $s_{5m-1} = 1$,for $m \geq 1$.

(v) If $s_0 = 1$,then $s_{5m} = 1$,for $m \geq 1$.

In contrast, suppose that

$$(-1 + s_{-4})(s_{-3} - 1)(-1 + s_{-2})(-1 + s_{-1})(-1 + s_0) \neq 0. \quad (4)$$

Next, it can be shown that

$$s_m \neq 1 \;\; for \;\; any \;\; m \geq 1.$$

Assume, by way of contradiction, that for some $M \geq 1$,

$$s_M = 1 \;\; and \;\; that \;\; s_m \neq 1, \;\; for \;\; -4m \leq M - 1. \quad (5)$$

Clearly,

$$1 = s_m = \frac{s_{m-1} s_{m-3} s_{m-4} s_{m-5} + s_{m-1} s_{m-3} + s_{m-1} s_{m-4} + s_{m-3} s_{m-4} + s_{m-5} + a}{s_{m-1} s_{m-3} s_{m-4} + s_{m-1} s_{m-3} s_{m-5} + s_{m-1} s_{m-4} s_{m-5} + s_{m-3} s_{m-4} s_{m-5} + 1 + a},$$

which implies $s_{M-1} = 1,\ or\ s_{M-3} = 1,\ or\ s_{M-4} = 1,\ or\ s_{M-5} = 1$.This contradicts with (5).

Lemma 1. *For any nontrivial positive solution* $\{s_m\}_{m=-4}^{\infty}$ *of Eq.*(1)*,these assertions are true for* $m \geq 0$

(a) $\dfrac{(s_{m+1}-1)(-1+s_m)(s_{m-2}-1)}{(-1+s_{m-3})(-1+s_{m-4})>0}.$

(b) $(s_{m+1}-s_m)(-1+s_m)<0.$

(c) $(s_{m+1}-s_{m-1})(-1+s_{m-1})<0.$

(d) $(s_{m+1}-s_{m-2})(s_{m-2}-1)<0.$

(e) $(s_{m+1}-s_{m-3})(-1+s_{m-3})<0.$

(f) $(s_{m+1}-s_{m-4})(s_{m-4}-1)<0.$

Proof. From Eq.(1),it follows that

$$s_{m+1}-1=\frac{(-1+s_m)(s_{m-2}-1)(-1+s_{m-3})(s_{m-4}-1)f(s_m,s_{m-2},s_{m-3},s_{m-4})}{s_m s_{m-2}s_{m-3}+s_m s_{m-2}s_{m-4}+s_m s_{m-3}s_{m-4}+s_{m-2}s_{m-3}s_{m-4}+1+a},$$

$$s_{m+1}-s_m=\frac{-(-1+s_m)z(s_m,s_{m-2},s_{m-3},s_{m-4})}{s_m s_{m-2}s_{m-3}+s_m s_{m-2}s_{m-4}+s_m s_{m-3}s_{m-4}+s_{m-2}s_{m-3}s_{m-4}+1+a}.$$

First, we analyze the semi-period characteristics of the non-trivial solutions of Eq.(1). It should be noted that the scope of our discussion here is limited to the strictly oscillatory solutions of Eq.(1).

3 Rule of cycle length

Lemma 2. *Let* $\{s_m\}_{m=-4}^{\infty}$*be a solution of Eq.*(1) *that is strictly oscillatory. Then, the successive occurrence pattern of the lengths of positive and negative semicycles for this solution is* $\cdots,1^-,2^+,1^-,1^+,1^-,1^+,2^-,4^+,3^-,2^+,2^-,1^+,5^-,1^+,1^-,3^+,\cdots$

Proof.According to first inequality of Lemma 1, the length of the negative semi-cycle is no more than 5, and the length of the positive semi-cycle is at most 4. Given the strictly oscillatory nature of this solution, for an integer that is not less than 0, one of the eight cases below will occur,

Case 1: $s_{i-4}>\bar{s}, s_{i-3}<\bar{s}, s_{i-2}>\bar{s}, s_{i-1}>\bar{s}$ and $s_i>\bar{s}$.

Case 2: $s_{i-4}>\bar{s}, s_{i-3}<\bar{s}, s_{i-2}>\bar{s}, s_{i-1}>\bar{s}$ and $s_i<\bar{s}$.

Case 3: $s_{i-4}>\bar{s}, s_{i-3}<\bar{s}, s_{i-2}>\bar{s}, s_{i-1}<\bar{s}$ and $s_i>\bar{s}$.

Case 4: $s_{i-4}>\bar{s}, s_{i-3}<\bar{s}, s_{i-2}>\bar{s}, s_{i-1}<\bar{s}$ and $s_i<\bar{s}$.

Case 5: $s_{i-4}>\bar{s}, s_{i-3}<\bar{s}, s_{i-2}<\bar{s}, s_{i-1}>\bar{s}$ and $s_i>\bar{s}$.

Case 6: $s_{i-4}>\bar{s}, s_{i-3}<\bar{s}, s_{i-2}<\bar{s}, s_{i-1}>\bar{s}$ and $s_i<\bar{s}$.

Case 7: $s_{i-4}>\bar{s}, s_{i-3}<\bar{s}, s_{i-2}<\bar{s}, s_{i-1}<\bar{s}$ and $s_i>\bar{s}$.

Case 8: $s_{i-4}>\bar{s}, s_{i-3}<\bar{s}, s_{i-2}<\bar{s}, s_{i-1}<\bar{s}$ and $s_i<\bar{s}$.

When Case 1 happens, based on the first inequality of Lemma 1, it is concluded that $s_{i+a}<\bar{s}$, a = 1, 4, 6, 8, 9, 14, 15, 16, 19, 20, 22, 23, 24, 25, 26, 28, 32, 35, 37 $\cdots$,

$s_{i+b}>\bar{s}$, b = 2, 3, 5, 7, 10, 11, 12, 13, 17, 18, 21, 27, 29, 30, 31, 33, 34, 36 $\cdots$.

This indicates that the rule for the successive appearance of the lengths of positive and negative semicycles of the solution of Eq.(1) is $1^-,2^+,1^-,1^+,1^-,1^+,2^-,4^+,3^-,2^+,2^-,1^+,5^-,1^+,1^-,3^+,1^-,2^+,1^-,1^+,1^-\cdots$.

If Case 2 occurs, then we have

$s_{i+c}<\bar{s}$, c = 2, 4, 5, 10, 11, 12, 15, 16, 18, 19, 20, 21, 22, 24, 28, 31, 33, 35 $\cdots$,

$s_{i+d}>\bar{s}$, d = 1, 3, 6, 7, 8, 9, 13, 14, 17, 23, 21, 25, 26, 27, 29, 30, 32, 34 $\cdots$.

This means that the rule for the sequential occurrence of the lengths of positive and negative semicycles of the solution of Eq.(1) is $1^+,1^-,1^+,2^-,4^+,3^-,2^+,2^-,1^+,5^-,1^+,1^-,3^+,1^-,2^+,1^-,1^+,1^-,1^+,2^-,4^+\cdots$.

when Case 3 comes up, then the first inequality of Lemma 1 tells us that

$s_{i+e}<\bar{s}$, e = 4, 5, 10, 11, 12, 15, 16, 18, 19, 20, 21, 22, 24, 28, 31, 33, 35, 36 $\cdots$,

$s_{i+f}>\bar{s}$, f = 6, 7, 8, 9, 13, 14, 17, 23, 25, 26, 27, 29, 30, 32, 34, 37 $\cdots$.

This means that the rule for the consecutive emergence of the numbers of terms of positive and negative semicycles of the solution of Eq.(1) still is $2^-,4^+,3^-,2^+,2^-,1^+,5^-,1^+,1^-,3^+,1^-,2^+,1^-,1^+,1^-,1^+,2^-,4^+\cdots$

By the same way and by using 2.3 we can find that the Case 4,5,6,7,8 have the priciple$1^-,2^+,1^-,1^+,1^-,1^+,2^-,4^+,3^-,2^+,2^-,1^+,5^-,1^+,1^-,3^+,1^-,2^+,1^-,1^+,1^-\cdots$.

4 Global asymptotic stability

In this part ,we derive the GAS of the unique positive fixed point.

Theorem 2. *The fixed point* $\bar{s}=1$ *of Eq.*(1) *is GAS.*

Proof. In fact, it is necessary to demonstrate that the fixed point $\overline{s} = 1$ is locally asymptotically stable (LAS) and a global attractor. The linearized equation of Eq.(1) at the fixed point $\overline{s} = 1$ is

$$s_{m+1} = 0\times s_m + 0\times s_{m-2} + 0\times s_{m-3} + 0\times s_{m-4}, \quad m \in N,$$

so the fixed point $\overline{s} = 1$ is stable.Now,we need to prove that

$$\lim_{m\to\infty} s_m = \overline{s} = 1. \tag{6}$$

If the solution$\{s_m\}_{m=-4}^{\infty}$ of Eq.(1) is a trivial solution,by Definition 2 it is clear that $\lim_{m\to\infty} s_m = \overline{s} = 1$. If the solution$\{s_m\}_{m=-4}^{\infty}$ of Eq.(1) is a nontrivial solution,then there are two cases.

If the solution is trivial, then equation (6) holds since $s_m = 1$ will eventually be satisfied.

First, assume the solution is non-trivial. Next, we may group the solution into two categories: non-oscillatory solution and oscillatory solution.

When a non-oscillatory solution appears, we will obtain a solution that is eventually positive or negative. Which is to say, there exists an integer M such that $s_m < 1$ for $m \geq M$. According to the second inequality of Lemma 1, the solution is bounded and monotonic. Then, $\lim_{m\to\infty} s_m$= U exists and finite. By taking the limit of Eq.(1), we have

$$U = \frac{U^4 + 3U^2 + U + \gamma}{4U^3 + 1 + \gamma}.$$

Consequently, U = 1.This indicates that equation (6) is applicable to nonoscillatory solutions.

Hence,it is adequate to establish that equation (6) is applicable to oscillatory solutions. This means the oscillatory solution takes place.

Now,let $\{s_m\}$ be a sequence that is strictly oscillating about the positive fixed point of Eq.(1). According to Lemma 2, we can infer that the rule for the consecutive lengths of positive and negative semi-cycles is$\cdots, 1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-, 4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+, \cdots$. For the sake of simplicity, consider a non-negative integer i, We define$\{s_i\}^-$as the terms of a single-length positive semicycle. After that comes $\{s_{i+1}, s_{i+2}\}^+$, a negative semicycle with a length of two. Then, a length-one negative semicycle and a length-one positive semicycle follow, and so on. In other words, the pattern of the lengths of successive positive and negative semi-cycles can be described periodically as:

$\{s_{i+31m}\}^-$, $\{s_{i+31m+1}, s_{i+31m+2}\}^+$, $\{s_{i+31m+3}\}^-$, $\{s_{i+31m+4}\}^+$,
$\{s_{i+31m+5}\}^-$, $\{s_{i+31m+6}\}^+$, $\{s_{i+31m+7}, s_{i+31m+8}\}^-$,
$\{s_{i+31m+9}, s_{i+31m+10}, s_{i+31m+11}, s_{i+31m+12}\}^+$,
$\{s_{i+31m+13}, s_{i+31m+14}, s_{i+31m+15}\}^-$,
$\{s_{i+31m+16}, s_{i+31m+17}\}^+$,
$\{s_{i+31m+18}, s_{i+31m+19}\}^-$, $\{s_{i+31m+20}\}^+$,
$\{s_{i+31m+21}, s_{i+31m+22}, s_{i+31m+23}, s_{i+31m+24}, s_{i+31m+25}\}^-$,
$\{s_{i+31m+26}\}^+$,$\{s_{i+31m+27}\}^-$,
$\{s_{i+31m+28}, s_{i+31m+29}, s_{i+31m+30}\}^+$,m=0,1,2,$\cdots$

From Lemma 1,we can infer that the following results are valid:

(A)

$$\begin{aligned} &s_{i+31m} < s_{i+31m+3} < s_{i+31m+5} < s_{i+31m+7} < s_{i+31m+8} \\ &< s_{i+31m+13} < s_{i+31m+14} < s_{i+31m+15} < s_{i+31m+18} \\ &< s_{i+31m+19} < s_{i+31m+21} < s_{i+31m+22} < s_{i+31m+23} \\ &< s_{i+31m+24} < s_{i+31m+25} < s_{i+31m+27} < s_{i+31(m+1)}, \\ &m = 0, 1, 2, \cdots \end{aligned}$$

(B)

$$\begin{aligned} &s_{i+31m+1} > s_{i+31m+2} > s_{i+31m+4} > s_{i+31m+6} > s_{i+31m+9} \\ &s_{i+31m+10} > s_{i+31m+11} > s_{i+31m+12} > s_{i+31m+16} \\ &s_{i+31m+17} > s_{i+31m+20} > s_{i+31m+26} > s_{i+31m+28} \\ &s_{i+31m+29} > s_{i+31m+30} > s_{i+31(m+1)+1}, \\ &m = 0, 1, 2, \cdots \end{aligned}$$

So, from (B) one can see that $\{s_{i+31m+1}\}_{m=0}^{\infty}$ is decreasing and bounded below by 1. So, the $Y = \lim_{m\to\infty} s_{i+31m+1}$exist and is finite.

Moreover, from (B), we can obtain

$$\begin{aligned} Y &= \lim_{m\to\infty} s_{i+31m+1} = \lim_{m\to\infty} s_{i+31m+2} = \lim_{m\to\infty} s_{i+31m+4} \\ &= \lim_{m\to\infty} s_{i+31m+6} = \lim_{m\to\infty} s_{i+31m+9} = \lim_{m\to\infty} s_{i+31m+10} \\ &= \lim_{m\to\infty} s_{i+31m+11} = \lim_{m\to\infty} s_{i+31m+12} = \lim_{m\to\infty} s_{i+31m+16} \\ &= \lim_{m\to\infty} s_{i+31m+17} = \lim_{m\to\infty} s_{i+31m+20} = \lim_{m\to\infty} s_{i+31m+26} \end{aligned}$$

$$= \lim_{m\to\infty} s_{i+31m+28} = \lim_{m\to\infty} s_{i+31m+29} = \lim_{m\to\infty} s_{i+31m+30}$$

Similarly, by (A)one can see that $\{s_{i+31m}\}_{m=0}^{\infty}$ is increasing and bounded above by 1. So, the $Q = \lim_{m\to\infty} s_{i+31m}$ exist and is finite.

Furthermore, One can get from (A)

$$Q = \lim_{m\to\infty} s_{i+31m} = \lim_{m\to\infty} s_{i+31m+3} = \lim_{m\to\infty} s_{i+31m+5}$$
$$= \lim_{m\to\infty} s_{i+31m+7} = \lim_{m\to\infty} s_{i+31m+8} = \lim_{m\to\infty} s_{i+31m+13}$$
$$= \lim_{m\to\infty} s_{i+31m+14} = \lim_{m\to\infty} s_{i+31m+15} = \lim_{m\to\infty} s_{i+31m+18}$$
$$= \lim_{m\to\infty} s_{i+31m+19} = \lim_{m\to\infty} s_{i+31m+21} = \lim_{m\to\infty} s_{i+31m+22}$$
$$= \lim_{m\to\infty} s_{i+31m+23} = \lim_{m\to\infty} s_{i+31m+24} = \lim_{m\to\infty} s_{i+31m+25}$$
$$= \lim_{m\to\infty} s_{i+31m+27}$$

Now, it is sufficient to show that Y = Q = 1.

Noting that

$$s_{i+31m+17} = \frac{F\left(s_{i+31m+12}, s_{i+31m+13}, s_{i+31m+14}, s_{i+31m+16}\right)}{H\left(s_{i+31m+12}, s_{i+31m+13}, s_{i+31m+14}, s_{i+31m+16}\right)}, \quad (7)$$

$$s_{i+31m+18} = \frac{F\left(s_{i+31m+13}, s_{i+31m+14}, s_{i+31m+15}, s_{i+31m+17}\right)}{H\left(s_{i+31m+13}, s_{i+31m+14}, s_{i+31m+15}, s_{i+31m+17}\right)}. \quad (8)$$

where

$$F(x,y,z,w) = xyzw + xy + xz + yz + w + a,$$

$$H(x,y,z,w) = xyz + xyw + xzw + yzw + 1 + a.$$

Taking limits on both sides of the Eq.(7) and Eq.(8) respectively, we get

$$Y = \frac{Y^2Q^2 + 2YQ + Q^2 + Y + a}{2YQ^2 + 2Y^2Q + 1 + a}, \quad (9)$$

$$Q = \frac{YQ^3 + 2YQ + Q^2 + Q + a}{3YQ^2 + Q^3 + 1 + a}. \quad (10)$$

Solving the two equations we can derive Y = Q = 1. Hence, $\lim_{m\to\infty} s_{i+31m+k} = 1, k = 0,1,2...,30.$

Thus,this completes the proof.

5 Numerical examples

To verify the theoretical outcomes, three instances are provided to illustrate the soundness of the theoretical analysis.

Example 1. Consider the Eq.(1) with the initial values $s_{-4} = 0.01$, $s_{-3} = 0.5$, $s_{-2} = 0.1$, $s_{-1} = 1.5$, $s_0 = 1.0$, and a=1 ,it is clear that the fixed point $\bar{s} = 1$ is GAS (see Figure 1).

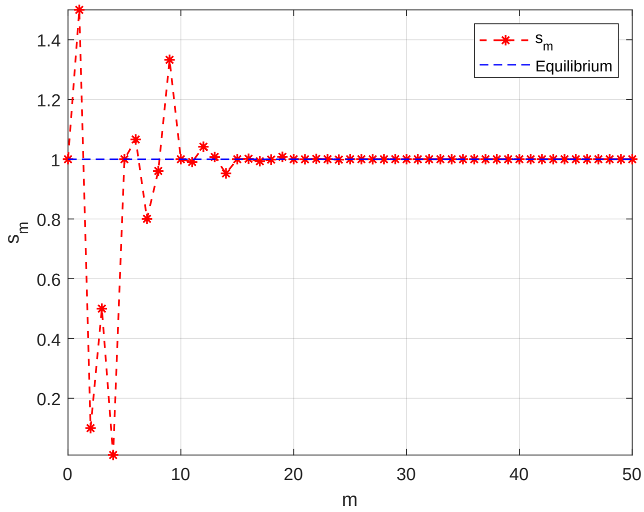

Figure 1. The fixed point $\bar{s} = 1$ is GAS.

Example 2. Consider the Eq.(1) with the initial values $s_{-4} = 7$, $s_{-3} = 6$, $s_{-2} = 10$, $s_{-1} = 8$, $s_0 = 9$, and a=0.001 ,it is clear that the fixed point $\bar{s} = 1$ is GAS (see Figure 2).

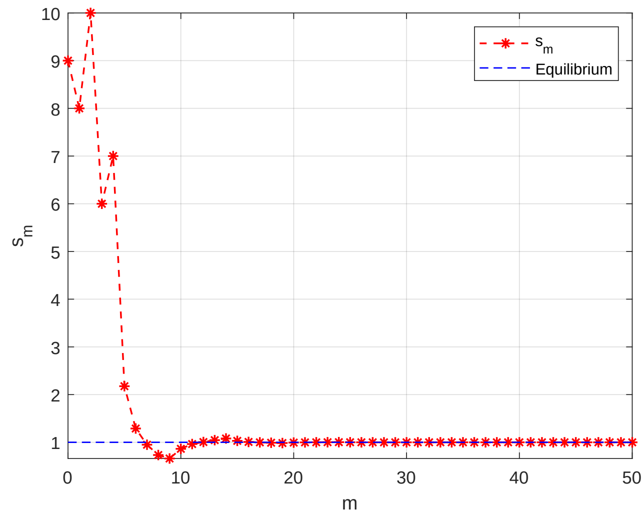

Figure 2. The fixed point $\bar{s} = 1$ is GAS.

Example 3. Consider the Eq.(1) with the initial values $s_{-4} = 0.7$, $s_{-3} = 0.6$, $s_{-2} = 10$, $s_{-1} = 0.8$, $s_0 = 0.1$

, and a=0.25 ,evidently, the fixed point $\bar{s} = 1$ is GAS (see Figure 3).

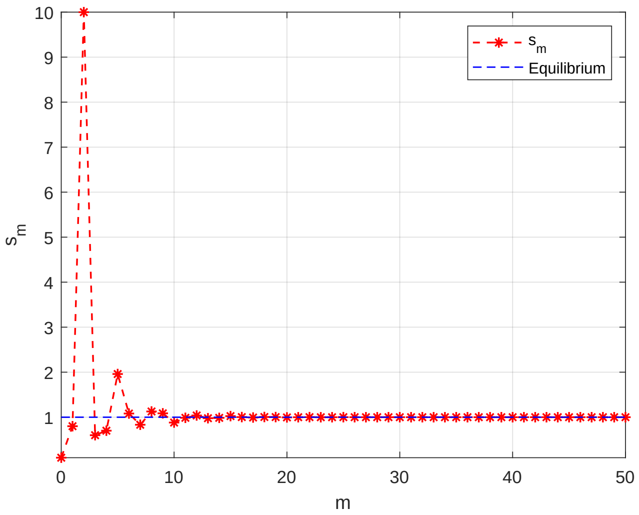

Figure 3. The fixed point $\bar{s} = 1$ is GAS.

6 Conclusion

The present paper researches on the structural rules of the trajectory and GAS of a fifth-order nonlinear difference equation via semi-cycle analysis. The key findings reveal that for non-trivial solutions, the successive lengths of positive and negative semi-cycles follow a periodic pattern with a prime period of 31, specifically $1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-, 4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+$, $1^-, 3^+$ etc. This rule was systematically derived through case analysis of oscillatory solutions, ensuring the periodicity and consistency of semi-cycle lengths.

The GAS of the positive fixed point $\bar{s} = 1$ was established by proving its local stability and demonstrating that all non-trivial solutions converge to $\bar{t}$ through monotonic or oscillatory behaviors. The semi-cycle analysis method played a critical role in identifying the convergence mechanisms, supported by theoretical lemmas and limit analysis. Numerical examples further validated the theoretical results, showing that solutions asymptotically approach the equilibrium regardless of initial conditions, confirming the robustness of the stability conclusion. This research contributes to the understanding of complex dynamics in high-order nonlinear systems, providing a framework for analyzing trajectory structures and stability in similar equations. Future work may extend these methods to equations with different orders or parametric variations.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

AI Use Statement

The authors declare that no generative AI was used in the preparation of this manuscript.

Ethical Approval and Consent to Participate

Not applicable.

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