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RESEARCH ARTICLE

Proof of Morse's Lemma

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Abstract

In this paper, we present an important property of smooth function germs of orbit tangent space, and apply it to discuss the proof of Morse's Lemma.

Keywords: function germ, Jacobian ideal, diffeomorphism, Morse germ, Morse's Lemma.

1 Introduction

Mather had given the necessary and sufficient conditions for the finite determinacy of smooth function germs in [1]. The characteristics of finite determinacy for smooth function germs are established.

Finite determinacy of smooth function germs has always been a very active project in the research of singularity theory. Pei and the authors discussed the finite determinacy of high codimension smooth function germs and gave an important property of smooth function germs of orbit tangent space in [2, 3]. In this paper, we apply it to discuss the proof of Morse's Lemma.

In recent years, there are a large number of literatures on the study of finite determinacy of smooth function germs. Wall showed the necessary and sufficient conditions for the finite determinacy of smooth mapping germs in [4]. Kushner and Leme, Sun and Wilson gave the relationship between the mapping germs of relative stability and the finite relative determinacy in [5, 6]. In addition, Liu, Luan, Shi and Pei provided the definitions and determination methods of finite determinacy and infinite relative determinacy for smooth function germs with certain boundary conditions in [7–9]. This theory has numerous applications in mathematics and the natural sciences, see [10–15].

The structure of this paper is as follows. In Section 2, we present some basic concepts and notations. In Section 3, we present our main results and proofs.

All undefined terms and symbols could be found in [16].

2 Related Work

Let E_n be a ring of C^∞ smooth function germs at $0 \in \mathbb{R}^n$ and let M_n denote the unique maximal ideal of E_n. Set M_n^k be the k-th power of M_n. The notation $J(f) = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \rangle_{E_n}$ denotes the Jacobian ideal of the smooth function germ f. Here (t, x)=$(t, x_1, x_2, \ldots, x_n)$ $\in \mathbb{R} \times \mathbb{R}^n$.
(Definition 2.1)

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Definition 2.1. *Let $I_1 = \langle f_1, f_2, \ldots, f_r \rangle_{E_n}$ and $I_2 = \langle g_1, g_2, \ldots, g_r \rangle_{E_n}$ be finitely generated ideals in E_n. Two ideals I_1 and I_2 are R-equivalent, if there exists an invertible matrix $[u_{ij}]_{r\times r}$ in E_n, such that*

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \end{pmatrix} = [u_{ij}]_{r\times r} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_r \end{pmatrix}.$$

Definition 2.2. (*Definition 2.2*) *Let $f, g \in E_n$. Two function germs f and g are said to be isomorphic (i.e., right equivalent) if there exists a local diffeomorphism germ $\Phi : (\mathbb{R}^n, 0) \to \mathbb{R}^n$ such that $g = f \circ \Phi$.*

Definition 2.3. (*Definition 2.3*) *Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a C^∞ real function germ and k be a positive integer. We say f is k-determined if all the Taylor polynomial germs which have the same order k with f in E_n are right equivalent to f.*

Definition 2.4. (*Proposition 2.4*) *Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a C^∞ germ . If $f \in M_n^2$ and the projection of f into M_n^2/M_n^3 is a non-degenerate quadratic form, then we say f is a Morse germ.*

In order to prove the Morse's Lemma, we will introduce the following proposition.
(Proposition 2.5)

Proposition 2.1. [10] *Let*

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x) \frac{\partial}{\partial x_i}$$

be a C^∞ vector field on an open neighborhood of $\mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}^n$, $t \in [0, 1]$. There exists an open set U containing $[0, 1] \times \{0\}$, which makes the following system of differential equations

$$\begin{cases} \frac{d\Phi_1(t,x)}{dt} = X_1(\Phi_1(t, x), \ldots, \Phi_n(t, x)) \\ \frac{d\Phi_2(t,x)}{dt} = X_2(\Phi_1(t, x), \ldots, \Phi_n(t, x)) \\ \vdots \quad \vdots \quad\quad \vdots \\ \frac{d\Phi_n(t,x)}{dt} = X_n(\Phi_1(t, x), \ldots, \Phi_n(t, x)) \end{cases}$$

have a unique solution with the initial condition

$$\begin{pmatrix} \Phi_1(0, x) \\ \Phi_2(0, x) \\ \vdots \\ \Phi_n(0, x) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Here, $\Phi_t : (U_0, x) \to (U_t, \Phi(t, x))$ is a local diffeomorphism.

3 Proof of Morse's Lemma

In this section, we present our main results and proofs.

Theorem 3.1. (*Theorem 3.1*) [6] *Let $h(x) \in M_n^k$ be small enough, then $M_n^k \subset M_n \cdot J(f)$ if and only if $M_n^k \subset M_n \cdot J(f + \tau h)$, where $\tau \in [0, 1]$.*

Theorem 3.2. (*Theorem 3.2*) *Let $f(x) = Q(x) + r(x)$ be a Morse germ in E_n, where $Q(x)$ be a non-degenerate quadratic form. Set $Q(x) = \frac{1}{2}x^T Ax, x = (x_1, x_2, \ldots, x_n)^T$ and $r(x) \in M_n^3$. We have*

(1) $M_n = J(Q(x))$,

(2) $M_n = J(Q(x) + tr(x)), t \in [0, 1]$.

Proof. Proof. Since $Q(x) = \frac{1}{2}x^T Ax, x = (x_1, x_2, \ldots, x_n)^T$. Then the partial derivatives of $Q(x)$ are as follows.

$$\begin{cases} \frac{\partial Q(x)}{\partial x_1} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{\partial Q(x)}{\partial x_2} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad \vdots \quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ \frac{\partial Q(x)}{\partial x_n} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n. \end{cases}$$

It can be also written as

$$\begin{pmatrix} \frac{\partial Q(x)}{\partial x_1} \\ \frac{\partial Q(x)}{\partial x_2} \\ \vdots \\ \frac{\partial Q(x)}{\partial x_n} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Let's write

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

The quadratic form $Q(x)$ is nondegenerate. One then has $\det A \neq 0$. As $M_n = \langle x_1, x_2, \ldots, x_n \rangle_{E_n}$. By the Definition 2.1, one has $M_n = \left\langle \frac{\partial Q(x)}{\partial x_1}, \frac{\partial Q(x)}{\partial x_2}, \ldots, \frac{\partial Q(x)}{\partial x_n} \right\rangle_{E_n}$. Since $J(Q(x)) = \left\langle \frac{\partial Q(x)}{\partial x_1}, \frac{\partial Q(x)}{\partial x_2}, \ldots, \frac{\partial Q(x)}{\partial x_n} \right\rangle_{E_n}$. Thus we have $M_n = J(Q(x))$.

Next we prove the condition (2). Since

$$\begin{cases} \frac{\partial(Q(x)+tr(x))}{\partial x_1} = \frac{\partial Q(x)}{\partial x_1} + t\frac{\partial r(x)}{\partial x_1} \\ \frac{\partial(Q(x)+tr(x))}{\partial x_2} = \frac{\partial Q(x)}{\partial x_2} + t\frac{\partial r(x)}{\partial x_2} \\ \quad\vdots \quad\quad \vdots \quad \vdots \quad\quad \vdots \\ \frac{\partial(Q(x)+tr(x))}{\partial x_n} = \frac{\partial Q(x)}{\partial x_n} + t\frac{\partial r(x)}{\partial x_n} \end{cases}$$

and $r(x) \in M_n^3$, one has $\frac{\partial r(x)}{\partial x_i} \in M_n^2\,(i = 1, 2, \ldots, n)$. By the condition (1), this shows that

$$\frac{\partial r(x)}{\partial x_i} = \sum_{i=1}^{n} m_{ij}(x)\frac{\partial r(x)}{\partial x_i}\,.$$

Here $m_{ij}(x)$, $\frac{\partial Q(x)}{\partial x_i} \in M_n\,(i, j = 1, 2, \ldots, n)$. Its matrix form is as follows.

$$\begin{pmatrix} \frac{\partial(Q(x)+tr(x))}{\partial x_1} \\ \frac{\partial(Q(x)+tr(x))}{\partial x_2} \\ \vdots \\ \frac{\partial(Q(x)+tr(x))}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial Q(x)}{\partial x_1} + t\frac{\partial r(x)}{\partial x_1} \\ \frac{\partial Q(x)}{\partial x_2} + t\frac{\partial r(x)}{\partial x_2} \\ \vdots \\ \frac{\partial Q(x)}{\partial x_n} + t\frac{\partial r(x)}{\partial x_n} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial Q(x)}{\partial x_1} + t\sum_{i=1}^{n} m_{ij}(x)\frac{\partial r(x)}{\partial x_1} \\ \frac{\partial Q(x)}{\partial x_2} + t\sum_{i=1}^{n} m_{ij}(x)\frac{\partial r(x)}{\partial x_2} \\ \vdots \\ \frac{\partial Q(x)}{\partial x_n} + t\sum_{i=1}^{n} m_{ij}(x)\frac{\partial r(x)}{\partial x_n} \end{pmatrix}$$
$$= \begin{pmatrix} 1+tm_{11}(x) & tm_{12}(x) & \cdots & tm_{1n}(x) \\ tm_{21}(x) & 1+tm_{22}(x) & \cdots & tm_{2n}(x) \\ \vdots & \vdots & \vdots & \vdots \\ tm_{n1}(x) & tm_{n2}(x) & \cdots & 1+tm_{nn}(x) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial Q(x)}{\partial x_1} \\ \frac{\partial Q(x)}{\partial x_2} \\ \vdots \\ \frac{\partial Q(x)}{\partial x_n} \end{pmatrix}.$$

Since $m_{ij}(x) \in M_n$, then $m_{ij}(0) = 0\ (i, j = 1, 2, \ldots, n)$. So we obtain

$$\det\begin{pmatrix} 1+tm_{11}(0) & tm_{12}(0) & \cdots & tm_{1n}(0) \\ tm_{21}(0) & 1+tm_{22}(0) & \cdots & tm_{2n}(0) \\ \vdots & \vdots & \vdots & \vdots \\ tm_{n1}(0) & tm_{n2}(0) & \cdots & 1+tm_{nn}(0) \end{pmatrix} \neq 0$$

By the Definition 2.1, one has

$$M_n = \left\langle \frac{\partial Q(x)}{\partial x_1} + t\frac{\partial r(x)}{\partial x_1}, \frac{\partial Q(x)}{\partial x_2} + t\frac{\partial r(x)}{\partial x_2}, \ldots, \frac{\partial Q(x)}{\partial x_n} + t\frac{\partial r(x)}{\partial x_n} \right\rangle_{E_n}.$$

That is

$$M_n = J(Q(x) + tr(x)).$$

It suffices to show that

$$M_n = J(Q(x)) = J(Q(x) + tr(x)).$$

□

Corollary 3.1. (*Corollary 3.3*) *Let $Q(x)$ be a non-degenerate quadratic form and $r(x) \in M_n^3$. If $X_i(t, x) \in M_n^2\,(i = 1, 2, \ldots, n)$, then the algebraic equation*

$$-h(x) = \sum_{i=1}^{n} X_i(t, x) \cdot \left(\frac{\partial Q(x)}{\partial x_i} + t\frac{\partial r(x)}{\partial x_i}\right)$$

is solvable. Where $X_i(t, x) \in M_n^2$, $i = 1, 2, \ldots, n$, $t \in [0, 1]$.

Proof. Proof. By the Theorem 3.2, we have

$$M_n = \left\langle \frac{\partial Q(x)}{\partial x_1} + t\frac{\partial r(x)}{\partial x_1}, \frac{\partial Q(x)}{\partial x_2} + t\frac{\partial r(x)}{\partial x_2}, \ldots, \frac{\partial Q(x)}{\partial x_n} + t\frac{\partial r(x)}{\partial x_n} \right\rangle_{E_n}.$$

Since $r(x) \in M_n^3 \subset M_n$, for any $t \in [0, 1]$, there exists $X_i(t, x) \in M_n^2\,(i = 1, 2, \ldots, n)$ such that

$$-h(x) = \sum_{i=1}^{n} X_i(t, x)\cdot\left(\frac{\partial Q(x)}{\partial x_i} + t\frac{\partial r(x)}{\partial x_i}\right), \quad X_i(t, x) \in M_n^2,$$
$$i = 1, 2, \ldots, n, \quad t \in [0, 1].$$

□

Lemma 3.1. (*Lemma 3.4*) *Let $F(t, x) = Q(x) + tr(x)$, where $t \in [0, 1]$ and $Q(x)$ be a non-degenerate quadratic form about $x_1, x_2, \ldots, x_n$. If $r(x) \in M_n^3$, then there exists a vector field*

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(t, x) \cdot \frac{\partial}{\partial x_i}$$

such that $X \cdot F = 0$.

Proof. Proof. Substituting $F(t,x) = Q(x) + tr(x)$ into $X \cdot F$, one obtains

$$X \cdot F = \frac{\partial F}{\partial t} + \sum_{i=1}^{n} X_i(t,x) \cdot \frac{\partial F}{\partial x_i}$$

$$= \frac{\partial(Q(x)+tr(x))}{\partial t} + \sum_{i=1}^{n} X_i(t,x) \cdot \frac{\partial(Q(x)+tr(x))}{\partial x_i}$$

$$= r(x) + \sum_{i=1}^{n} X_i(t,x) \cdot \left(\frac{\partial Q(x)}{\partial x_i} + t\frac{\partial r(x)}{\partial x_i}\right)$$

$$= 0.$$

That is

$$-r(x) = \sum_{i=1}^{n} X_i(t,x) \cdot \left(\frac{\partial Q(x)}{\partial x_i} + t\frac{\partial r(x)}{\partial x_i}\right),$$

$$X_i(t,x) \in M_n^2.$$

$$i = 1,2,\ldots,n, \quad t \in [0,1].$$

By the Corollary 3.3, the algebraic equation

$$-r(x) = \sum_{i=1}^{n} X_i(t,x) \cdot \left(\frac{\partial Q(x)}{\partial x_i} + t\frac{\partial r(x)}{\partial x_i}\right)$$

is solvable. Here $X_i(t,x) \in M_n^2$, $i = 1,2,\ldots,n$, $t \in [0,1]$. Thus there exists a vector field

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(t,x) \cdot \frac{\partial}{\partial x_i}$$

such that $X \cdot F = 0$.

□

Theorem 3.3 (Morse's Lemma)**.** *Let $f(x) = Q(x)+r(x)$ be a Morse germ in M_n^2, where $Q(x)$ is a non-degenerate quadratic form. If $r(x) \in M_n^3$, then $f(x)$ is isomorphic to $Q(x)$.*

Proof. Proof. Let $F(t,x) = Q(x) + tr(x)$, where $t \in [0,1]$ and $Q(x)$ be a non-degenerate quadratic form about $x_1, x_2, \ldots, x_n$. If $r(x) \in M_n^3$, then there exists a vector field

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(t,x) \cdot \frac{\partial}{\partial x_i}$$

such that $X \cdot F = 0$. That is

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{n} X_i(t,x) \cdot \frac{\partial F}{\partial x_i} = 0.$$

By the Proposition 2.5, one has

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{n} \frac{d\Phi_i(t,x)}{dt} \cdot \frac{\partial F}{\partial x_i} = 0.$$

It implies

$$\frac{d}{dt}(F \circ \Phi(t,x)) = 0$$

and shows that $F(t, \Phi(t,x))$ is a constant for any $t \in [0,1]$. Thus for any $t_1, t_2 \in [0,1]$ and $t_1 \neq t_2$, it suffices to show that

$$F(t_1, \Phi(t_1,x)) = F(t_2, \Phi(t_2,x)).$$

Especially, the $t_1 = 0$ and $t_2 = 1$, we have

$$F(0, \Phi(0,x)) = F(1, \Phi(1,x)).$$

As the equalities $F(0,x) = F(1, \Phi(1,x))$ and $F(t,x) = Q(x) + tr(x)$ hold. One then has $F(0,x) = Q(x)$ and $F(1,x) = Q(x) + r(x)$.

By the above discussions, we get $(Q(x) + r(x)) \circ \Phi(1,x) = Q(x)$. It means that $f \circ \Phi(1,x) = Q(x)$. Therefore $f(x)$ is isomorphic to $Q(x)$.

□

4 Conclusion

In this paper, we have presented a detailed proof of Morse's Lemma using properties of the orbit tangent space and Jacobian ideals of smooth function germs. Our main approach relies on Theorem 3.2, which establishes the crucial relationship $M_n = J(Q(x)) = J(Q(x) + tr(x))$ for a Morse germ $f(x) = Q(x) + r(x)$ with non-degenerate quadratic part $Q(x)$ and higher-order terms $r(x) \in M_n^3$.

This proof demonstrates the power of singularity theory in understanding the local behavior of smooth functions. The condition $M_n = J(Q(x))$ characterizes the non-degeneracy of the critical point and is both necessary and sufficient for the existence of the required coordinate change. Our method, based on the properties of the orbit tangent space and Jacobian ideals, provides a clear algebraic framework for studying equivalence problems in singularity theory.

The techniques developed here can be extended to more general settings, such as functions with degenerate critical points (via splitting lemma) or map germs between manifolds. Future work could explore applications of these methods in differential topology, dynamical systems, and geometric analysis.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

AI Use Statement

The authors declare that no generative AI was used in the preparation of this manuscript.

Ethical Approval and Consent to Participate

Not applicable.

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