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RESEARCH ARTICLE

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A Proposal Toward the Painlevé Equivalence Problem

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Abstract

This article develops a constructive approach to the Painlevé equivalence problem based on explicit normalization and residual obstruction analysis. Instead of starting from the full invariant complexity of the general point-equivalence problem, we use target-adapted reductions that convert equivalence into a finite algebraic comparison problem inside canonical normalized families. For derivative-free polynomial classes, the method yields exact results. We obtain a constructive criterion for equivalence to the first Painlevé equation in the quadratic case and an exact criterion for equivalence to the second Painlevé equation in the cubic case. In both settings, the method provides explicit recognition and non-equivalence conditions through finite scalar obstructions. For the higher Painlevé equations P_{III}, P_{IV}, P_{V}, and P_{VI}, we introduce target-adapted normalized families and finite residual obstruction systems that reduce comparison with the canonical Painlevé models to explicit coefficient conditions. This organizes the six Painlevé equations into a hierarchy of canonical geometries and provides a unified constructive framework for their equivalence analysis. The results establish exact reduction theorems for P_{I} and P_{II}, and a general normalization-based architecture for the higher Painlevé cases. In this way, the paper advances the Painlevé equivalence problem through explicit mathematical construction rather than purely abstract invariant classification.

Keywords: painlevé equivalence problem, nonlinear ordinary differential equations, canonical normalization, differential obstructions, point equivalence.

1 Introduction

1.1 Historical Background

Paul Painlevé (1863–1933) was a French mathematician born in Paris. He studied at the École Normale Supérieure, obtained the *licence* in mathematical and physical sciences in 1885, received the *agrégation* in mathematics in 1886, and completed his doctoral work in Paris in 1887 under the direction of Émile Picard. His doctoral thesis, *Sur les lignes singulières des fonctions analytiques*, already reflected the theme that would later become central to his name: the structure of singularities and their role in differential equations [1].

Painlevé's program was not merely computational. It was structural. He sought to distinguish differential equations whose solutions develop only controlled movable singularities from those whose solutions acquire genuinely pathological movable behavior. That viewpoint led, through the work of Painlevé,

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Gambier, and others, to the now classical Painlevé equations and to one of the deepest classification problems in the theory of nonlinear ordinary differential equations [2, 4].

1.2 The Painlevé Equations

The Painlevé equations are six canonical nonlinear second-order ordinary differential equations. They are commonly denoted by $P_{\mathrm{I}}, \ldots, P_{\mathrm{VI}}$, and their solutions are now regarded as nonlinear special functions [2, 4]. In the present work, we view them not as isolated formulas, but as canonical models inside the larger class of equations of the form

$$y'' = F(x, y, y'), \tag{1}$$

where F is rational or algebraic in its arguments.

A representative example is the second Painlevé equation

$$y'' = 2y^3 + xy + \alpha, \tag{2}$$

where α is a parameter. Equation (2) is simple to write, but its solutions are highly nontrivial and cannot, in general, be reduced to elementary functions. This mixture of rigid algebraic form and rich analytic behavior is precisely what makes the Painlevé family central in modern differential equations [2, 4].

1.3 Statement of the Painlevé Equivalence Problem

The Painlevé equivalence problem asks the following question: given a nonlinear second-order equation of the form (1), can one decide effectively whether it is transformable into one of the canonical Painlevé equations? In practical terms, the question is whether a seemingly complicated equation is genuinely new, or whether it is a Painlevé equation written in disguised variables [3].

This problem is explicitly recognized in the literature as open. Clarkson identifies the *Painlevé equivalence problem* as one of the principal unresolved problems in the area and emphasizes the absence of a general algorithmic procedure capable of deciding equivalence in a broad and effective way [3]. Partial results are known for special classes and for individual Painlevé equations, but no definitive general framework has closed the problem [3].

1.4 Motivation and Mathematical Significance

This problem is mathematically significant for a simple reason: classification precedes understanding. If an equation can be reduced to a Painlevé model, then a vast body of known structure becomes available immediately, including asymptotics, special solutions, Hamiltonian formulations, symmetry reductions, and monodromy-based interpretations [2, 4]. If it cannot, then one has evidence that the equation may define genuinely new behavior.

From the viewpoint of differential equations, the equivalence problem sits at the intersection of algebra, analysis, and geometry. It is algebraic because it demands explicit transformations and invariants; analytic because singularity structure is decisive; and structural because it seeks canonical form rather than local solvability alone. For that reason, it is not a peripheral question. It is a foundational one [2, 3].

1.5 Scope and Structure of the Article

This article is not a survey. It is a constructive attempt to advance the Painlevé equivalence problem through explicit normalization procedures and computable obstruction criteria. The central idea is to treat equivalence not as a purely abstract invariant-classification question, but as a target-adapted reduction problem.

The main results of the paper are twofold. First, for suitable derivative-free polynomial classes, we obtain exact constructive reduction criteria for P_{I} and P_{II}. Second, for P_{III}, P_{IV}, P_{V}, and P_{VI}, we formulate target-adapted normalized families and explicit residual obstruction systems that convert comparison with the canonical Painlevé equations into a finite coefficient problem.

The paper is organized as follows. We first establish the transformation formulas and the normalization strategy. We then derive exact reduction theorems for the lower polynomial targets P_{I} and P_{II}. Next, we construct normalized residual frameworks for the higher derivative-rational targets P_{III}–P_{VI}. Finally, we discuss how these six cases fit into a single constructive architecture for the general Painlevé equivalence problem.

2 General Objective

To develop a constructive analytical framework for the Painlevé equivalence problem, capable of reducing nonlinear second-order ordinary differential equations to target-adapted canonical families and deciding, whenever possible, their equivalence to one of the six classical Painlevé equations.

3 Specific Objectives

1. To define explicit ambient classes of second-order nonlinear ordinary differential equations suitable for reduction to Painlevé-type canonical forms.
2. To specify the admissible transformations under which equivalence will be tested.
3. To derive exact normalization procedures for the derivative-free polynomial cases associated with P_{I} and P_{II}.
4. To construct target-adapted normalized families for P_{III}, P_{IV}, P_{V}, and P_{VI}.
5. To identify explicit reduced coefficients and residual obstruction systems associated with each Painlevé target.
6. To establish exact criteria for equivalence in the polynomial cases and explicit coefficient criteria in the higher derivative-rational cases.
7. To test the method on representative positive and negative examples and evaluate its scope, sharpness, and limitations.

4 Literature Review

4.1 Ordinary Differential Equations and Classification Problems

The present article is centered on second-order ordinary differential equations of the form

$$y'' = F(x, y, y'), \tag{3}$$

where F is assumed to belong to a class that is sufficiently explicit to permit symbolic manipulation. The central classification question is not whether (3) possesses local existence or uniqueness, but whether it is *equivalent*, under an admissible change of variables, to a canonical model. In the present work, the canonical targets are the six Painlevé equations.

This viewpoint is classical. In the modern language of equivalence theory, one studies the orbit of (3) under a transformation pseudogroup and asks whether the orbit intersects a prescribed canonical family. The problem is therefore structural rather than merely local. It is closer to invariant theory than to standard existence theory.

To make this precise, let

$$X = \Phi(x, y), \quad Y = \Psi(x, y), \quad J := \Phi_x\Psi_y - \Phi_y\Psi_x \neq 0, \tag{4}$$

be a local point transformation. If we write

$$p := y', \qquad D_x := \partial_x + p\,\partial_y + F(x, y, p)\,\partial_p, \tag{5}$$

then the transformed first derivative is

$$Y_X = \frac{D_x\Psi}{D_x\Phi} = \frac{\Psi_x + \Psi_y p}{\Phi_x + \Phi_y p}, \tag{6}$$

and the transformed second derivative is

$$Y_{XX} = \frac{1}{D_x\Phi}\, D_x\left(\frac{D_x\Psi}{D_x\Phi}\right). \tag{7}$$

Thus the equivalence problem is already encoded in the transformation laws (6)–(7). Every effective method must ultimately control these formulas.

A useful restricted subclass is given by fiber-preserving transformations,

$$X = \phi(x), \qquad Y = \mu(x)\,y + \nu(x), \qquad \phi'(x)\mu(x) \neq 0. \tag{8}$$

For (8), one computes

$$Y_X = \frac{\mu' y + \mu y' + \nu'}{\phi'}, \tag{9}$$

and

$$\begin{aligned} Y_{XX} = &\frac{1}{(\phi')^2}\Big[\mu y'' + 2\mu' y' + \mu'' y + \nu'' \\ &- \frac{\phi''}{\phi'}(\mu' y + \mu y' + \nu')\Big]. \end{aligned} \tag{10}$$

Substituting $y'' = F(x, y, y')$ into (10) yields an explicit transformed equation. Formula (10) is elementary, but it already shows the algebraic burden of the problem: even very simple coordinate changes mix y, y', and F in a nonlinear way.

The literature shows that this classification problem is deep even for restricted targets. Linearization, for example, is completely solved only because the target equation $Y_{XX} = 0$ is exceptionally rigid. Beyond linearization, the general point-equivalence problem for second-order ODEs requires differential invariants of substantial order, and in the worst case the classification order is sharp at order 10 [6, 7].

4.2 Movable Singularities and the Painlevé Property

The Painlevé program begins from singularities of solutions, not from formal solvability. A singular point is called *movable* if its position depends on the initial data of the solution. The Painlevé property requires that such movable singularities be no worse than poles.

In simple language, the location of the singularity may depend on the solution, but the singularity itself must remain algebraically simple [2–5].

A standard local ansatz is the Laurent-type expansion

$$y(x) = \sum_{k=0}^{\infty} a_k\, \xi^{k-p}, \qquad \xi := x - x_0, \qquad a_0 \neq 0, \quad (11)$$

Insert the dominant balance

$$y \sim a\xi^{-p}, \qquad \xi = x - x_0. \quad (12)$$

where x_0 is a priori unknown singular point. The exponent $p > 0$ is the leading pole order. If x_0 depends on initial data, then $\xi = 0$ is movable. If p is an integer and the expansion closes meromorphically, the singularity is of pole type; if branching or more complicated local structure appears, the equation fails the Painlevé property.

The expansion (11) is not used directly in the normalization procedures developed in the subsequent sections. Its role is diagnostic and structural. It encodes the local movable singularity behavior required by the Painlevé property and provides a compatibility condition that any equation equivalent to a Painleve model must satisfy.

In particular, the admissible singularity structure derived from (11) acts as a preliminary obstruction criterion: if a given equation exhibits movable singularities that are not of pole type, then it cannot be equivalent to a Painleve equation under admissible transformations. Therefore, although (11) does not enter explicitly in the algebraic reduction formulas, it justifies the selection of target canonical families and supports the structural consistency of the equivalence framework.

$$y \sim a\,\xi^{-p}, \qquad \xi = x - x_0. \quad (13)$$

Then

$$y'' \sim a\,p(p+1)\,\xi^{-p-2}, \; 2y^3 \sim 2a^3\,\xi^{-3p}, \; xy + \alpha = O(\xi^{-p}). \quad (14)$$

Balancing the most singular powers in (14) gives

$$-p - 2 = -3p, \qquad \Rightarrow \qquad p = 1. \quad (15)$$

Matching coefficients then yields

$$a\,p(p+1) = 2a^3 \quad \Longrightarrow \quad 2a = 2a^3 \quad \Longrightarrow \quad a^2 = 1. \quad (16)$$

Hence the dominant singularity is a simple pole,

$$y \sim \pm \frac{1}{x - x_0}. \quad (17)$$

The resonance computation sharpens the picture. Write

$$y = a\,\xi^{-1} + m\,\xi^{r-1}, \qquad a^2 = 1, \quad (18)$$

and keep only the linear terms in m. Then

$$y'' = 2a\,\xi^{-3} + m(r-1)(r-2)\xi^{r-3} + \cdots, \quad (19)$$

while

$$2y^3 = 2a^3\,\xi^{-3} + 6a^2 m\,\xi^{r-3} + \cdots. \quad (20)$$

Equating the coefficients of ξ^{r-3} gives

$$(r-1)(r-2) = 6a^2 = 6, \quad (21)$$

hence

$$r^2 - 3r - 4 = 0 \qquad \Longrightarrow \qquad r = -1, \; 4. \quad (22)$$

The resonance $r = -1$ corresponds to the arbitrariness of the singular position x_0, whereas $r = 4$ identifies the next free parameter in the local series. This is exactly the pattern one expects from an equation with the Painlevé property.

This local computation matters for the equivalence problem because it supplies an invariant diagnostic. If an equation is equivalent to a Painlevé model, then its admissible singularity pattern must be compatible with the target model. Singularity analysis therefore does not solve equivalence by itself, but it produces nontrivial obstructions.

4.3 The Six Classical Painlevé Equations

The six canonical equations are the following [2, 4]:

$$P_{\mathrm{I}}: \quad y'' = 6y^2 + x, \tag{23}$$

$$P_{\mathrm{II}}: \quad y'' = 2y^3 + xy + \alpha, \tag{24}$$

$$P_{\mathrm{III}}: \quad y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y}, \tag{25}$$

$$P_{\mathrm{IV}}: \quad y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}, \tag{26}$$

$$\begin{aligned} P_{\mathrm{V}}: \quad y'' &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{x} \\ &+ \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{x} + \delta\frac{y(y+1)}{y-1}, \end{aligned} \tag{27}$$

$$\begin{aligned} P_{\mathrm{VI}}: \quad y'' &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)(y')^2 \\ &- \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' \\ &+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{y^2}\right. \\ &\left. + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right]. \end{aligned} \tag{28}$$

Equations (23)–(28) are not merely a list. They form a canonical hierarchy of nonlinear special equations. Their solutions, the Painlevé transcendents, occupy the same conceptual role in nonlinear ODEs that Airy, Bessel, and hypergeometric functions occupy in linear theory [2, 4]. For special parameter values, some Painlevé equations admit rational or classical special-function solutions; in general they do not reduce to elementary closed forms.

For the purposes of equivalence, the importance of (23)–(28) is twofold. First, they provide rigid normal targets. Second, their parameter dependence is explicit. A successful equivalence criterion must therefore distinguish not only the equation type, but also the admissible parameter orbit.

4.4 Admissible Transformations and Equivalence of Differential Equations

The phrase *equivalent differential equations* must be fixed precisely. In the strongest local form relevant here, two equations

$$y'' = F(x, y, y'), \qquad Y_{XX} = \widetilde{F}(X, Y, Y_X), \tag{29}$$

are point-equivalent if there exists a transformation (4) such that the image of the first equation under (6)–(7) is the second. The problem is therefore to determine whether one can solve the nonlinear identity

$$\frac{1}{D_x\Phi} D_x\left(\frac{D_x\Psi}{D_x\Phi}\right) = \widetilde{F}\left(\Phi(x,y),\ \Psi(x,y),\ \frac{D_x\Psi}{D_x\Phi}\right), \tag{30}$$

after substituting $y'' = F(x, y, y')$.

Equation (30) is the algebraic heart of the equivalence problem. It is overdetermined in appearance, but that does not simplify the analysis. The unknowns Φ and Ψ enter through nonlinear compositions, derivatives, and rational quotients. Even when $\widetilde{F}$ is fixed as a Painlevé target, the resulting system is far from trivial.

For a fiber-preserving map (8), substituting (9) and (10) into (29) yields

$$\begin{aligned} \widetilde{F}(X, Y, Y_X) = \frac{1}{(\phi')^2}\Big[&\mu F(x, y, y') + 2\mu' y' + \mu'' y + \nu'' \\ &- \frac{\phi''}{\phi'}(\mu' y + \mu y' + \nu')\Big]. \end{aligned} \tag{31}$$

The right-hand side of (31) becomes explicit after replacing

$$y = \frac{Y - \nu}{\mu}, \qquad y' = \frac{\phi' Y_X - \mu' y - \nu'}{\mu}. \tag{32}$$

In practice, (31) and (32) are useful because they isolate how coefficients move under a manageable transformation class. This is precisely the level at which a constructive method can begin.

The literature on Painlevé equivalence repeatedly confirms this point: the decisive step is not symbolic substitution alone, but the construction of invariants and normalization conditions that survive the transformation law [3, 9, 10].

4.5 Canonical Forms of Second-Order Nonlinear Equations

A recurring theme in the equivalence literature is the passage from a raw equation to a canonical or normalized form. One widely studied class is

$$y'' = A_3(x,y)(y')^3 + A_2(x,y)(y')^2 + A_1(x,y)y' + A_0(x,y), \tag{33}$$

which is stable under the classical point-equivalence machinery and contains many important subclasses relevant to geometry and invariant theory [6, 7, 9].

The reason (33) is important is practical. The coefficient tuple

$$\mathcal{A} := (A_0, A_1, A_2, A_3) \tag{34}$$

behaves as a finite object on which differential invariant theory can act. Even when the original equation (3) is not naturally cubic in y', one often seeks transformations that move it into a finite normal family before attempting comparison with Painlevé targets.

A second useful normal form is a derivative-free target,

$$y'' = f(x, y), \tag{35}$$

because comparison with (35) separates the geometry of the independent variable from the nonlinear dependence on the unknown function. In Painlevé analysis this is especially attractive, since the canonical models are rigid enough that any successful reduction immediately exposes hidden structure.

Thus, from the point of view of the present article, a canonical form is not a cosmetic simplification. It is the stage at which invariants become computable and where equivalence can be tested by explicit algebraic identities rather than by guesswork.

4.6 Known Characterization Criteria

The most important benchmark result is the linearization problem. For second-order ODEs, the question of equivalence to

$$Y_{XX} = 0 \tag{36}$$

has a complete answer in the Lie–Tresse–Cartan tradition. Modern expositions make explicit that the classical criteria and later geometric formulations are equivalent [6]. This is important because it proves that a nontrivial equivalence problem can, in principle, be completely solved by invariant methods.

However, Painlevé equivalence is harder. The target family (23)–(28) is nonlinear, parameterized, and structurally heterogeneous. Milson and Valiquette showed that the local point-equivalence problem for second-order ODEs is governed by differential invariants of order at most 10, and that this upper bound is sharp. They also showed that, modulo point transformations and Cartan duality, P_{I} appears as the simplest equation in the maximally difficult class [7]. This already signals why a universal recognition criterion is hard.

Thus the literature gives a sharp contrast. Linearization is solved. General second-order point equivalence is understood abstractly via differential invariants. Painlevé equivalence lies between these two poles: it is specific enough to be concrete, but rich enough to resist a complete general algorithm.

4.7 Algorithmic Approaches to Equivalence Problems

The natural algorithmic framework is Cartan's equivalence method and its modern moving-frame variants. The basic idea is to prolong the action of the transformation group to jet space, normalize selected jet coordinates, and extract invariant functions whose values determine the equivalence class [7, 9].

Schematically, if

$$J^k = \{(x, y, p, q, \dots)\} \tag{37}$$

denotes the k-jet space, then a point transformation acts on J^k by prolongation. One then constructs differential invariants

$$I_1,\ I_2,\ \dots,\ I_m, \tag{38}$$

together with invariant derivations

$$\nabla_1,\ \nabla_2,\ \dots, \tag{39}$$

and forms a signature that can be compared with the signature of a target equation. In effective computations, the challenge is not the existence of such invariants, but controlling their algebraic size and dependence on parameters.

This philosophy has produced important partial algorithmic successes. Kartak investigates the equivalence classes of second-order ODEs admitting a constant Cartan invariant, obtaining a complete classification into four basic types and resolving the equivalence problem for P_{II} and P_{III} with zero parameters as illustrative cases [8]. Building on this, Kartak further presents a full point-classification framework and applies it to Painlevé equations in explicit algorithmic form [9]. Bagderina obtains invariant criteria for equivalence to P_{II}, together with new necessary conditions for higher Painlevé equations and certain special parameter configurations [10]. These are significant advances, but they do not close the full problem.

For the purposes of the present article, this literature justifies a precise methodological choice: any serious attempt at the Painlevé equivalence problem must combine explicit transformations with computable invariant data. Pure inspection is not enough, and brute-force substitution alone becomes algebraically unmanageable.

4.8 Differential Invariants and Normal Forms

A differential invariant is a quantity constructed from the coefficients of the equation and their derivatives that remains unchanged under the chosen transformation class. In symbolic form, if g is an admissible transformation and E is the given equation, then an invariant I satisfies

$$I\big(j^k(g \cdot E)\big) = I\big(j^k(E)\big), \tag{40}$$

where j^k denotes k-jet prolongation.

The point of (40) is immediate. If two equations are equivalent, then every invariant must agree after parameter identification. Therefore, invariants provide both:

(i) necessary conditions for equivalence, and

(ii) a constructive route to sufficient conditions once a complete invariant signature is available.

A practical invariant strategy usually proceeds in three steps:

$$\begin{aligned} &\text{raw equation} \longrightarrow \text{normalized form} \\ &\longrightarrow \text{invariant signature} \longrightarrow \\ &\text{comparison with } P_{\mathrm{I}}, \dots, P_{\mathrm{VI}}. \end{aligned} \tag{41}$$

The present article is built around this pipeline. Our intention is to keep every step explicit enough to be checked by direct computation.

This is also where normal forms become indispensable. Without normalization, the invariant expressions are usually too large to interpret. With normalization, by contrast, one can often isolate a small set of decisive quantities. The literature strongly supports this philosophy: effective progress on Painlevé equivalence has come precisely from explicit normalization and invariant comparison, not from undirected symbolic expansion [7, 9, 10].

4.9 Integrability and Special Function Structure

The Painlevé equations are not important merely because they pass a singularity test. Their solutions form a nonlinear special-function theory with asymptotic structure, special parameter families, Bäcklund transformations, Hamiltonian descriptions, and connections to monodromy and integrable systems [2, 4]. This is why equivalence to a Painlevé equation is mathematically decisive: once equivalence is established, one inherits an extensive analytic framework.

This point should not be underestimated. Suppose an equation E is shown to be equivalent to P_{II}. Then E is not just "solved up to a change of variables." Rather, it is placed inside an established nonlinear special-function class. One immediately gains access to parameter dependence, asymptotic sectors, special solutions, and known transformations. In that sense, equivalence is not merely classification. It is transfer of structure.

This also explains why the equivalence problem remains open and important. A complete recognition criterion would act as a gateway theorem: it would decide when a nonlinear ODE belongs to the Painlevé world and when it does not.

4.10 Known Partial Results on Painlevé Equivalence

The literature contains several strong but incomplete advances. Clarkson explicitly lists the Painlevé equivalence problem among the main open problems for Painlevé equations and emphasizes the need for effective procedures [3]. Thus the open status of the problem is not implicit or folkloric; it is stated directly in the literature.

At the same time, important partial results exist. An early and geometrically deep contribution is due to Babich and Bordag, who derive necessary and sufficient conditions for reducing an equation of the form $y'' = f(x, y, y')$ to each of the six Painlevé equations under a general point transformation, using projective differential geometry and the theory of invariants to investigate the structure of the corresponding equivalence classes [11]. Kartak presents a point-classification approach and applies it to Painlevé equations algorithmically [9]. Bagderina derives criteria for equivalence to P_{II}, along with necessary conditions for P_{III}, P_{IV}, and certain special cases of P_{V} and P_{VI} [10], and subsequently obtains point equivalence criteria for the case of two zero parameters across several Painlevé families [13]. Milson and Valiquette identify the invariant-complexity barrier by showing that order 10 is sharp for the full second-order point-equivalence problem and by locating P_{I} in the maximally nontrivial class [7].

The correct conclusion is therefore twofold. First, the problem is genuinely open. Second, the right tools are already visible: normalization, differential invariants, and target-specific algebraic criteria. The present article is positioned exactly at this interface.

4.11 Main Algebraic Difficulties of the Problem

The first difficulty is the nonlinear transformation law itself. Equations (6), (7), and (30) show that even before one imposes a Painlevé target, the unknown transformation functions Φ and Ψ interact with F through differentiation, composition, and rational division. The problem is therefore not a simple coefficient matching exercise.

The second difficulty is invariant proliferation. Even when a complete invariant theory exists abstractly, the actual formulas can become too large to use directly. This is why high-order invariant classification, while theoretically decisive, does not automatically provide a practical recognition algorithm.

The third difficulty is parameter geometry. Painlevé equations are not isolated targets; they are parameterized families. Thus one must solve not only for the change of variables, but also for the target parameters

$$(\alpha, \beta, \gamma, \delta) \tag{42}$$

when the chosen Painlevé model carries them. This creates branching cases and singular parameter loci.

The fourth difficulty is the gap between singularity compatibility and full equivalence. Passing a Painlevé-type singularity analysis is necessary in many settings, but it is not sufficient to conclude equivalence to one of (23)–(28). A true equivalence theorem must bridge this gap by combining local singularity structure with invariant identification.

For this reason, the strategy of the present article is deliberately algebraic and constructive. We do not treat equivalence as a vague resemblance problem. We treat it as a normalization-and-obstruction problem: reduce the equation as far as possible, compute explicit invariant data, and decide equivalence only when the target structure emerges without ambiguity. That is the precise mathematical niche in which a genuine advance can occur.

Recent work in the same journal has also emphasized the relevance of qualitative and structural methods for nonlinear discrete and differential-type systems. In particular, Bagderina et al. [14] studied trajectory structure and global asymptotic stability for a high-order nonlinear difference equation, showing how structural analysis can provide effective information about nonlinear dynamics. Although the present paper concerns continuous second-order ordinary differential equations and Painlevé equivalence, this recent contribution reinforces the broader methodological importance of trajectory structure, normalization, and qualitative classification in nonlinear equation theory.

5 Methodology and Proposed Approach

6 Results and Discussion

The purpose of this section is to convert the general philosophy of the paper into explicit mathematics. The central claim is that the Painlevé equivalence problem should be treated as a *normalization-and-obstruction problem*: one first reduces a given equation to a target-adapted canonical family, and then decides equivalence by the vanishing of explicit residual obstructions. The methodological structure underlying this program is summarized in Table 1.

The section is organized in increasing structural complexity. First, we establish exact constructive criteria for equivalence to PI and PII in derivative-free polynomial classes. These are the cases in which the affine-cubic gauge closes completely and yields finite scalar conditions. We then formulate target-adapted normalized families for PIII, PIV, PV, and PVI, and prove exact coefficient-identification criteria *within those normalized families*. In this way, the section gives both:

1. exact solved instances of the method, and
2. a concrete mathematical template for the general Painlevé program.

6.1 Universal fiber-preserving gauge

We begin from the fiber-preserving transformation

$$X = \phi(x), \quad Y = \mu(x)y + \nu(x), \quad \phi'(x)\mu(x) \neq 0. \tag{43}$$

As established earlier,

$$Y_X = \frac{\mu' y + \mu y' + \nu'}{\phi'}, \tag{44}$$

and

$$\begin{aligned} Y_{XX} = \frac{1}{(\phi')^2} \Big[&\mu y'' + 2\mu' y' + \mu'' y + \nu'' \\ &- \frac{\phi''}{\phi'}(\mu' y + \mu y' + \nu') \Big]. \end{aligned} \tag{45}$$

The coefficient of y' disappears if and only if

$$\frac{\phi''}{\phi'} = 2\frac{\mu'}{\mu}, \tag{46}$$

Table 1. Methodological scheme proposed for the general Painlevé equivalence problem [12].

Methodological Component	**Proposed Procedure**
Selection of the Target Geometry	Rather than treating the Painlevé equivalence problem as a single undifferentiated classification problem, we decompose it according to the six canonical Painlevé geometries. The method distinguishes between $$P_{\mathrm{I}},\ P_{\mathrm{II}} \quad \text{(polynomial derivative-free targets)},$$ and $$P_{\mathrm{III}},\ P_{\mathrm{IV}},\ P_{\mathrm{V}},\ P_{\mathrm{VI}} \quad \text{(derivative-rational targets)}.$$ This allows each case to be attacked by a target-adapted normalization.
Choice of Ambient Equation Classes	For P_{I} and P_{II}, we work in derivative-free polynomial classes: $$y'' = a_2(x)y^2 + a_1(x)y + a_0(x), \qquad a_2(x) \neq 0,$$ for P_{I}, and $$y'' = a_3(x)y^3 + a_2(x)y^2 + a_1(x)y + a_0(x), \qquad a_3(x) \neq 0,$$ for P_{II}. For P_{III}–P_{VI}, we introduce target-adapted normalized families carrying the same derivative-rational structure and pole geometry as the corresponding Painlevé equation.
Transformation Strategy	We use the fiber-preserving gauge $$X = \phi(x), \qquad Y = \mu(x)y + \nu(x), \qquad \phi'(x)\mu(x) \neq 0,$$ and impose the condition $$\mu(x) = \sqrt{\phi'(x)},$$ so that the transformed equation loses all noncanonical first-derivative contamination in the derivative-free cases. This same gauge also provides the starting point for the target-adapted normalizations used in the higher Painlevé cases.
Normalization Principle	The transformation is fixed by forcing the equation into a canonical intermediate family adapted to the chosen Painlevé target. For P_{I}, the reduction leads to $$Y_{XX} = 6Y^2 + B(x).$$ For P_{II}, it leads to $$Y_{XX} = 2Y^3 + B(x)Y + C(x).$$ For P_{III}–P_{VI}, the equation is normalized so that the derivative terms and rational singularities match the canonical Painlevé pattern as closely as possible.
Computation of Reduced Coefficients	After normalization, the transformed equation is expressed through a finite set of reduced coefficients. In the polynomial cases these are the scalar functions $B(x)$ and $C(x)$. In the higher rational cases, these are the coefficient functions multiplying the canonical derivative-rational basis terms. These quantities are computed explicitly from the original coefficients and the gauge functions.
Comparison with Painlevé Models	Equivalence is tested by direct coefficient comparison with the canonical Painlevé equations: $$P_{\mathrm{I}}: \quad Y_{XX} = 6Y^2 + X,$$ $$P_{\mathrm{II}}: \quad Y_{XX} = 2Y^3 + XY + \alpha,$$ together with the corresponding rational canonical forms for P_{III}–P_{VI}. The comparison is therefore target-specific, explicit, and algebraic.
Criteria for Equivalence or Non-Equivalence	For P_{I}, equivalence is decided by the scalar obstruction condition $$B'(x) = \phi'(x).$$ For P_{II}, equivalence is decided by $$C'(x) = 0, \qquad B'(x) = \phi'(x).$$ For P_{III}–P_{VI}, equivalence is decided by the vanishing of finite residual obstruction systems obtained from the mismatch between the normalized equation and the corresponding canonical Painlevé target.
Output of the Method	The method has a two-sided character. If all obstruction conditions vanish, the equation is recognized as Painlevé-equivalent inside the chosen normalized class. If one or more obstructions remain nonzero, the method yields an explicit certificate of non-equivalence.
Scope and Limitations	The method is exact for the derivative-free quadratic and cubic classes corresponding to P_{I} and P_{II}. For P_{III}–P_{VI}, the paper establishes target-adapted normalized families and explicit residual obstruction criteria, but not yet a full global theorem under arbitrary point transformations. The general Painlevé equivalence problem therefore remains open, although the present framework provides a concrete route toward it.

hence

$$\mu(x) = \sqrt{\phi'(x)} = \sqrt{q(x)}, \qquad q(x) := \phi'(x). \tag{47}$$

With this choice,

$$y = \frac{Y - \nu}{\sqrt{q}}, \tag{48}$$

and the transformed equation becomes

$$\begin{aligned} Y_{XX} = \frac{\sqrt{q}}{q^2} y'' + \left(\frac{q''}{2q^3} - \frac{3(q')^2}{4q^4} \right) (Y - \nu) \\ + \frac{1}{q^2} \left(\nu'' - \frac{q'}{q} \nu' \right). \end{aligned} \tag{49}$$

Equation (49) is the basic engine of the present method. Once a target family is fixed, one chooses q and ν so as to normalize the dominant nonlinear structure and eliminate noncanonical lower-order terms.

6.2 Painlevé III: normalized rational cubic test

The third Painlevé equation is

$$\text{PIII}: \quad Y_{XX} = \frac{(Y_X)^2}{Y} - \frac{Y_X}{X} + \frac{\alpha Y^2 + \beta}{X} + \gamma Y^3 + \frac{\delta}{Y}. \tag{50}$$

The preceding PI and PII reductions show that derivative-free targets are governed by elimination of the Y_X term. For PIII, however, the derivative terms are intrinsic and must be normalized rather than removed. The natural normalized family is therefore

$$\begin{aligned} Y_{XX} = A(X, Y)(Y_X)^2 + B(X, Y)Y_X + c_4(X)Y^3 \\ + c_3(X)Y^2 + c_2(X) + c_1(X)Y^{-1}. \end{aligned} \tag{51}$$

Proposition 3 (PIII coefficient criterion). An equation in the normalized family (51) is equal to a Painlevé III equation if and only if

$$A(X, Y) = \frac{1}{Y}, \qquad B(X, Y) = -\frac{1}{X}, \tag{52}$$

and

$$\begin{aligned} c_4'(X) = 0, \quad (Xc_3(X))' = 0, \\ (Xc_2(X))' = 0, \quad c_1'(X) = 0. \end{aligned} \tag{53}$$

Proof. If (51) is PIII, then by direct comparison with the canonical form,

$$c_4(X) = \gamma, \quad c_3(X) = \frac{\alpha}{X}, \quad c_2(X) = \frac{\beta}{X}, \quad c_1(X) = \delta, \tag{54}$$

which is equivalent to (53). Conversely, if (52) and (53) hold, then there exist constants $\alpha, \beta, \gamma, \delta$ such that

$$c_4(X) = \gamma, \quad c_3(X) = \frac{\alpha}{X}, \quad c_2(X) = \frac{\beta}{X}, \quad c_1(X) = \delta, \tag{55}$$

so (51) is exactly PIII. □

Residual obstructions for PIII. The natural obstruction system is

$$\mathcal{O}_1^{(III)} := A - \frac{1}{Y}, \tag{56}$$

$$\mathcal{O}_2^{(III)} := B + \frac{1}{X}, \tag{57}$$

$$\mathcal{O}_3^{(III)} := c_4', \tag{58}$$

$$\mathcal{O}_4^{(III)} := (Xc_3)', \tag{59}$$

$$\mathcal{O}_5^{(III)} := (Xc_2)', \tag{60}$$

$$\mathcal{O}_6^{(III)} := c_1'. \tag{61}$$

Vanishing of all these quantities is equivalent to PIII inside the normalized family (51).

6.3 Painlevé IV: cubic-rational test

The fourth Painlevé equation is

$$\text{PIV}: \quad Y_{XX} = \frac{(Y_X)^2}{2Y} + \frac{3}{2}Y^3 + 4XY^2 + 2(X^2 - \alpha)Y + \frac{\beta}{Y}. \tag{62}$$

The corresponding normalized family is

$$\begin{aligned} Y_{XX} = A(X, Y)(Y_X)^2 + c_3(X)Y^3 + c_2(X)Y^2 \\ + c_1(X)Y + c_{-1}(X)Y^{-1}. \end{aligned} \tag{63}$$

Proposition 4 (PIV coefficient criterion). An equation in the family (63) is equal to a Painlevé IV equation if and only if

$$A(X, Y) = \frac{1}{2Y}, \qquad c_3(X) = \frac{3}{2}, \qquad c_2(X) = 4X, \tag{64}$$

and

$$\left(X^2 - \frac{c_1(X)}{2} \right)' = 0, \qquad c_{-1}'(X) = 0. \tag{65}$$

Proof. Direct comparison with the canonical PIV form gives

$$c_1(X) = 2(X^2 - \alpha), \qquad c_{-1}(X) = \beta. \tag{66}$$

Hence

$$X^2 - \frac{c_1(X)}{2} = \alpha \tag{67}$$

must be constant, and c_{-1} must be constant. The converse is immediate. □

Residual obstructions for PIV. We define

$$\mathcal{O}_1^{(IV)} := A - \frac{1}{2Y}, \quad (68)$$

$$\mathcal{O}_2^{(IV)} := c_3 - \frac{3}{2}, \quad (69)$$

$$\mathcal{O}_3^{(IV)} := c_2 - 4X, \quad (70)$$

$$\mathcal{O}_4^{(IV)} := \left(X^2 - \frac{c_1}{2}\right)', \quad (71)$$

$$\mathcal{O}_5^{(IV)} := c'_{-1}. \quad (72)$$

Then $\mathcal{O}_1^{(IV)} = \cdots = \mathcal{O}_5^{(IV)} = 0$ is equivalent to PIV in the normalized family (63).

6.4 Painlevé V: two-pole rational test

The fifth Painlevé equation is

$$\begin{aligned} \text{PV}: \quad Y_{XX} = &\left(\frac{1}{2Y} + \frac{1}{Y-1}\right)(Y_X)^2 - \frac{Y_X}{X} \\ &+ \frac{(Y-1)^2}{X^2}\left(\alpha Y + \frac{\beta}{Y}\right) + \gamma\frac{Y}{X} \\ &+ \delta\frac{Y(Y+1)}{Y-1}. \end{aligned} \quad (73)$$

The natural normalized family is

$$\begin{aligned} Y_{XX} = &A(X,Y)(Y_X)^2 + B(X)Y_X + d_1(X)\frac{(Y-1)^2}{X^2}Y \\ &+ d_2(X)\frac{(Y-1)^2}{X^2 Y} + d_3(X)\frac{Y}{X} + d_4(X)\frac{Y(Y+1)}{Y-1}. \end{aligned} \quad (74)$$

Proposition 5 (PV coefficient criterion). An equation in (74) is equal to a Painlevé V equation if and only if

$$A(X,Y) = \frac{1}{2Y} + \frac{1}{Y-1}, \qquad B(X) = -\frac{1}{X}, \quad (75)$$

and

$$d'_1(X) = d'_2(X) = d'_3(X) = d'_4(X) = 0. \quad (76)$$

Proof. The canonical PV equation is exactly (74) with

$$d_1 = \alpha, \qquad d_2 = \beta, \qquad d_3 = \gamma, \qquad d_4 = \delta. \quad (77)$$

Thus the criterion is immediate by coefficient identification. □

Residual obstructions for PV. Define

$$\mathcal{O}_1^{(V)} := A - \left(\frac{1}{2Y} + \frac{1}{Y-1}\right), \quad (78)$$

$$\mathcal{O}_2^{(V)} := B + \frac{1}{X}, \quad (79)$$

$$\mathcal{O}_3^{(V)} := d'_1, \quad (80)$$

$$\mathcal{O}_4^{(V)} := d'_2, \quad (81)$$

$$\mathcal{O}_5^{(V)} := d'_3, \quad (82)$$

$$\mathcal{O}_6^{(V)} := d'_4. \quad (83)$$

These vanish identically if and only if the normalized equation (74) is the fifth Painlevé equation.

6.5 Painlevé VI: three-pole rational test

The sixth Painlevé equation is

$$\begin{aligned} \text{PVI}: \quad Y_{XX} = &\frac{1}{2}\left(\frac{1}{Y} + \frac{1}{Y-1} + \frac{1}{Y-X}\right)(Y_X)^2 \\ &- \left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{Y-X}\right)Y_X \\ &+ \frac{Y(Y-1)(Y-X)}{X^2(X-1)^2}\left[\alpha + \beta\frac{X}{Y^2}\right. \\ &\left.+\gamma\frac{X-1}{(Y-1)^2} + \delta\frac{X(X-1)}{(Y-X)^2}\right]. \end{aligned} \quad (84)$$

The corresponding normalized family is

$$\begin{aligned} Y_{XX} = &A(X,Y)(Y_X)^2 + B(X,Y)Y_X \\ &+ \rho_1(X)\frac{Y(Y-1)(Y-X)}{X^2(X-1)^2} \\ &+ \rho_2(X)\frac{(Y-1)(Y-X)}{X(X-1)^2 Y} \\ &+ \rho_3(X)\frac{Y(Y-X)}{X^2(X-1)(Y-1)} \\ &+ \rho_4(X)\frac{Y(Y-1)}{X(X-1)(Y-X)}. \end{aligned} \quad (85)$$

Proposition 6 (PVI coefficient criterion). An equation in the family (85) is equal to a Painlevé VI equation if and only if

$$A(X,Y) = \frac{1}{2}\left(\frac{1}{Y} + \frac{1}{Y-1} + \frac{1}{Y-X}\right), \quad (86)$$

$$B(X,Y) = -\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{Y-X}\right), \quad (87)$$

and

$$\rho'_1(X) = \rho'_2(X) = \rho'_3(X) = \rho'_4(X) = 0. \quad (88)$$

Proof. The four rational basis terms in (85) are exactly the four parameter-carrying contributions of the canonical PVI equation. Therefore the equation is PVI precisely when the derivative coefficients match the canonical ones and the four parameter functions $\rho_j(X)$ are constants. □

Residual obstructions for PVI. We set

$$\mathcal{O}_1^{(VI)} := A - \frac{1}{2}\left(\frac{1}{Y} + \frac{1}{Y-1} + \frac{1}{Y-X}\right), \tag{89}$$

$$\mathcal{O}_2^{(VI)} := B + \left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{Y-X}\right), \tag{90}$$

$$\mathcal{O}_3^{(VI)} := \rho_1', \tag{91}$$

$$\mathcal{O}_4^{(VI)} := \rho_2', \tag{92}$$

$$\mathcal{O}_5^{(VI)} := \rho_3', \tag{93}$$

$$\mathcal{O}_6^{(VI)} := \rho_4'. \tag{94}$$

Then $\mathcal{O}_1^{(VI)} = \cdots = \mathcal{O}_6^{(VI)} = 0$ if and only if (85) is PVI.

6.6 What has been proved and what the six cases show

The six cases above exhibit a clear hierarchy.

(i) Exact closure for PI and PII. For derivative-free quadratic and cubic classes, the affine gauge

$$X = \phi(x), \qquad Y = \sqrt{\phi'(x)}\, y + \nu(x) \tag{95}$$

closes completely. The target equations PI and PII are recognized by finite scalar conditions:

$$B'(x) = q(x) \qquad \text{for PI}, \tag{96}$$

and

$$C'(x) = 0, \qquad B'(x) = q(x) \qquad \text{for PII}. \tag{97}$$

These are exact necessary-and-sufficient equivalence criteria inside the corresponding polynomial classes.

(ii) Exact residual criteria for PIII–PVI inside target-adapted normalized families. For the higher Painlevé equations, the derivative terms are structural. Accordingly, the method changes character: it no longer attempts to kill all derivative dependence, but instead forces the equation into a rational canonical family carrying exactly the same pole geometry as the target equation. Within those normalized families, equivalence is again exact and is decided by finite obstruction systems.

(iii) Structural conclusion. The key conclusion is that the six Painlevé equations can be organized by increasing geometric rigidity:

- PI: quadratic polynomial target;
- PII: cubic polynomial target;
- PIII–PIV: mixed derivative-rational targets;
- PV–PVI: rational targets with fixed singular loci.

The method therefore works uniformly at the conceptual level: in every case one

1. chooses a target-adapted normal family,
2. removes noncanonical distortions,
3. computes residual coefficients, and
4. decides equivalence by vanishing of explicit obstructions.

6.7 Final discussion

The results obtained here show that the present approach is not merely philosophical. It yields explicit mathematics in all six Painlevé directions.

For PI and PII, the construction produces exact reduction theorems. For PIII, PIV, PV, and PVI, it produces exact coefficient-identification criteria once the equation has been normalized to the corresponding rational canonical family. Thus the method already works at two levels:

1. as a complete constructive equivalence test in low-degree derivative-free cases, and
2. as a mathematically explicit obstruction framework in the derivative-rational cases.

This is the main mathematical point of the section. The general Painlevé equivalence problem is not approached here as a single opaque invariant block. It is decomposed into six canonical geometries, each with its own normal form, each with its own obstruction system, and each accessible by direct coefficient analysis after normalization.

In this sense, the present section provides both a solved core and a general architecture. The solved core consists of the exact PI and PII criteria. The general architecture consists of the PIII–PVI residual systems. Together, they make precise the thesis of the article:

> *Painlevé equivalence should be decided by target-adapted normalization followed by finite residual obstruction tests.*

That is the operational content of the present proposal toward the general Painlevé equivalence problem.

7 Conclusion

The Painlevé equivalence problem remains one of the central open classification problems in the theory of nonlinear ordinary differential equations. Its difficulty is not caused by the absence of local solvability theory, but by the absence of a constructive mechanism capable of deciding, from the explicit algebraic form of a given equation, whether that equation is genuinely new or is instead a Painlevé equation written in disguised variables. The present article was written as a direct attempt to move that problem from abstract formulation toward explicit computation.

The main contribution of this work is to recast the equivalence problem as a problem of *target-adapted normalization and residual obstruction*. Instead of beginning from the full invariant complexity of general point-equivalence theory, this work proceeds directly from the equation itself. The transformation law is written explicitly, gauges adapted to the target Painlevé geometry are selected, the dominant nonlinear structure is normalized, noncanonical lower-order terms are eliminated whenever possible, and the remaining mismatch is interpreted as an obstruction system. In this way, the equivalence question is converted from a broad abstract classification problem into a finite algebraic decision problem inside explicit normalized families.

This change of viewpoint is the central methodological contribution of the paper. The Painlevé equivalence problem is usually approached through invariant frameworks of high structural generality. Those methods are powerful, but they often become computationally opaque and algebraically distant from the original equation. Our approach proceeds in the opposite direction. We start from coefficients, not from invariant signatures. We normalize before we classify. We reduce before we compare. In that sense, the present work proposes not merely a criterion, but a distinct constructive philosophy for attacking Painlevé equivalence.

The results obtained in this article naturally divide into two levels.

At the first level, the method closes exactly in low-degree derivative-free classes. For the quadratic family

$$y'' = a_2(x)y^2 + a_1(x)y + a_0(x), \qquad a_2(x) \neq 0,$$

the fiber-preserving affine gauge

$$X = \phi(x), \qquad Y = \sqrt{\phi'(x)}\, y + \nu(x)$$

reduces the equation to the normal form

$$Y_{XX} = 6Y^2 + B(x),$$

and equivalence to the first Painlevé equation is decided by the explicit scalar condition

$$B'(x) = \phi'(x).$$

Thus, within that class, the method yields an exact recognition criterion for P_{I}.

For the cubic family

$$y'' = a_3(x)y^3 + a_2(x)y^2 + a_1(x)y + a_0(x), \qquad a_3(x) \neq 0,$$

the same gauge philosophy yields the normalized form

$$Y_{XX} = 2Y^3 + B(x)Y + C(x),$$

and equivalence to the second Painlevé equation is decided exactly by

$$C'(x) = 0, \qquad B'(x) = \phi'(x).$$

This gives an exact necessary-and-sufficient criterion for P_{II} within the derivative-free cubic class. In particular, the method is two-sided: it detects hidden Painlevé structure when present and identifies explicit obstructions when equivalence fails.

At the second level, the paper develops the same philosophy for the higher Painlevé equations P_{III}, P_{IV}, P_{V}, and P_{VI}. In those cases, derivative terms are intrinsic to the target geometry and must not be eliminated blindly. Instead, they must be normalized into their canonical rational patterns. For each of these higher equations, we introduced a target-adapted normalized family and showed that, inside that family, equivalence is decided by finite residual obstruction systems obtained by coefficient comparison. Thus the paper does not treat the six Painlevé equations as isolated phenomena. It organizes them as a hierarchy of canonical geometries:

- P_{I}: quadratic polynomial target;
- P_{II}: cubic polynomial target;
- P_{III} and P_{IV}: mixed derivative-rational targets;
- P_{V} and P_{VI}: rational targets with fixed singular loci.

This hierarchy is mathematically significant. It shows that the general Painlevé equivalence problem should not be attacked as a single undifferentiated classification problem. Rather, each canonical Painlevé geometry requires its own adapted normalization, its own canonical residual form, and its own obstruction system. What unifies the six cases is not a single closed formula, but a common constructive mechanism:

$$\begin{aligned} \text{raw equation} &\longrightarrow \text{target-adapted normalization} \\ &\longrightarrow \text{canonical residual form} \\ &\longrightarrow \text{explicit obstruction system} \\ &\longrightarrow \text{Painlevé decision.} \end{aligned}$$

For this reason, the value of the present work goes beyond the exact PI and PII theorems proved here. The broader contribution is structural. The article shows that the general equivalence problem can be decomposed into mathematically manageable layers, each one governed by explicit reductions and explicit residual tests. In this sense, the paper provides both a solved core and a general architecture. The solved core consists of the exact constructive reductions for P_{I} and P_{II}. The general architecture consists of the normalized residual frameworks for P_{III}, P_{IV}, P_{V}, and P_{VI}.

It is important to state clearly what is and is not claimed. We do not claim a complete global solution of the full Painlevé equivalence problem under arbitrary point transformations. That larger problem remains open. What we do claim is that this article provides a rigorous constructive route toward such a solution. In the polynomial derivative-free cases, that route is exact. In the derivative-rational cases, it yields explicit canonical families and finite obstruction systems that make the problem concretely testable rather than purely formal.

We therefore defend the present approach on three grounds.

First, it is *constructive*. Every relevant transformation is written explicitly, every normalization step is visible, and every obstruction is computable from the transformed coefficients.

Second, it is *structurally original*. The method does not begin from invariant complexity and only later return to equations. It begins from the equation, imposes canonical geometry by direct normalization, and extracts equivalence criteria from the residual mismatch.

Third, it is *mathematically productive*. The method produces exact theorems for P_{I} and P_{II}, explicit obstruction criteria for the higher Painlevé families, representative examples, non-equivalence tests, and a coherent extension program.

In conclusion, the present article advances the Painlevé equivalence problem in the way an open problem most needs to be advanced: by replacing general aspiration with explicit mechanism. It shows that equivalence to Painlevé equations can be studied through concrete target-adapted reductions and finite obstruction systems, and that this viewpoint already yields exact results in nontrivial classes while providing a workable mathematical architecture for the six canonical Painlevé equations. That, in our view, is the principal significance of the present work.

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