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RESEARCH ARTICLE

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Strategy Dynamics of Three-Strategy Snowdrift Game Induced by Reward Strategy and Payoff Delay

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Abstract

In the evolutionary game, the introduction of the third or more specific strategies to study cooperative evolution has attracted widespread attention recently. The main feature of this paper is to adopt a special reward strategy as the third choice to observe the evolution dynamics. This strategy can improve the utilization of resources and help cooperators and defectors get greater benefits. We investigate the evolutionary game dynamics of a three-strategy snowdrift game with special reward strategy and payoff delay. First, for the nondelay system, we discuss the dynamic properties of strategies, including the existence, stability and bistability of equilibrium states. Furthermore for the time-delay system, we explore the existence conditions of Hopf bifurcation with payoff delay as a parameter. Then we get the direction and stability of Hopf bifurcation through theoretical derivations and numerical simulations. The results show that (i) cooperators, defectors and rewarders can coexist stably; (ii) as the reward increases, the defective strategy gradually disappears, which helps to promote cooperation; (iii) the stable equilibrium becomes unstable when the payoff delay is large, and Hopf bifurcation and stable periodic oscillation appears; (iv) when the payoff delay is large enough, the defective strategy disappears. Our work may promote the prosperity of cooperation among biological populations.

Keywords: evolutionary game theory, reward, three-strategy, time delay, hopf bifurcation.

1 Introduction

In real life, the snowdrift game is widely used in microbial systems to human populations. It happens when defectors can also obtain the benefits obtained by cooperative behavior [1]. For example, yeast cells producing invertase cooperate only when they compete with cheating cells that do not produce invertase-this corresponds to the best strategy of the snowdrift game[2]. Likewise, the sentry behavior of mammals may be controlled by the interaction of snowdrift games. Clutton-Brock et al. [3] found that guarding may be the best behavior of individuals if no other animals are guarding.

In addition, cooperation is ubiquitous in biological systems, but the conflict between individual and group

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rationalities often does not help the development of cooperation. Evolutionary game theory is widely used in biological systems [4–6]. Game theory research on cooperative evolution is mostly based on replicator dynamics. And through replicator dynamics behavior, we see the evolution of the cooperation and defection strategies. Similarly, researchers have been interested in how to increase the proportion of cooperative strategy in snowdrift game [7], such as spatial structural snowdrift game [8], network snowdrift game [9], and asynchronous snowdrift game [10]. Besides, many factors can affect the evolution of cooperation. For instance, Ji et al. [11] found that fixing the parameters of personal information and environmental information in the snowdrift game can maximize the cooperation ratio. Improving the complex interaction of players can increase the frequency of cooperative strategy in snowdrift game[12].

How to promote cooperation in multi-strategy evolutionary games is also the focus of researchers' attention, such as in the market [13], biological population [14], etc. Punishment and reward are both ways to increase cooperation. Chowdhury et al. [15] introduced a competitive strategy involving policing and free space into the prisoner's dilemma. This approach facilitates the coexistence of diverse strategies, thereby contributing to the maintenance of diversity within the social-ecological environment. In a network evolutionary game model with four strategies [16], cooperators, rewards, and loners finally stably coexist in a population with a spatial structure. In addition, reward is one of the important ways to promote cooperation in different game types. The n-person snowdrift game is an important extended model of snowdrift game [17]. It was found that the high cost of reward would not be conducive to the universality of reward [18]. Pan et al. [19] analyzed the influence of the special cooperative strategy of both penalty and reward on cooperation. Reward strategy rewards cooperative strategy by sacrificing some of its own benefits, and thus promote cooperation. In some ecosystems, the introduction of alien species may increase the available resources and promote mutual symbiosis between species [20]. On this basis, this paper considers a special reward strategy, which aims to improve the benefits of cooperation strategy and defection strategy, so as to optimize the utilization of resources. Specifically, when the cooperation strategy or defection strategy meets the special reward strategy, the participants' income increases on the original basis. This mechanism can enhance the efficiency of resource utilization in the ecosystem.

It takes some time for all living things to react, and the interaction between individuals may affect future outcomes, therefore, considering time delay is indispensable [21–23]. Since the introduction of time delay in replication dynamics [24], considering time delay in evolutionary game models is more and more common. In [25], the function of stochasticity and time delay on the social and biological models are studied. Alfaro and Sanjuan [26] discussed the effect of punishment time delay on cooperation in the game and found that punishment time delay is not conducive to cooperation. In an evolutionary population with dependent strategy delay, the delay affects the disappearance of internal steady state or the emergence of new steady state [27]. Therefore, considering time delay in the three-strategy snowdrift game is necessary.

The remainder of this paper is structured as follows. Section 2 shows the model of the three-strategy snowdrift game without payoff delay and analyzes the existence and stability of the equilibrium states. And the conditions of bistable phenomenon and the transcritical bifurcation experienced by the system are discussed. In section 3, we construct a time-delay system and discusses the existence of Hopf bifurcation. The direction, periodicity, and stability of the periodic solution of Hopf bifurcation are obtained in section 4. In Section 5, numerical simulations are conducted to validate the preceding theoretical derivations. Section 6 presents the summary and prospect.

2 The three-strategy snowdrift game

Based on the snowdrift game, Li et al. [34] considered that when the cooperator meets the defector, the defector obtains $b\alpha$ instead of the entire profit b, then the cooperator obtains the benefit of $b(1-\alpha)-c$, and the payoff matrix is

$$\begin{array}{cc} & \begin{array}{cc} C & \quad\quad D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} \frac{b-c}{2} & b\,(1-\alpha)-c \\ b\alpha & 0 \end{pmatrix} \end{array}, \tag{1}$$

where α is the ratio of the defector's payoff, and C and D stand for cooperation strategy and defection strategy respectively. Here we assume that benefit b is greater than cost c. Then the ratio of the cooperator's payoff is $1-\alpha$, and the cost of choosing cooperation is c.

Rewarding cooperators is a valid way to improve cooperation in evolutionary game theory. A common example is that client fish can stabilize the cooperative behavior of cleaner fish by indirectly rewarding them through repeated interactions in which ectoparasites are removed [28]. Here, we consider a special reward strategy R, which can improve the utilization of resources and increase the benefits of both parties when they meet cooperators and defectors. Specifically, when the cooperator encounters the reward strategy, the cooperator and the rewarder get a part of the income u. When the defector meets the reward strategy, the defector gets an extra income u, but the reward strategy gets nothing. The encounter between the third strategy and the original strategy provides additional incentives for participants, thus promoting the optimal utilization of environmental resources. Now the payoff matrix is

$$\begin{array}{c} \\ C \\ D \\ R \end{array} \begin{array}{c} \begin{array}{ccc} C & D & R \end{array} \\ \begin{pmatrix} \frac{b-c}{2} & b(1-\alpha)-c & \frac{b-c}{2}+u \\ b\alpha & 0 & u \\ \frac{b-c}{2}+u & 0 & 0 \end{pmatrix} \end{array}. \tag{2}$$

The proportion of cooperators at time t is $x(t)$, the proportion of defectors at time t is $y(t)$, and the proportion of rewarders at time t is $z(t)$ in the infinite population.

2.1 Nondelay model

According to the payoff matrix (2), we get the benefits of cooperators, defectors and rewarders as follows

$$\begin{aligned} \pi_C(t) &= \frac{b-c}{2}x(t) + (b(1-\alpha)-c)\,y(t) \\ &\quad + \left(\frac{b-c}{2}+u\right) z(t), \qquad (3) \\ \pi_D(t) &= b\alpha x(t) + uz(t), \qquad (4) \\ \pi_R(t) &= \left(\frac{b-c}{2}+u\right) x(t). \end{aligned}$$

The three-strategy snowdrift game system is described by

$$\begin{cases} \dot{x}(t) = x(t)\left(\pi_C(t) - \bar{f}(t)\right), \\ \dot{y}(t) = y(t)\left(\pi_D(t) - \bar{f}(t)\right), \\ \dot{z}(t) = z(t)\left(\pi_R(t) - \bar{f}(t)\right), \end{cases} \tag{5}$$

where the average payoff $\bar{f}(t)$ is $\bar{f}(t) = x(t)\pi_C(t) + y(t)\pi_D(t) + z(t)\pi_R(t)$.

Combining equations (3)-(5), the system (5) is rewritten in a high-order form as follows

$$\begin{cases} \dot{x}(t) = x(t)\Big[\frac{b-c}{2}x^2(t) + 2ux^2(t) + uy^2(t) \\ \qquad +3ux(t)y(t) + \frac{b-c}{2}y - \alpha by(t) - 2uy(t) \\ \qquad -(b-c)x(t) - 3ux(t) + \frac{b-c}{2} + u\Big], \\ \dot{y}(t) = y(t)\Big[\frac{b-c}{2}x^2(t) + 2ux^2(t) + uy^2(t) \\ \qquad +3ux(t)y(t) - 3ux(t) + \alpha bx(t) \\ \qquad -(b-c)x(t) - 2uy(t) + u\Big]. \end{cases} \tag{6}$$

2.2 Results without time delay

The system (6) exhibits several complex equilibrium points, including: $E_1(1,0,0)$, $E_2(0,1,0)$, $E_3(0,0,1)$, $E_4\left(\frac{2(b-c-\alpha b)}{b-c}, \frac{-b+c+2\alpha b}{b-c}, 0\right)$, $E_5\left(\frac{b-c+2u}{b-c+4u}, 0, \frac{2u}{b-c+4u}\right)$ and $E_6(x^*, y^*, z^*)$, where

$$\begin{aligned} x^* &= \frac{4u(b-c-\alpha b)}{4\alpha^2b^2 - 4\alpha b^2 + 4\alpha bc - 4\alpha bu + (b-c)^2 + 4u(b-c)}, \\ y^* &= \frac{2\alpha b^2 - 2\alpha bc + 4\alpha bu - 4bu + 4cu - (b-c)^2}{4\alpha^2b^2 - 4\alpha b^2 + 4\alpha bc - 4\alpha bu + (b-c)^2 + 4u(b-c)}, \\ z^* &= \frac{4\alpha^2b^2 - 6\alpha b^2 + 6\alpha bc - 4\alpha bu + 4bu - 4cu + 2(b-c)^2}{4\alpha^2b^2 - 4\alpha b^2 + 4\alpha bc - 4\alpha bu + (b-c)^2 + 4u(b-c)}. \end{aligned}$$

E_1, E_2 and E_3 always exist. Next, we consider the conditions for the existence of E_4, E_5 and E_6.

- Existence of equilibria:

(I) $E_4\left(\frac{2(b-c-\alpha b)}{b-c}, \frac{-b+c+2\alpha b}{b-c}, 0\right)$ exists if $0 < \frac{2(b-c-\alpha b)}{b-c} < 1$ and $0 < \frac{-b+c+2\alpha b}{b-c} < 1$. Under the conditions of snowdrift game, E_5 always exists.

(II) $E_5\left(\frac{b-c+2u}{b-c+4u}, 0, \frac{2u}{b-c+4u}\right)$ exists if $0 < \frac{b-c+2u}{b-c+4u} < 1$ and $0 < \frac{2u}{b-c+4u} < 1$. Under the condition of snowdrift game, E_6 always exists.

(III) $E_6(x^*, y^*, z^*)$ exists if $0 < x^*, y^*, z^* < 1$, i.e.

$$\begin{aligned} \kappa_1 :&\quad 4\alpha^2b^2 - 4\alpha b^2 + 4\alpha bc + (b-c)^2 > 0, \\ \kappa_2 :&\quad 2\alpha b^2 - 2\alpha bc + 4\alpha bu - 4bu + 4cu - (b-c)^2 > 0, \\ \kappa_3 :&\quad 4\alpha^2b^2 - 6\alpha b^2 + 2\alpha bc + 2(b-c)^2 + 8u(b-c) \\ &\quad -8\alpha bu > 0. \end{aligned}$$

Then let

$$\begin{cases} F(x,y) = \Big[\frac{b-c}{2}x^2(t) + 2ux^2(t) + uy^2(t) + 3ux(t)y(t) + \frac{b-c}{2}y \\ \quad -\alpha by(t) - 2uy(t) - (b-c)x(t) - 3ux(t) + \frac{b-c}{2} + u\Big], \\ G(x,y) = \Big[\frac{b-c}{2}x^2(t) + 2ux^2(t) + uy^2(t) + 3ux(t)y(t) \\ \quad -3ux(t) + \alpha bx(t) - (b-c)x(t) - 2uy(t) + u\Big]. \end{cases} \tag{7}$$

Next, the Jacobian matrix of system (6) is

$$J = \begin{pmatrix} F(x,y) + x(t)\frac{\partial F(x,y)}{\partial x} & x(t)\frac{\partial F(x,y)}{\partial y} \\ y(t)\frac{\partial G(x,y)}{\partial x} & G(x,y) + y(t)\frac{\partial G(x,y)}{\partial y} \end{pmatrix}, \tag{8}$$

where

$$\begin{cases} \frac{\partial F(x,y)}{\partial x} = (b-c)x(t) + 4ux(t) + 2uy(t) - b \\ \quad + c - 3u, \\ \frac{\partial F(x,y)}{\partial y} = 2uy(t) + 3ux(t) + \frac{b-c}{2} - \alpha b - 2u, \\ \frac{\partial G(x,y)}{\partial x} = (b-c)x(t) + 4ux(t) + 3uy(t) - b \\ \quad + c - 3u + \alpha b, \\ \frac{\partial G(x,y)}{\partial y} = 2uy(t) + 3ux(t) - 2u. \end{cases} \tag{9}$$

Theorem 1. *(1) The equilibrium $E_1(1,0,0)$ is unstable.*
(2) The boundary equilibrium $E_2(0,1,0)$ is unstable.
(3) The boundary equilibrium $E_3(0,0,1)$ is unstable.
(4) $E_4\left(\frac{2(b-c-\alpha b)}{b-c}, \frac{-b+c+2\alpha b}{b-c}, 0\right)$ is stable when κ_4 is true.
(5) $E_5\left(\frac{b-c+2u}{b-c+4u}, 0, \frac{2u}{b-c+4u}\right)$ is stable when κ_2 is not true.
(6) $E_6(x^*, y^*, z^*)$ is stable when $j_{61}j_{64} - j_{62}j_{63} > 0$ and $j_{61} + j_{64} < 0$, where the values of j_{61}, j_{62}, j_{63}, j_{64} are given in equation (13) below.

Proof. (1) The eigenvalues of the Jacobian matrix at E_1 are $\lambda_{11} = u > 0$ and $\lambda_{12} = -\frac{b-c}{2} + \alpha b$, so $E_1(1,0,0)$ is always unstable.

(2) The eigenvalues of the Jacobian matrix at E_2 are $\lambda_{21} = b - c - \alpha b$ and $\lambda_{22} = 0$. Hence $E_2(0,1,0)$ is unstable.

(3) The eigenvalues of the Jacobian matrix at E_3 are $\lambda_{31} = \frac{b-c}{2} + u$ and $\lambda_{32} = u > 0$. Therefore, $E_3(1,0)$ is unstable.

(4) The Jacobian matrix at $E_4\left(\frac{2(b-c-\alpha b)}{b-c}, \frac{-b+c+2\alpha b}{b-c}, 0\right)$ is

$$J_4 = \begin{pmatrix} j_{41} & j_{42} \\ j_{43} & j_{44} \end{pmatrix}, \tag{10}$$

where

$$j_{41} = -\frac{2(b-c-\alpha b)\left(2\alpha b^2 - 2\alpha bc + 2\alpha bu - (b-c)^2 - 2bu + 2cu\right)}{(b-c)^2},$$

$$j_{42} = -\frac{(b-c-\alpha b)\left(2\alpha b^2 + 2\alpha bc + 4\alpha bu - (b-c)^2 - 4bu + 4cu\right)}{(b-c)^2},$$

$$j_{43} = -\frac{(b-c+2u)\left(2\alpha^2 b^2 - 3\alpha b^2 + 3\alpha bc + (b-c)^2\right)}{(b-c)^2},$$

$$j_{44} = -\frac{2u\left(2\alpha^2 b^2 - 3\alpha b^2 + 3\alpha bc + (b-c)^2\right)}{(b-c)^2}.$$

Due to $\lambda_4^2 - (j_{41} + j_{44})\lambda_4 + (j_{41}j_{44} - j_{42}j_{43}) = 0$, we get $\lambda_{41} = \frac{2\alpha^2 b^2 - 3\alpha b^2 + 3\alpha bc + (b-c)^2}{b-c}$ and $\lambda_{42} = \frac{2\alpha^2 b^2 - 3\alpha b^2 + 3\alpha bc + (b-c)^2 + 2u(b-c-\alpha b)}{b-c}$. E_4 is stable when $\kappa_4 : 2\alpha^2 b^2 - 3\alpha b^2 + 3\alpha bc + (b-c)^2 + 2u(b-c-\alpha b) < 0$ holds.

(5) The Jacobian matrix at $E_5\left(\frac{b-c+2u}{b-c+4u}, 0, \frac{2u}{b-c+4u}\right)$ is

$$J_5 = \begin{pmatrix} -\frac{u(b-c+2u)}{b-c+4u} & \Lambda_1 \\ 0 & \Lambda_2 \end{pmatrix}, \tag{11}$$

where $\Lambda_1 = -\frac{(b-c+2u)\left(2\alpha b^2 + 4u^2 - 2\alpha bc + 8\alpha bu - 6bu + 6cu - (b-c)^2\right)}{(b-c+4u)^2}$ and $\Lambda_2 = \frac{2\alpha b^2 - 2\alpha bc + 4\alpha bu - 4bu + 4cu - (b-c)^2}{2(b-c+4u)}$. Therefore $\lambda_{51} = -\frac{u(b-c+2u)}{b-c+4u} < 0$ and $\lambda_{52} = \Lambda_2$. E_5 is stable when κ_2 is not true.

(6) The Jacobian matrix at $E_6(x^*, y^*, z^*)$ is

$$J_6 = \begin{pmatrix} j_{61} & j_{62} \\ j_{63} & j_{64} \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} j_{61} &= \frac{3}{2}(b-c)x^* + 6ux^{*2} + 2uy^{*2} + 6ux^*y^* - 6ux^* \\ &\quad + \frac{b-c}{2}y^* - \alpha by^* - 2uy^* + 2(b-c)x^* + \frac{b-c}{2} + u, \\ j_{62} &= 4ux^*y^* + 3ux^{*2} + \frac{b-c}{2}x^* - \alpha bx^* - 2ux^*, \\ j_{63} &= (b-c)x^*y^* + 4ux^*y^* + 3y^{*2} - by^* + cy^* \\ &\quad - 3uy^* + \alpha by^*, \\ j_{64} &= \frac{(b-c)}{2}x^{*2} + 2ux^{*2} + 5uy^{*2} + 6ux^*y^* - 3ux^* \\ &\quad + \alpha bx^* - 4uy^* - (b-c)x^* + u. \end{aligned}$$

Table 1. Existence and stability of equilibria.

Equilibria	Existence	Stability
E_1	always	unstable
E_2	always	unstable
E_3	always	unstable
E_4	$0 < \frac{2(b-c-\alpha b)}{b-c} < 1$ and $0 < \frac{-b+c+2\alpha b}{b-c} < 1$	κ_4
E_5	always	κ_2 is not true
E_6	κ_1–κ_3	$j_{61}j_{64} - j_{62}j_{63} > 0$ and $j_{61} + j_{64} < 0$

The characteristic equation for matrix (12) is

$$\lambda_6^2 - (j_{61} + j_{64})\,\lambda_6 + (j_{61}j_{64} - j_{62}j_{63}) = 0. \qquad (13)$$

The two roots of the equation (13) are less than 0 when $j_{61}j_{64} - j_{62}j_{63} > 0$ and $j_{61} + j_{64} < 0$. Thus $E_6\,(x^*, y^*, z^*)$ is stable. □

Table 1 summarizes the above results.

When u is less than 0, the third strategy competes with the other two strategies, which would damage the income of cooperators and defectors. In this case, the following is the bistable theorem.

Theorem 2. *When $\frac{b-c}{2} + u < 0$ and κ_5 are true, E_3 and E_4 are stable.*

2.3 Bifurcation analysis

Theorem 3. *(1) When parameter $u = \bar{u} = \frac{\alpha b(3b-3c-2\alpha b)-(b-c)^2}{2(b-c-\alpha b)}$, the system experiences a transcritical bifurcation at E_4.*
(2) The system undergoes a transcritical bifurcation at E_5 when parameter $\alpha = \bar{\alpha} = \frac{(b-c)^2+4bu-4cu}{2b^2-2bc+4bu}$.

Proof. (1) According to J_4, when $u = \bar{u}$, $\lambda_{42} = 0$ and the other eigenvalue $\lambda_{41} = -\frac{u(b-c+2u)}{b-c+4u} < 0$. And the eigenvectors corresponding to the zero eigenvalues of matrix J_5 and J_5^{T} are

$$V_1 = \begin{pmatrix} 1 \\ \Lambda_3 \end{pmatrix},\ W_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $\Lambda_3 = -\frac{(c-b+\alpha b)(c-b+2\alpha b)^2}{2\alpha b(2\alpha^2 b^2-3\alpha b^2+3\alpha bc+b^2-2bc+c^2)}$.

According to (6), we get

$$F_u(E_4, \bar{u}) = \begin{pmatrix} x\left(2x^2 + y^2 + 3xy - 2y - 3x + 1\right) \\ y\left(2x^2 + y^2 + 3xy - 2y - 3x + 1\right) \end{pmatrix}_{(E_4)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$DF_u(E_4, \bar{u})V_1 = \begin{pmatrix} x\,(4x+3y-3) & x\,(3x+2y-2) \\ y\,(4x+3y-3) & y\,(3x+2y-2) \end{pmatrix}_{(E_4)} \begin{pmatrix} 1 \\ \Lambda_3 \end{pmatrix} = \begin{pmatrix} x_4^2\,(1+\Lambda_3) \\ x_4 y_4\,(1+\Lambda_3) \end{pmatrix},$$

and

$$D^2F(E_4, \bar{u})(V_1, V_1) = \begin{pmatrix} \Lambda_4 \\ \Lambda_5 \end{pmatrix},$$

where

$$\begin{aligned} x_4 &= \frac{2\,(b-c-\alpha b)}{b-c},\ y_4 = \frac{-b+c+2\alpha b}{b-c}, \\ \Lambda_4 &= 3\,(b-c)\,x_4 + 2\bar{u}x_4 - 2\,(b-c) \\ &\quad +2\left(4\bar{u}y_4 + \frac{b-c}{2} - \alpha b - 2\bar{u} + 6\bar{u}x_4\right)\Lambda_3 + 4\bar{u}x_5\Lambda_3^2, \\ \Lambda_5 &= (b-c+4\bar{u})\,y_4 + 2\left[(b-c)\,x_4 + \bar{u}\right. \\ &\quad \left. +2\bar{u}y_4 + \alpha b - b + c\right]\Lambda_3 + (6\bar{u}y_4 + 2\bar{u})\,\Lambda_3^2. \end{aligned}$$

Then

$$\begin{cases} W_1^{\mathrm{T}}F_u(E_4, \bar{u}) = 0, \\ W_1^{\mathrm{T}}DF_u(E_4, \bar{u})V_1 = x_4^2\,(1+\Lambda_3) \neq 0, \\ W_1^{\mathrm{T}}D^2F(E_4, \bar{u})(V_1, V_1) = \Lambda_4 \neq 0. \end{cases}$$

Therefore, when parameter $u = \bar{u} = \frac{\alpha b(3b-3c-2\alpha b)-(b-c)^2}{2(b-c-\alpha b)}$, the system experiences a transcritical bifurcation at E_4.

(2) According to J_5, when $\alpha = \bar{\alpha}$, one eigenvalue is 0 and the other $\lambda_{51} = -\frac{u(b-c+2u)}{b-c+4u} < 0$. Then, the eigenvectors corresponding to the zero eigenvalues of matrix J_5 and J_5^{T} are

$$V_2 = \begin{pmatrix} 1 \\ -\frac{(b-c+4u)(b-c+2u)}{2u(3b-3c+4u)} \end{pmatrix},\ W_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

According to (6), we get

$$F_\alpha(E_5, \bar{\alpha}) = \begin{pmatrix} -bxy \\ bxy \end{pmatrix}_{(E_5)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$DF_{\alpha}(E_5,\bar{\alpha})V_2 = \begin{pmatrix} -by & -bx \\ by & bx \end{pmatrix}_{(E_5)} \begin{pmatrix} 1 \\ -\frac{(b-c+4u)(b-c+2u)}{2u(3b-3c+4u)} \end{pmatrix} = \begin{pmatrix} \frac{b(b-c+2u)^2}{2u(3b-3c+4u)} \\ -\frac{b(b-c+2u)^2}{2u(3b-3c+4u)} \end{pmatrix},$$

and

$$D^2F(E_5,\bar{\alpha})(V_2,V_2) = \begin{pmatrix} \Lambda_6 \\ (\bar{\alpha}b-u)v_2+\left(\frac{6u(b-c+2u)}{b-c+4u}-4u\right)v_2^2 \end{pmatrix},$$

where $\Lambda_6 = b - c + \left(b-c-2\alpha b-4u+\frac{12u(b-c+2u)}{b-c+4u}\right)v_2 + \frac{(b-c+4u)(b-c+2u)^3}{u(3b-3c+4u)^2}$ and $v_2 = -\frac{(b-c+4u)(b-c+2u)}{2u(3b-3c+4u)}$. Then

$$\begin{cases} W_2^{\mathrm{T}}F_{\alpha}(E_5,\bar{\alpha}) = 0, \\ W_2^{\mathrm{T}}DF_{\alpha}(E_5,\bar{\alpha})V_2 = \dfrac{b\,(b-c+2u)^2}{2u\,(3b-3c+4u)} \neq 0, \\ W_2^{\mathrm{T}}D^2F(E_5,\bar{\alpha})(V_2,V_2) = \Lambda_6 \neq 0. \end{cases}$$

Therefore, when parameter $\alpha = \bar{\alpha} = \frac{(b-c)^2+4bu-4cu}{2b^2-2bc+4bu}$, the system undergoes a transcritical bifurcation at E_5. □

3 Model and results with time delay

Next, we examine the effect of time delay on the three-strategy snowdrift game. During the game, players can not get benefits in time, which takes some time, so we assume that the benefits of players depend on the population ratio at $(t-\tau)$. The payoffs for cooperators, defectors, and rewarders are defined as follows:

$$\begin{aligned} \pi_{C_2}(t) &= \frac{b-c}{2}x(t-\tau) + (b(1-\alpha)-c)\,y(t-\tau) \\ &\quad + \left(\frac{b-c}{2}+u\right)z(t-\tau), \\ \pi_{D_2}(t) &= b\alpha x(t-\tau) + uz(t-\tau), \\ \pi_{R_2}(t) &= \left(\frac{b-c}{2}+u\right)x(t-\tau). \end{aligned} \tag{14}$$

Then the average payoff $\bar{f}_2(t)$ is $\bar{f}(t) = x(t)\,\pi_{C_2}(t) + y(t)\,\pi_{D_2}(t) + z(t)\,\pi_{R_2}(t)$.

Therefore, the complex high-order time-delay system is getting by

$$\begin{cases} \dot{x}(t) = x(t)\left[(1-x(t))\left(\pi_{C_2}(t)-\pi_{R_2}(t)\right) - y(t)\left(\pi_{D_2}(t)-\pi_{R_2}(t)\right)\right], \\ \dot{y}(t) = y(t)\left[(1-y(t))\left(\pi_{D_2}(t)-\pi_{R_2}(t)\right) - x(t)\left(\pi_{C_2}(t)-\pi_{R_2}(t)\right)\right], \end{cases} \tag{15}$$

where

$$\begin{aligned} \pi_{C_2}(t)-\pi_{R_2}(t) &= -\left(\frac{b-c}{2}+2u\right)x(t-\tau) \\ &\quad + \left(\frac{b-c}{2}-\alpha b-u\right)y(t-\tau) + \frac{b-c}{2}+u, \\ \pi_{D_2}(t)-\pi_{R_2}(t) &= \left(\alpha b-\frac{b-c}{2}-2u\right)x(t-\tau) \\ &\quad - uy(t-\tau)+u. \end{aligned} \tag{16}$$

Linearizing the system (15) at the internal equilibrium results in the following

$$\begin{cases} \dot{x}(t) = n_1x(t)+n_2y(t)+n_3x(t-\tau)+n_4y(t-\tau), \\ \dot{y}(t) = n_5x(t)+n_6y(t)+n_7x(t-\tau)+n_8y(t-\tau), \end{cases} \tag{17}$$

where

$$\begin{aligned} n_1 &= -x^*\left[-\left(\frac{b-c}{2}+2u\right)x^* + \left(\frac{b-c}{2}-\alpha b-u\right)y^* + \frac{b-c}{2}+u\right], \\ n_2 &= -x^*\left[\left(\alpha b-\frac{b-c}{2}-2u\right)x^* - uy^* + u\right], \\ n_3 &= -x^*(1-x^*)\left(\frac{b-c}{2}+2u\right) - x^*y^*\left(\alpha b-\frac{b-c}{2}-2u\right), \\ n_4 &= x^*(1-x^*)\left(\frac{b-c}{2}-\alpha b-u\right) + ux^*y^*, \\ n_5 &= -y^*\left[-\left(\frac{b-c}{2}+2u\right)x^* + \left(\frac{b-c}{2}-\alpha b-u\right)y^* + \frac{b-c}{2}+u\right], \\ n_6 &= -y^*\left[\left(\alpha b-\frac{b-c}{2}-2u\right)x^* - uy^* + u\right], \\ n_7 &= y^*(1-y^*)\left(\alpha b-\frac{b-c}{2}-2u\right) + x^*y^*\left(\frac{b-c}{2}+2u\right), \\ n_8 &= -uy^*(1-y^*) - x^*y^*\left(\frac{b-c}{2}-\alpha b-u\right). \end{aligned}$$

Then the characteristic equation of the system (17) is obtained by

$$\lambda^2 - \left(m_1+m_2e^{-\lambda\tau}\right)\lambda + m_3e^{-\lambda\tau} + m_4e^{-2\lambda\tau} + m_5 = 0, \tag{18}$$

where

$$\begin{aligned} m_1 &= n_1+n_6,\ m_2 = n_3+n_8, \\ m_3 &= n_1n_8+n_3n_6-n_2n_7-n_4n_5, \\ m_4 &= n_3n_8-n_4n_7,\ m_5 = n_1n_6-n_2n_5. \end{aligned}$$

Let $\lambda = i\omega\ (\omega > 0)$, we have

$$\begin{cases} \left(-\omega^2 + m_4 + m_5\right)\cos(\omega\tau) + m_1\omega\sin(\omega\tau) = -m_3, \\ \left(-\omega^2 - m_4 + m_5\right)\sin(\omega\tau) - m_1\omega\cos(\omega\tau) = m_2\omega. \end{cases} \tag{19}$$

Equations (19) lead to

$$\cos(\omega\tau) = \frac{s_4\omega^2 + s_5}{\omega^4 + s_1\omega^2 + s_2}, \tag{20}$$

$$\sin(\omega\tau) = \frac{-m_2\omega^3 + s_3\omega}{\omega^4 + s_1\omega^2 + s_2}, \tag{21}$$

where

$$\begin{aligned} s_1 &= m_1^2 - 2m_5,\ s_2 = m_5^2 - m_4^2, \\ s_3 &= m_2m_5 + m_2m_4 - m_1m_3, \\ s_4 &= m_3 - m_1m_2,\ s_5 = m_3m_4 - m_3m_5. \end{aligned}$$

Due to eqautions (20) and (21), we get

$$\omega^8 + s_{11}\omega^6 + s_{12}\omega^4 + s_{13}\omega^2 + s_{14} = 0, \tag{22}$$

where $s_{11} = 2s_1 - m_2^2$, $s_{12} = s_1^2 + 2s_2 + 2m_2s_3 - s_4^2$, $s_{13} = 2s_1s_2 - s_3^2 - 2s_4s_5$ and $s_{14} = s_2^2 - s_5^2$.

Let $\bar{\omega} = \omega$, we have

$$\bar{\omega}^4 + s_{11}\bar{\omega}^3 + s_{12}\bar{\omega}^2 + s_{13}\bar{\omega} + s_{14} = 0. \tag{23}$$

$(\kappa_6) : s_{14} < 0,\ s_{13} < 0.$

Since (κ_6) holds, then according to Descart sign law, equation (23) has at least one positive root, and then we use ω_i to represent the positive roots of equation (23). Calculating equation (20) results in

$$\tau_i^j = \frac{1}{\omega_i}\arccos\frac{s_4\omega_i^2 + s_5}{\omega_i^4 + s_1\omega_i^2 + s_2} + \frac{2j\pi}{\omega_i}, \tag{24}$$
$$i = 1, 2, \ldots l,\ j = 0, 1, 2, \ldots.$$

Then let $\tau_0 = \min \tau_i^j$ and $\omega_0 = \omega(\tau_0)$.

From equation (19), we have

$$\begin{cases} H_2\left[\frac{\mathrm{d}(\mathrm{Re}\,\lambda)}{\mathrm{d}\,\tau}\right]_{\tau=\tau_0} + H_1\frac{\mathrm{d}\,\omega_0}{\mathrm{d}\,\tau}_{\tau=\tau_0} = H_3, \\ -H_1\left[\frac{\mathrm{d}(\mathrm{Re}\,\lambda)}{\mathrm{d}\,\tau}\right]_{\tau=\tau_0} + H_2\frac{\mathrm{d}\,\omega_0}{\mathrm{d}\,\tau}_{\tau=\tau_0} = H_4, \end{cases} \tag{25}$$

where

$$\begin{aligned} H_1 =& -2\omega_0\cos(\omega_0\tau_0) - (-\omega_0^2 + m_4 + m_5)\tau_0\sin(\omega_0\tau_0) \\ &+ m_1\sin(\omega_0\tau_0) + m_1\omega_0\tau_0\cos(\omega_0\tau_0), \\ H_2 =& -2\omega_0\sin(\omega_0\tau_0) - (-\omega_0^2 + m_5 - m_4)\tau_0\cos(\omega_0\tau_0) \\ &- m_1\cos(\omega_0\tau_0) + m_1\omega_0\tau_0\sin(\omega_0\tau_0) - m_2, \\ H_3 =& (-\omega_0^2 + m_4 + m_5)\omega_0\sin(\omega_0\tau_0) - m_1\omega_0^2\cos(\omega_0\tau_0), \\ H_4 =& (\omega_0^2 + m_4 - m_5)\omega_0\cos(\omega_0\tau_0) - m_1\omega_0^2\sin(\omega_0\tau_0). \end{aligned}$$

Thus we can get

$$\frac{\mathrm{d}\,(\mathrm{Re}\,\lambda)}{\mathrm{d}\,\tau} = \frac{H_3H_2 - H_4H_1}{H_1^2 + H_2^2}. \tag{26}$$

Theorem 4. *For system (15) suppose $H_3H_2 - H_4H_1 > 0$ and κ_6 hold.*
(1) E_6 is stable when $\tau \in (0, \tau_0)$.
(2) E_6 is unstable when $\tau \in (\tau_0, +\infty)$.
(3) Hopf bifurcation occurs at E_6 when $\tau = \tau_0$.

4 Stability and direction of Hopf bifurcation

This section explores various properties of Hopf bifurcation, including its direction, stability, and changes in periodic behavior [29].

We let $\dot{x}(t) = x(t) - x^*$, $\dot{y}(t) = y(t) - y^*$ and $\tau = \tau_0 + \eta$, then the delay is normalized using the scale $t \mapsto \left(\frac{t}{\tau}\right)$. Consequently, system (15) can be expressed as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \tau\left[L_1\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + L_2\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right], \tag{27}$$

where

$$L_1 = \begin{pmatrix} n_1 & n_2 \\ n_5 & n_6 \end{pmatrix},\ L_2 = \begin{pmatrix} n_3 & n_4 \\ n_7 & n_8 \end{pmatrix}.$$

The values of h_1 and h_2 are

$$\begin{aligned} h_1 =& g_{11}x(t)x(t-1) + g_{12}x(t)y(t-1) \\ &+ g_{13}x^2(t)y(t-1) + g_{14}x^2(t)x(t-1) \\ &+ g_{15}x(t)y(t)y(t-1) + g_{16}y(t)x(t-1) \\ &+ g_{17}y(t)y(t-1), \\ h_2 =& g_{21}x(t)x(t-1) + g_{22}x(t)y(t-1) \\ &+ g_{23}y^2(t)x(t-1) + g_{15}y^2(t)y(t-1) \\ &+ g_{25}x(t)y(t)y(t-1) + g_{26}y(t)x(t-1) \\ &+ g_{27}y(t)y(t-1), \end{aligned}$$

where

$$g_{11} = -(1-2x^*)\left(\frac{b-c}{2}+2u\right) - y^*\left(\alpha b - 2u - \frac{b-c}{2}\right),$$
$$g_{12} = (1-2x^*)\left(\frac{b-c}{2} - \alpha b - u\right) - uy^*,$$
$$g_{13} = -\left(\frac{b-c}{2} - \alpha b - u\right),\ g_{14} = \frac{b-c}{2} + 2u,$$
$$g_{15} = u,\ g_{16} = -\left(\alpha b - 2u - \frac{b-c}{2}\right)x^*,\ g_{17} = ux^*,$$
$$g_{21} = \left(\frac{b-c}{2}+2u\right)y^*,\ g_{22} = -\left(\frac{b-c}{2} - \alpha b - u\right)y^*,$$
$$g_{23} = -\left(\alpha b - 2u - \frac{b-c}{2}\right),\ g_{25} = -\left(\frac{b-c}{2} - \alpha b - u\right),$$
$$g_{26} = (1-2y^*)\left(\alpha b - 2u - \frac{b-c}{2}\right) + \left(\frac{b-c}{2}+2u\right)x^*,$$
$$g_{27} = -u(1-2y^*) - \left(\frac{b-c}{2} - \alpha b - 2u\right)x^*.$$

We denote

$$L_\eta(\varphi) = (\tau+\eta)\times(L_1\varphi(0) + L_2\varphi(-1)),$$

for $\varphi = (\varphi_1,\varphi_2)^T \in C\left([-1,0],\mathbb{R}^2\right)$. There exists a bounded variational function $r(\psi,\eta)$ in accordance with the Riesz representation theorem, such that

$$L_\eta(\varphi) = \int_{-1}^{0} dr(\psi,\eta)\varphi(\eta),\ \varphi\in\left([-1,0],R^2\right).$$

Then we choose

$$r(\psi,\eta) = \begin{cases}(\tau_0+\eta)(L_1+L_2),\psi=0,\\ (\tau_0+\eta)L_2, -1<\psi<0\\ 0,\psi=-1.\end{cases} \tag{28}$$

Define

$$F(\eta)\varphi(u) = \begin{cases}\frac{d\varphi(u)}{du}, u\in[-1,0),\\ \int_{-1}^{0} dr(\theta,\eta)\varphi(\theta), u=0,\end{cases}$$

and

$$R(\eta)\varphi(u) = \begin{cases}0, u\in[-1,0),\\ \eta(n,\varphi), u=0.\end{cases}$$

One has

$$\eta(\psi,\varphi) = (\tau^*+\psi)\left(h_1', h_2'\right)^T,$$

where

$$\begin{aligned} h_1' &= g_{11}x(0)x(-1) + g_{12}x(0)y(-1)\\ &+ g_{13}x^2(0)y(-1) + g_{14}x^2(0)x(-1)\\ &+ g_{15}x(0)y(0)y(-1) + g_{16}y(0)x(-1)\\ &+ g_{17}y(0)y(-1),\\ h_2' &= g_{21}x(0)x(-1) + g_{22}x(0)y(-1)\\ &+ g_{23}y^2(0)x(-1) + g_{15}y^2(0)y(-1)\\ &+ g_{25}x(0)y(0)y(-1) + g_{26}y(0)x(-1)\\ &+ g_{27}y(0)y(-1).\end{aligned}$$

Then system (15) becomes

$$\dot{X}_t = F(\eta)X_t + R(\eta)X_t,$$

where $X_t = X(t+\rho) \in C$, $\rho\in[-1,0]$. For $\phi \in C\left([0,1],\mathbb{R}^{2*}\right)$, define

$$F^*\phi(v) = \begin{cases}-\frac{d\phi(v)}{dv}, v\in(-1,0],\\ \int_{-1}^{0}\phi(-\theta)dr(\theta,0), v=0,\end{cases}$$

and

$$\langle\phi(v),\varphi(u)\rangle = \bar{\phi}(0)\varphi(0) - \int_{-1}^{0}\int_{\theta=0}^{\psi}\bar{\phi}(\theta-\psi)dr(\psi,0)\phi(\theta)d\theta. \tag{29}$$

Since $\pm i\omega_0\tau_0$ is the eigenvalue of $F(0)$ and F^*, and let $Q(0) = (1,\delta_2)^T$ and $Q^*(v) = P(1,\delta_2^*)e^{i\omega_0\tau_0 v}$ $(v\in[0,1])$ be the eigenvector of F_0 and F^*. Thus we get

$$\delta_2 = \frac{n_5 + n_7e^{-i\omega_0\tau_0}}{i\omega^* - n_6 - n_8e^{-i\omega_0\tau_0}},$$

and

$$\delta_2^* = -\frac{n_2 + n_4e^{i\omega_0\tau_0}}{i\omega^* + n_6 + n_8e^{i\omega_0\tau_0}}.$$

By $\langle Q^*(v), Q(u)\rangle = 1$,

$$G = \left[1 + \bar{\delta}_2\delta_2^* + \left(n_3 + n_4\bar{\delta}_2 + n_7\delta^* + n_8\bar{\delta}_2\delta_2^*\right)\tau_0e^{i\omega_0\tau_0}\right]^{-1}.$$

One has

$$\begin{aligned} G_{20} &= 2\tau_0\bar{G}\left(K_{11} + \bar{\delta}_2^*K_{21}\right),\\ G_{11} &= \tau_0\bar{G}\left(K_{12} + \bar{\delta}_2^*K_{22}\right),\\ G_{02} &= 2\tau_0\bar{G}\left(K_{13} + \bar{\delta}_2^*K_{32}\right),\\ G_{21} &= 2\tau_0\bar{G}\left(K_{14} + \bar{\delta}_2^*K_{42}\right),\end{aligned}$$

where

$$K_{11} = g_{11}e^{-i\omega_0\tau_0} + g_{12}\delta_2 e^{-i\omega_0\tau_0} + g_{16}\delta_2 e^{-i\omega_0\tau_0} + g_{17}\delta_2^2 e^{-i\omega_0\tau_0},$$

$$K_{21} = g_{21}e^{-i\omega_0\tau_0} + g_{22}\delta_2 e^{-i\omega_0\tau_0} + g_{26}\delta_2 e^{-i\omega_0\tau_0} + g_{27}\delta_2^2 e^{-i\omega_0\tau_0},$$

$$K_{12} = g_{11}\left(e^{-i\omega_0\tau_0} + e^{i\omega_0\tau_0}\right) + g_{12}\left(\bar{\delta}_2 e^{i\omega_0\tau_0} + \delta_2 e^{-i\omega_0\tau_0}\right) + g_{16}\left(\delta_2 e^{i\omega_0\tau_0} + \bar{\delta}_2 e^{-i\omega_0\tau_0}\right) + g_{17}\left(\delta_2\bar{\delta}_2 e^{i\omega_0\tau_0} + \bar{\delta}_2\delta_2 e^{-i\omega_0\tau_0}\right),$$

$$K_{22} = g_{21}\left(e^{-i\omega_0\tau_0} + e^{i\omega_0\tau_0}\right) + g_{22}\left(\bar{\delta}_2 e^{i\omega_0\tau_0} + \delta_2 e^{-i\omega_0\tau_0}\right) + g_{26}\left(\delta_2 e^{i\omega_0\tau_0} + \bar{\delta}_2 e^{-i\omega_0\tau_0}\right) + g_{27}\left(\delta_2\bar{\delta}_2 e^{i\omega_0\tau_0} + \bar{\delta}_2\delta_2 e^{-i\omega_0\tau_0}\right),$$

$$K_{13} = g_{11}e^{i\omega_0\tau_0} + g_{12}\bar{\delta}_2 e^{i\omega_0\tau_0} + g_{16}\bar{\delta}_2 e^{i\omega_0\tau_0} + g_{17}\bar{\delta}_2^2 e^{i\omega_0\tau_0},$$

$$K_{32} = g_{21}e^{i\omega_0\tau_0} + g_{22}\bar{\delta}_2 e^{i\omega_0\tau_0} + g_{26}\bar{\delta}_2 e^{i\omega_0\tau_0} + g_{27}\bar{\delta}_2^2 e^{i\omega_0\tau_0},$$

$$\begin{aligned} K_{14} = {} & g_{11}\left(W_{11}^{(1)}(-1) + \frac{W_{20}^{(1)}}{2}(-1)e^{-2i\omega_0\tau_0} + \frac{W_{20}^{(1)}}{2}(0)e^{i\omega_0\tau_0} + W_{11}^{(1)}(0)e^{-i\omega_0\tau_0}\right) \\ & + g_{12}\left(W_{11}^{(2)}(-1) + \frac{W_{20}^{(2)}}{2}(-1)e^{-2i\omega_0\tau_0} + \frac{W_{20}^{(1)}}{2}(0)\bar{\delta}_2 e^{i\omega_0\tau_0} + W_{11}^{(1)}(0)\delta_2 e^{-i\omega_0\tau_0}\right) \\ & + g_{13}\left(\bar{\delta}_2 e^{i\omega_0\tau_0} + \delta_2 e^{-i\omega_0\tau_0}\right) + g_{14}\left(e^{i\omega_0\tau_0} + e^{-i\omega_0\tau_0}\right) + g_{15}\left(\delta_2\bar{\delta}_2 e^{i\omega_0\tau_0} + \bar{\delta}_2\delta_2 e^{-i\omega_0\tau_0}\right) \\ & + g_{16}\left(\delta_2 W_{11}^{(1)}(-1) + \bar{\delta}_2\frac{W_{20}^{(1)}}{2}(-1)e^{-2i\omega_0\tau_0} + \frac{W_{20}^{(2)}(0)}{2}e^{i\omega_0\tau_0} + W_{11}^{(2)}(0)e^{-i\omega_0\tau_0}\right) \\ & + g_{17}\left(\delta_2 W_{11}^{(2)}(-1) + \bar{\delta}_2\frac{W_{20}^{(2)}}{2}(-1)e^{-2i\omega_0\tau_0} + \bar{\delta}_2\frac{W_{20}^{(2)}}{2}(0)e^{i\omega_0\tau_0} + \delta_2 W_{11}^{(2)}(0)e^{-i\omega_0\tau_0}\right). \end{aligned}$$

$$\begin{aligned} K_{42} = {} & g_{21}\left(W_{11}^{(1)}(-1) + \frac{W_{20}^{(1)}}{2}(-1)e^{-2i\omega_0\tau_0} + \frac{W_{20}^{(1)}}{2}(0)e^{i\omega_0\tau_0} + W_{11}^{(1)}(0)e^{-i\omega_0\tau_0}\right) \\ & + g_{22}\left(W_{11}^{(2)}(-1) + \frac{W_{20}^{(2)}}{2}(-1)e^{-2i\omega_0\tau_0} + \frac{W_{20}^{(1)}}{2}(0)\bar{\delta}_2 e^{i\omega_0\tau_0} + W_{11}^{(1)}(0)\delta_2 e^{-i\omega_0\tau_0}\right) \\ & + g_{23}\left(\bar{\delta}_2 e^{i\omega_0\tau_0} + \delta_2 e^{-i\omega_0\tau_0}\right) + g_{15}\left(e^{i\omega_0\tau_0} + e^{-i\omega_0\tau_0}\right) \\ & + g_{25}\left(\delta_2\bar{\delta}_2 e^{i\omega_0\tau_0} + \bar{\delta}_2\delta_2 e^{-i\omega_0\tau_0}\right) \\ & + g_{26}\left(\delta_2 W_{11}^{(1)}(-1) + \bar{\delta}_2\frac{W_{20}^{(1)}}{2}(-1)e^{-2i\omega_0\tau_0} + \frac{W_{20}^{(2)}(0)}{2}e^{i\omega_0\tau_0} + W_{11}^{(2)}(0)e^{-i\omega_0\tau_0}\right) \\ & + g_{27}\left(\delta_2 W_{11}^{(2)}(-1) + \bar{\delta}_2\frac{W_{20}^{(2)}}{2}(-1)e^{-2i\omega_0\tau_0} + \bar{\delta}_2\frac{W_{20}^{(2)}}{2}(0)e^{i\omega_0\tau_0} + \delta_2 W_{11}^{(2)}(0)e^{-i\omega_0\tau_0}\right), \end{aligned}$$

$$W_{20}(\psi) = \frac{iG_{20}Q(0)\,e^{i\omega_0\tau_0\psi}}{\omega_0\tau_0} + \frac{i\bar{G}_{02}\bar{Q}(0)\,e^{-i\omega_0\tau_0\psi}}{3\omega_0\tau_0} + \Omega_1 e^{2i\omega_0\tau_0\psi},$$

$$W_{11}(\psi) = \frac{-iG_{11}Q(0)\,e^{i\omega_0\tau_0\psi}}{\omega_0\tau_0} + \frac{i\bar{G}_{11}\bar{Q}(0)\,e^{-i\omega_0\tau_0\psi}}{\omega_0\tau_0} + \Omega_2.$$

The values of Ω_1 and Ω_2 are given by

$$\Omega_1 = 2\begin{pmatrix} 2i\omega_0 - n_1 - n_3 e^{-2i\omega_0\tau_0} & -n_2 - n_4 e^{-2i\omega_0\tau_0} \\ -n_5 - n_7 e^{-2i\omega_0\tau_0} & 2i\omega_0 - n_6 - n_8 e^{-2i\omega_0\tau_0} \end{pmatrix}^{-1} \begin{pmatrix} K_{11} \\ K_{21} \end{pmatrix},$$

$$\Omega_2 = -\begin{pmatrix} n_1 + n_3 & n_2 + n_4 \\ n_5 + n_7 & n_6 + n_8 \end{pmatrix}^{-1} \begin{pmatrix} K_{12} \\ K_{22} \end{pmatrix}.$$

Finally, we obtain

$$N_1(0) = \left(G_{11}G_{20} - 2\left|G_{11}\right|^2 - \frac{\left|G_{02}\right|^2}{3}\right)\frac{i}{2\omega_0\tau_0} + \frac{1}{2}G_{21},$$

$$v_1 = -\frac{\mathrm{Re}\left[N_1(0)\right]}{\mathrm{Re}\left[\lambda'(\tau_0)\right]},$$

$$k = 2\,\mathrm{Re}\left[N_1(0)\right],$$

$$R_1 = -\frac{\mathrm{Im}\left[N_1(0)\right] + \eta_1\,\mathrm{Im}\left[\lambda'(\tau_0)\right]}{\omega_0\tau_0}.$$

Theorem 5. *For system*(15)*, the results are presented as follows.*
(1) *If* $v_1 > 0$ ($v_1 < 0$)*, the Hopf bifurcation is supercritical* (*subcritical*)*.*
(2) *If* $k < 0$ ($k > 0$)*, the periodic solution is stable* (*unstable*)*.*
(3) *If* $R_1 > 0$ ($R_1 < 0$)*, the period increases* (*decreases*)*.* ■

5 Numerical simulations

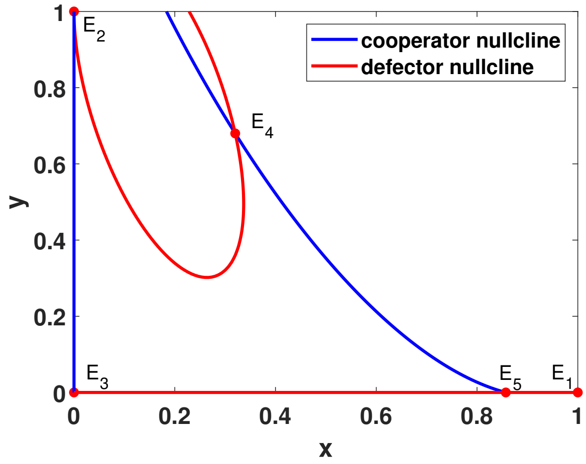
In this section, we perform a series of numerical simulations to corroborate the earlier findings and examine their biological implications.

5.1 Model with no time delay

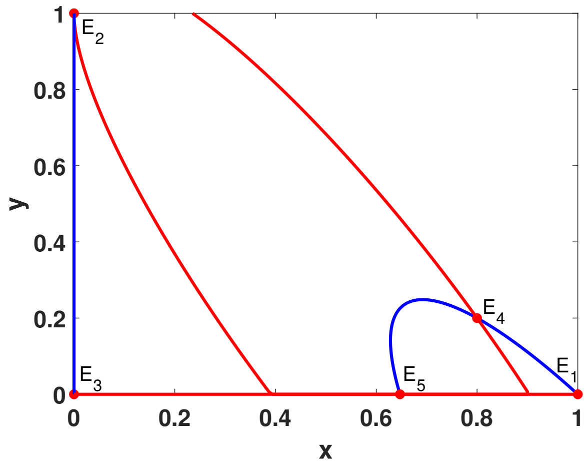
At first, Figure 1 illustrates the existence of the equilibrium points of the nondelay model (6) under the condition of satisfying the snowdrift game, and we can clearly see whether the equilibrium points exist. Then, according to these parameters, the stability of the equilibria is judged as shown in Figures 2, 3 and 4, and Theorem 1 is also verified. Figures 2, 3 and 4 show stable equilibrium points E_4, E_5 and E_6 respectively. In Figure 2, it is evident that regardless of the initial values, the system ultimately stabilizes at the equilibrium E_4. Only the cooperation and defection strategy coexist, and the reward strategy disappears. Reward strategy, as the third strategy, does not affect the coexistence of cooperative strategy and defection strategy in the population.

Figure 3 also shows the stable boundary equilibrium E_5. At this time, the cooperative strategy coexists with the reward strategy, and the defective strategy disappears. At this time, the greater the reward u, the more players choose the reward strategy. The stable internal equilibrium E_6 can be seen in Figure 4. The three strategies coexist, and the reward strategy can always play a role.

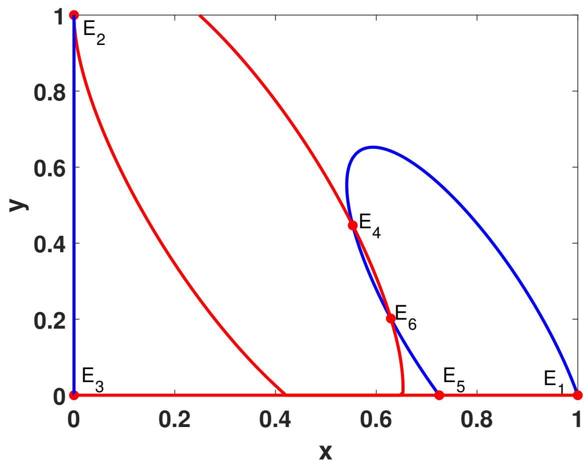
Then, according to Theorem 2, under the assumption that u is less than 0, the bistable phenomenon of model (6) is shown in Figure 5. Reward strategy reduces the utilization of resources by cooperation strategy and defection strategy, and loses the original benefits of cooperators and defectors. At this time, the final stable state of the nondelay model depends on the initial value, and different initial values lead to different stable equilibria E_3 and E_4. We find that when the model is stable in E_3, the cooperation strategy and defection strategy disappear and no longer exist, only the reward strategy exists stably. When the model is stable at E_4, the reward strategy disappears, and



(a)

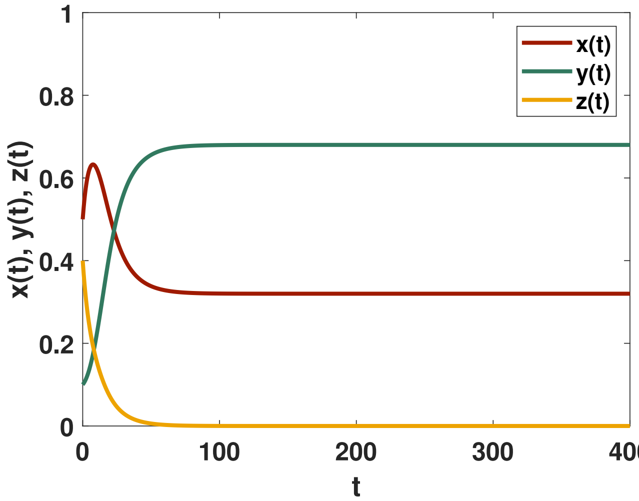


(b)

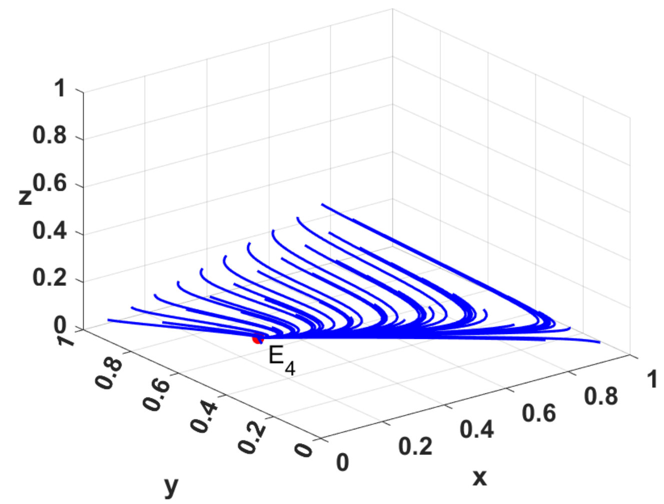


(c)

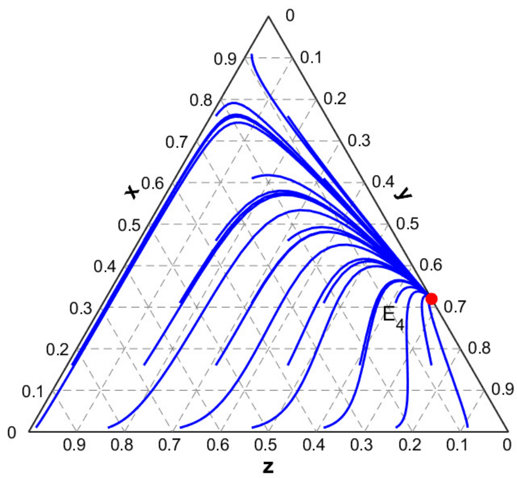
Figure 1. Existence of equilibria for nondelay model (6). Parameters: (a) $b = 2.1$, $c = 1.1$, $\alpha = 0.4$, $u = 0.1$; (b) $b = 3$, $c = 1$, $\alpha = 0.4$, $u = 1.2$; (c) $b = 3.1$, $c = 1.3$, $\alpha = 0.42$, $u = 0.55$. The parameters of Figures 2 3 4 are the same as (a) (b) (c).



(a)

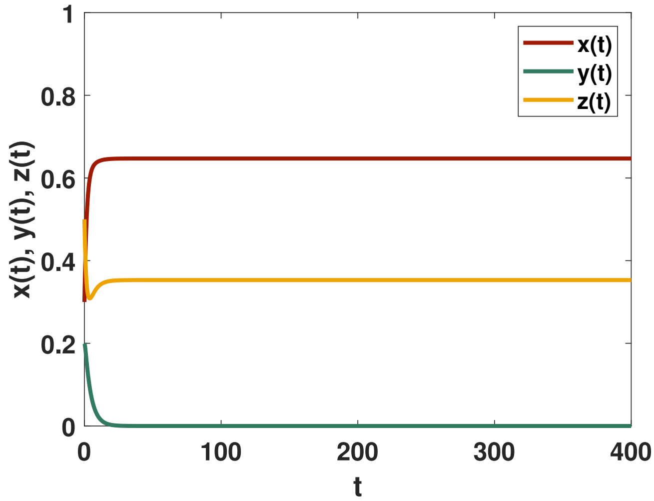


(b)

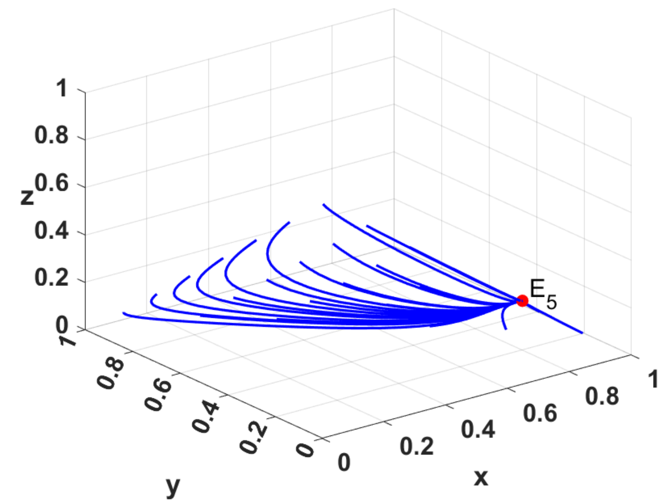


(c)

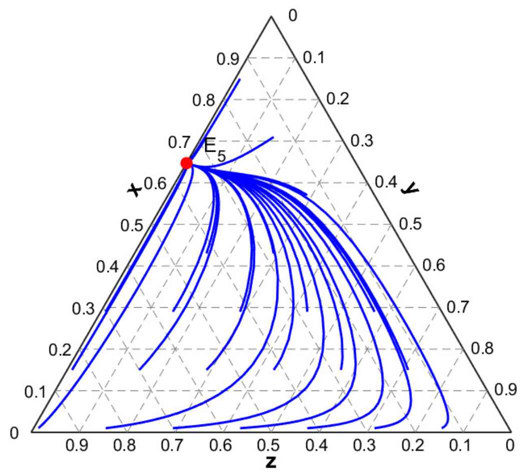
Figure 2. E_4 is stable.



(a)

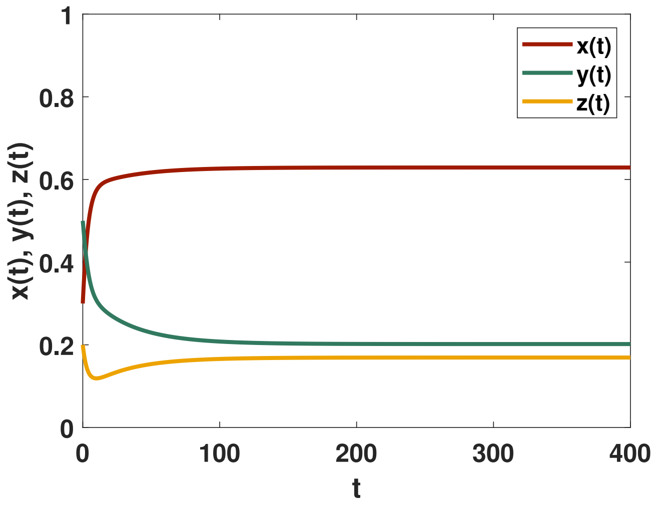


(b)

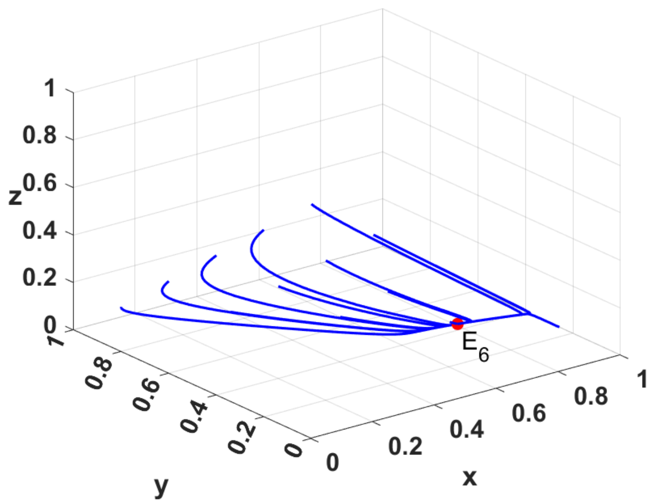


(c)

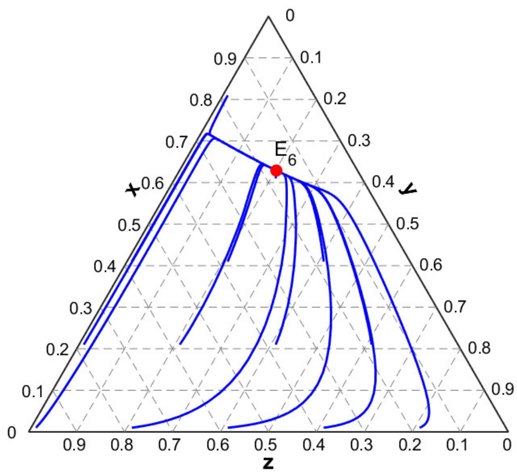
Figure 3. E_5 is stable.



(a)

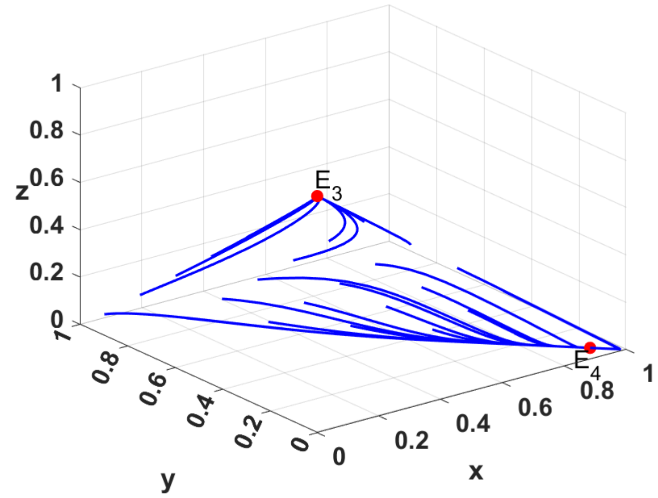


(b)

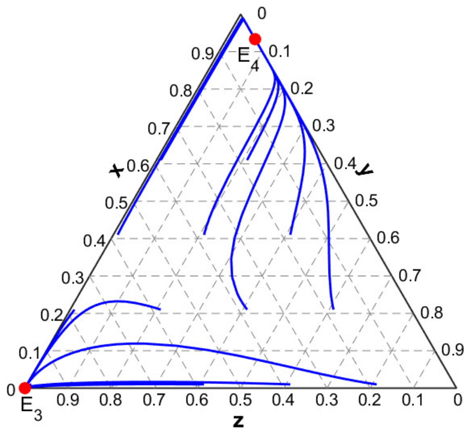


(c)

Figure 4. E_6 is stable.



(a)



(b)

Figure 5. Phase diagram in xyz-space of nondelay system (6) at E_3 and E_4. $b = 4$, $c = 1$, $\alpha = 0.4$, $u = -3$.

the cooperation strategy and defection strategy coexist. Therefore, different initial values have great influence on the final state of the strategy. These results further demonstrate that, when all other conditions remain unchanged, u no longer acts as a rewarding factor, and the payoff of cooperators no longer increases when interacting with the third strategy. Consequently, two distinct phenomena emerge simultaneously: one in which only the third strategy persists, and another in which cooperation and defection coexist, thereby enriching the dynamical behavior of the system.

Figure 6(a) demonstrates the change trend of the three strategies with the change of parameter u. The nondelay model (6) stabilizes at E_4 at the beginning, then transcritical bifurcation occurs at the critical

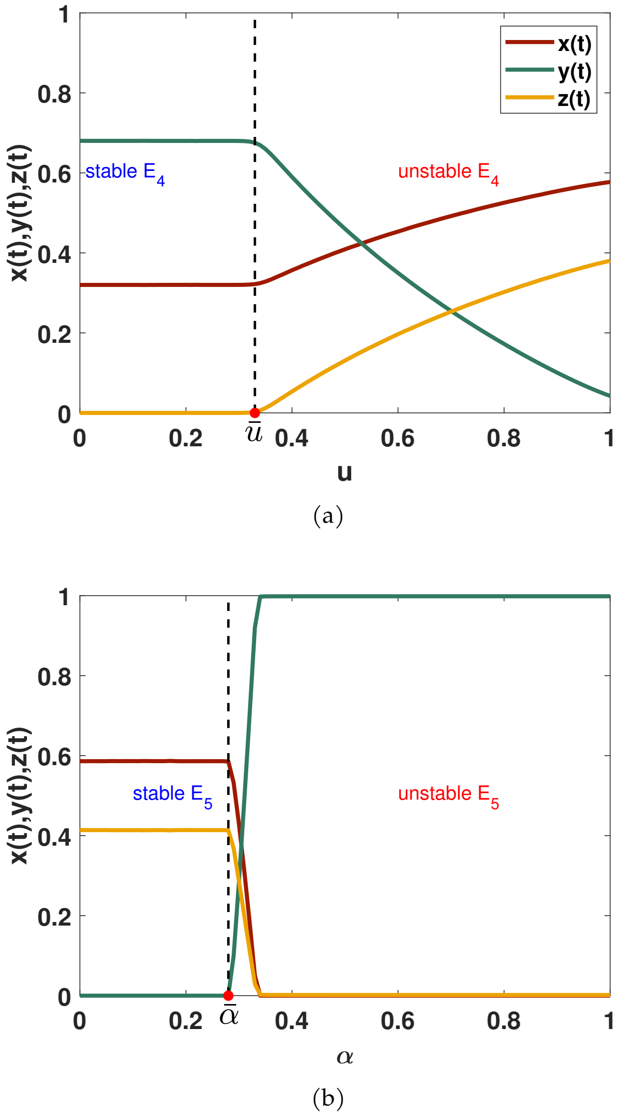


Figure 6. Effect of parameters u and α on the nondelay model (6). The remaining parameters in (a) and (b) are the same as those in (b) and (c) of Figure 1, respectively.

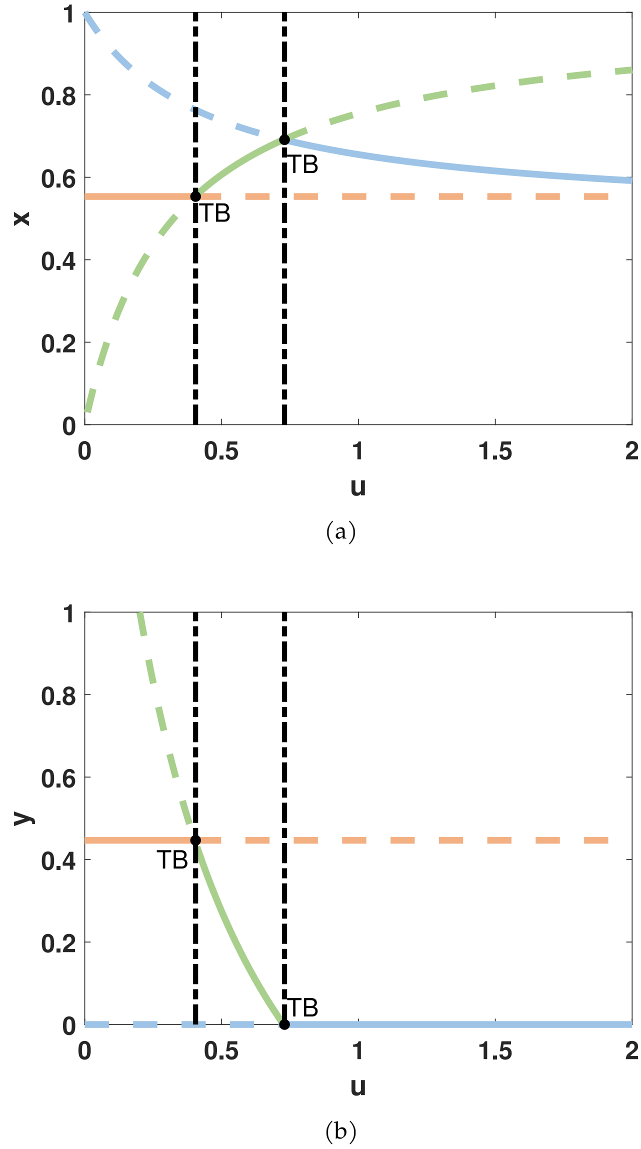
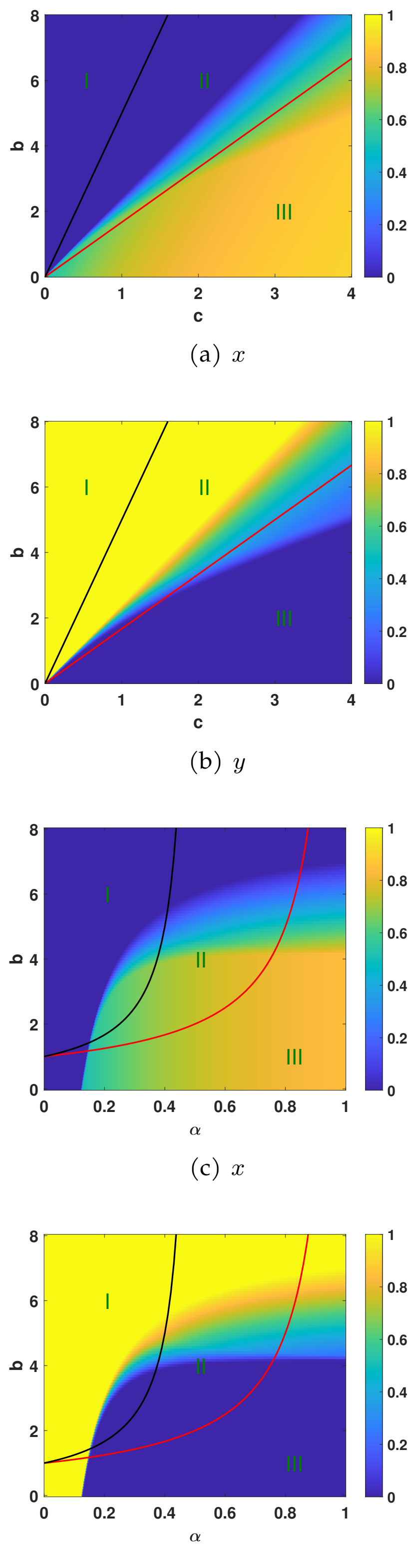


Figure 7. Effect of parameter u on the stability of equilibria. Blue, orange and green lines represent the equilibrium point E_4, E_6 and E_5, respectively, where the solid line denotes stable equilibrium, while the dotted line represents unstable equilibrium.

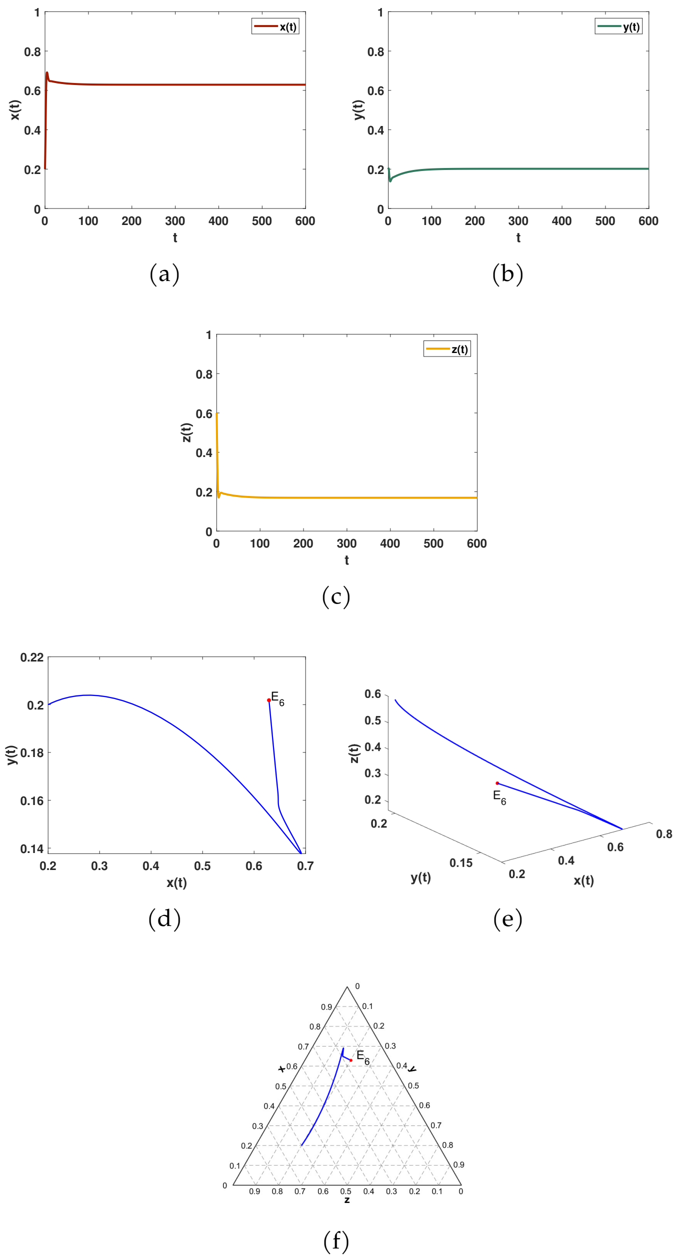
value, and the nondelay model (6) unstabilizes at E_4 after bifurcation. As the reward increases, the reward strategy gains prominence, leading to a decline in the dominance of the betrayal strategy, which subsequently begins to diminish. The results show that the parameter u can affect the dominance of the strategy, from the coexistence of cooperation and betrayal to the emergence of reward strategy. The effect of the parameter α on the model (6) is illustrated in Figure 6(b). At first, the model is stable in a stable equilibrium E_5. With the increase of α, the transcritical bifurcation occurs and the proportion of the payoff obtained by the defector when he meets the collaborator increases, and E_5 is no longer stable. Defection strategy begins to emerge as the parameter α increases.

Keep the parameters b, c, and α in Fig. 1(c) unchanged, and change the reward u to observe the stability of equilibria in Figure 7. Since the boundary equilibrium points E_1, E_2 and E_3 are always unstable, blue, orange and green lines represent the equilibrium point E_4, E_6 and E_5, respectively, where the solid line denotes stable equilibrium, while the dotted line represents unstable equilibrium. With the increasing of u, it changes from stable equilibrium E_4 to stable equilibrium E_6 and then to stable equilibrium E_5. The change of reward u leads to the existence and stability of different strategies. Initially, as the reward increases, the number of players opting for the cooperative strategy rises, while those



(a) x (b) y (c) x (d) y

Figure 8. Effect of parameters on variation of x and y. Region II satisfies the snowdrift game conditions, and region III satisfies the prisoner's dilemma conditions. Parameters: (a)(b) $u = 0.5$, $\alpha = 0.4$; (c)(d) $c = 1$, $u = 0.8$.

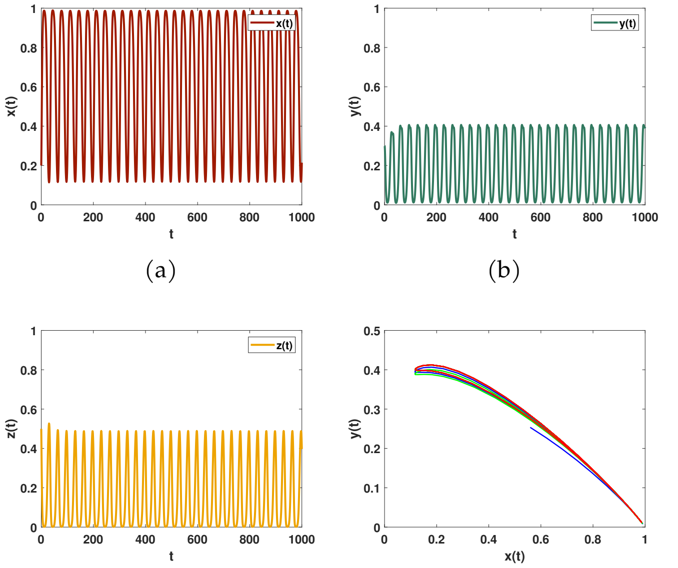


(a) (b) (c) (d) (e) (f)

Figure 9. Temporal dynamics and phase-portrait of delay system (15) when $\tau = 1.5$.

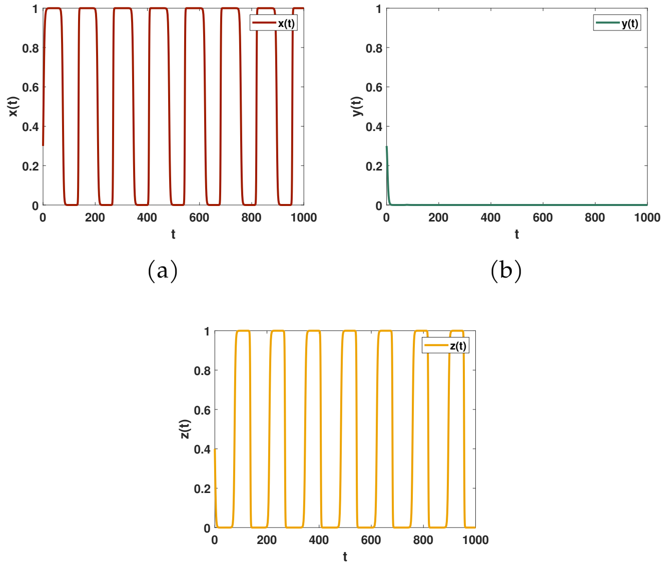
choosing the betrayal strategy gradually diminish. Although further increases in the reward lead to a decrease in the proportion of players adopting cooperation compared to earlier levels, cooperation remains the dominant strategy.

Next, in Figure 8, we change the values of two parameters simultaneously to observe the change in system (6). In Figures 8(a) and (b), we choose parameters $\alpha = 0.4$ and $u = 1$, where blue to yellow represents small values to large values. The black and the red lines meet the conditions of snowdrift game. Therefore, we can clearly find that when the betrayer meets the cooperator, the increase in the cost of cooperation leads to a higher proportion of



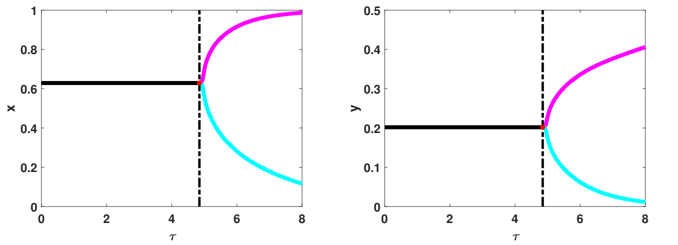
(a) (b) (c) (d)

Figure 10. Temporal dynamics and phase-portrait of delay system (15) when $\tau = 8$.



(a) (b) (c)

Figure 12. Temporal dynamics and phase-portrait of delay system (15) when τ is large enough to be 40.



(a) (b)

Figure 11. Bifurcation diagrams of x and y.

cooperative players when the payoff of the betrayer remains unchanged. When the cost of cooperation remains the same and the income of traitors increases, the number of cooperative players decreases, while the number of defective players shows the opposite trend. Then we control $c = 1$ and $u = 0.8$ and change α and b. Figures 8(c) and (d) show that with the simultaneous increase of α and b, the income of defectors increases obviously, at this time, the number of cooperators decreases, while those with defects increase. Under the condition that α remains unchanged, the increase of b also leads to an increase in the income of defectors, which leads to a decrease in the proportion of cooperative players and an increase in the proportion of defectors.

5.2 Model with time delay

For time-delay system (15), the parameters we choose are $b = 3.1$, $c = 1.3$, $\alpha = 0.42$, $u = 0.55$. Through calculation, we can get $\omega_0 = 0.3181,\ \tau_0 = 5.0350,\ E_6 = (x^*, y^*, z^*) = (0.6290, 0.2018, 0.1692)$.

The ratio of cooperative players is finally stable at x^*, and the ratio of defective players is stable at y^* when $\tau = 1.5$ (see Figure 9). When $\tau = 8$, the equilibrium E_6 is unstable, and the cooperation and defection strategies oscillate in Figure 10. Based on the previous derivations, we obtain the following results through computational simulations.

$$N_1(0) = -10.6115 - 4.7722i,\ \frac{d\,(\mathrm{Re}\,\lambda)}{d\tau} = 0.0278 > 0,$$

$$\upsilon_1 = 381.7410,\ k = -21.2230,\ R_1 = 13.6660.$$

By Theorem 4, since $\upsilon_1 > 0$, the hopf bifurcation is supercritical, and since $k < 0$, the bifurcating periodic solution is stable as shown in Figure 10(d). The bifurcating period increases since $R_1 > 0$.

By comparing Figure 9 and Figure 10, equilibrium E_6's stability is affected by time delay, which is stable as shown in Figure 9. In contrast, Figure 10 produces an oscillating dynamic behavior, and the stable E_6 becomes unstable, resulting in a stable limit cycle. At this time, the proportion of collaborators, defectors and rewarders is not constant but periodic oscillation occurs. The bifurcation diagram Figure 11 is obtained, and hopf bifurcation occurs at $\tau_0 = 5.0350$, where τ_0 is the critical time delay.

The black line in Figure 11 proposes that E_6 is stable at this time, just like in Figure 9. The purple and blue curves propose that the equilibrium point is unstable, which accords with the situation of a large time delay in Figure 12. Increasing the width between the two curves shows that increasing time delay increases the amplitude of the periodic solution. Figure 12 sets $\tau = 40$ and explores the influence on the strategy when τ is large enough. In addition, the ratios of cooperation strategy and reward strategy oscillate between 0 and 1, and the oscillation amplitude increases, while the ratio of betrayal strategy tends toward 0. Defectors disappear, and collaborators and rewarders oscillate.

6 Conclusion

This paper presents a three-strategy snowdrift game model that incorporates a special reward strategy and accounts for payoff delay. The stable state of evolution includes three situations: cooperators and defectors coexist, cooperators and rewarders coexist, and all three strategies coexist and are stable. The behavior that players enjoy the benefits without paying the cost in the game is called free-riders. Rewarders have made great contributions to curb the free-riders behavior of defectors. Therefore, there is a circumstance in which cooperators and rewarders coexist. Bistable phenomenon appears when the third strategy brings not rewards but damages benefits, and the final stable state of the nondelay model depends on the initial states of the strategies. Then we analyze the effect of single parameter and double parameter on the strategies. In the snowdrift game, when the reward increases, the reward strategy appears, and finally it is stable in the state where the cooperative strategy and the reward strategy coexist, and the defective strategy disappears.

Time delay is a factor that cannot be ignored in biological systems, we investigate the time-delay system of snowdrift game and find payoff delay affects the stability of dynamic performance. When the payoff delay is small, that is, the time required for players to obtain payoff is short, it does not change the stability of the equilibrium. When the payoff delay is large, the cooperative strategy oscillates periodically, indicating that the proportion of cooperators changes periodically. Hopf bifurcation occurs with the increase of payoff delay. The stable equilibrium becomes unstable and the stable periodic solution appears. Then, the periodic properties of Hopf bifurcation are discussed and verified by numerical simulations. In addition, a large enough payoff delay has a great influence on the system. When the player gains profits for too long, the player does not choose to defect again.

This paper discusses the influence of reward strategy, the defector's earning rate, and payoff delay on the cooperator in the game. However, some mutations also affect the stability of the system [30] or accidents[31]. In the future, on the basis of promoting cooperation, we will continue to consider the multi-strategy game model and also consider the influence of random noise and spatial diffusion on the game [32, 33].

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

AI Use Statement

The authors declare that no generative AI was used in the preparation of this manuscript.

Ethical Approval and Consent to Participate

Not applicable.

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