ICJK
Journal of Mathematics and Interdisciplinary Applications
http://dx.doi.org/10.62762/JMIA.2026.627569

RESEARCH ARTICLE

On Multi-Group Chickenpox Mathematical Epidemic Models with Effective Vaccination and Re-Infection Challenges

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Abstract

A class of multi-age group SEQ_3IHR chickenpox epidemic models with effective vaccination in a different subpopulation size, is considered, in this paper. It is assumed that individual members of the subpopulations who recovered from the disease, may be re-infected by the disease. The models comprise of six infectious classes of subclasses and three distinct infected stages of subclasses. We establish that the global dynamics of the epidemic models are determined entirely by the basic reproduction number. In this paper, we determine that the disease-free and endemic equilibrium states, and carry out a stability analysis of the disease-free as well as the endemic equilibrium points of the model. The findings show that age heterogeneity and vaccination rate play prominent roles in the determination of epidemic spread and control of chickenpox disease in the population and also serve as the factors that influenced the basic reproduction number. We found numerically that re-infections have little or no influence on the basic reproduction number but contribute tremendously to the reduction and/or increase in the number of individuals in incubation, prodromal and active stages, as well as treatment compartment of chickenpox infection over time.

Keywords: chickenpox, multi-age group SEQ_3IHR, mathematical epidemic models, effective vaccination, re-infection, subpopulation.

1 Introduction

The virus disease known as chickenpox (or varicella), is a highly contagious disease caused by the varicella-zoster virus. Chickenpox is a member of the herpesvirus family, and common among humans worldwide. Usually, individuals will only get chickenpox once in their lifetime. After recovery from the chickenpox infection, the body develops immunity to the virus. But, it is imperative to note that after chickenpox recovery, the virus becomes inactive in the nerve cells of the hosts, which may be reactivated later in future, and may lead to a painful condition known as shingles (or herpes zoster). The virus is an airborne and can be contracted and spread through coughs or sneezes from an infected person. Individuals of any age group can contract chickenpox disease. It is

Submitted: 23 March 2026
Accepted: 29 April 2026
Published: 11 May 2026

Vol. 2, **No.** 2, 2026.
10.62762/JMIA.2026.627569

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Citation
Nkeki, C. I., & Omoregie, R. (2026). On Multi-Group Chickenpox Mathematical Epidemic Models with Effective Vaccination and Re-Infection Challenges. *Journal of Mathematics and Interdisciplinary Applications*, 2(2), 91–111.

usually very serious in the following groups of persons: babies, children, adults, pregnant women, and persons with compromised or weak immune systems. It is important to note that anyone can contract chickenpox disease if they have not had it before, or if they are unvaccinated against varicella-zoster virus, or they have compromised immune systems.

There are primarily three categories of chickenpox infections, which include: (i) varicella, or primary infection, which is the initial outbreak of chickenpox disease. It is most common in children and is characterized by rash, fever and malaise, (ii) breakthrough varicella, which occurs among individuals who have been vaccinated against chickenpox disease, but still contract the disease, and (iii) herpes zoster or shingles, which is a reactivation of the varicella-zoster virus. That is, after the initial chickenpox infection, the virus remains inactive in nerve tissue of individuals, and can reactivate later in life, as such individual grow old or has weakened immune systems, see Medicover Hospitals [1].

Chickenpox has three distinct infection stages: the first stage is the incubation stage, which takes 7–21 days. At this stage, the virus enters the body and start to reproduce, without visible symptoms, the infected individual may feel normal, but has the tendency of infecting other susceptible persons. Second stage is the prodromal stage, which take 1–2 days. At this stage, there will be early symptoms, and it depends on how the immune system reacts to the varicella infection. The symptoms at this point, include mild fever (100°F to 102°F or higher than that, in some cases), headache, body pains, loss of appetite, fatigue and weakness. It is important to note that at this stage of the infection, it might be mild or absent in young children. The third is the active stage of the infection, which take 3–7 days. At this stage, chickenpox rash begins to develop, and spread around on the face, chest, back, and to the rest parts of the body. The itching can range from mild to severe, fever may persist or worsen as the rash continue to develop, see Sehattak [2]. There is also the final stage of chickenpox infection, which is the recovered stage.

The incidence of chickenpox varies among different age groups in a population, referred to multi-group models. Varicella disease commonly affects children less than 10 years old with the highest incidence among children between 1–4 years old. Hence, chickenpox multi-group models are epidemiological models that divide the given populations into subgroups, so as to study the spread of the disease within the subpopulation over time. In this paper, we categorize the population into different age groups of chickenpox infections. The subgroups considered in this paper are groups below one year of age, 1–4 years of age, 5–14 years, 15–44 years, and 45 years and above.

Several researchers have studied the spread dynamics of chickenpox via mathematical modelling approach. Guo *et al.* [3] considered a class of multi-group SIR epidemic models with varying subpopulation sizes. They established that the global dynamics are completely determined by the basic reproduction number. Sun [4] established the global stability of a multi-group epidemic model involving nonlinear incidence. They established that global dynamics of the model are determined entirely by the basic reproduction number, and they showed that, the basic reproduction number is a global threshold parameter, if it is less than or equal to one, the disease free equilibrium is globally stable and the disease dies out, and there is a unique endemic equilibrium, which is globally stable and the disease will persist in the population, if it is larger than one. Zhang and Guo [5] formulated a multi-stage SEIR model for infectious diseases with continuous age structure for each successive infectious stage during a long infective period. In their model, they described disease progression through multiple infectious stages as in the case of HIV, hepatitis B and hepatitis C. They carried out mathematical analysis of their model and showed that the global dynamics are completely determined by the basic reproductive number. They also ascertained that if the disease-free equilibrium is globally asymptotically stable and the disease dies out, and if a unique endemic equilibrium is globally asymptotically stable, and the disease persists at the endemic equilibrium.

Ayoola *et al.* [6] examined a mathematical modeling of a nonlinear differential equation to investigate the effect of vaccination on the spread of chickenpox in a population. They carried out the existence and uniqueness of the positive solution. The disease-free and endemic equilibrium states and the stability analysis of the disease-free and endemic equilibrium states of the model were studied. They found that the rate of vaccination and precaution for the spread of chickenpox serve as a factor that influenced the basic reproductive number. The forecasts made in their paper was via numerical simulation using the Laplace Adomian Decomposition method. Nkeki and Mbarie [7] considered a

mathematical model and stability analysis of the transmission dynamics of chickenpox in the presence of treatment, quarantine, precaution, and vaccination as control measures are presented. The control measures and vaccine efficacy on the spread of chickenpox were studied in their paper, to see the role of the control measures in influencing the basic reproduction number. It was found that adopting the four control measures plays a prominent role in reducing the chance of contracting and the spread of chickenpox in the population. Almuallem *et al.* [8] present a comprehensive mathematical analysis of the chickenpox transmission model, including positivity, existence, invariant region, and uniqueness of the solution. They introduced optimal control measures using two time-dependent control variables, which include, vaccination and quarantine. Ibrahim *et al.* [9] investigated the dynamics of chickenpox that incorporates intervention in the form of isolation and treatment. They considered the positivity and boundedness of the solution of their chickenpox model, also they ascertained the existence of disease and equilibria points. The effective reproduction number was computed using the next-generation operator method. They carried out sensitivity analysis and found that natural recovery and isolation are very sensitive in reducing the control reproduction number and effective contact rate was found to be very sensitive in increasing the effective reproduction number. Nkeki and Mbarie [10] examined mathematical modelling and stability analysis of chickenpox disease model in the presence of vaccination in a homogeneous population. The model was structured into eight compartments, which represents the actual structure of chickenpox transmission dynamics. The basic reproduction number and the stability analysis of their model were determined. In their numerical simulations, they determined the role of vaccination and recovery rate in the control and prevention of the spread of chickenpox in a public primary school in a Central City of China. It was found that individual students who are unvaccinated against chickenpox are exposed to higher risk of chickenpox infection at the incubation, prodromal and active stages of the infection periods. They also found that recovery, treatment and vaccination rates played vital roles in the control and prevention of the spread of chickenpox among the school children. They also established that vaccination against chickenpox will reduce the spread of the virus up to 90%, which is in line with WHO estimation of 87% to 95%. For the case of compartmental Disease Models with Heterogeneous Populations, see Mohapatra *et al.* [11].

The contribution of this paper is sevenfold: first, this paper consider SEQ_3IHR multi-group model for the transmission dynamics of chickenpox in a heterogeneous population with sub-infection stages: incubation, prodromal and active stages; second, effect of vaccination against chickenpox infection is studied; third, chickenpox re-infection among the sub-population is studied; fourth, the sub-group basic reproduction numbers for chickenpox model that captured the impact of having distinct incubation, prodromal and active stages of the spread of virus are determined; fifth, the stability analysis of SEQ_3IHR chickenpox models is studied; sixth, sensitivity analysis of the chickenpox basic reproduction number is carried out; finally, numerical simulations of our multi-group SEQ_3IHR chickenpox model, taking Nineveh Province of Iraq as a case study, is considered.

Highlight of the Paper:

- A multi-group SEQ_3IHR chickenpox epidemic models with effective vaccination in a different subpopulation sizes is considered;
- Three distinct infectious stages: incubation, prodromal and active are studied;
- The global dynamics of the epidemic models is established;
- Stability analysis of the disease-free and the endemic equilibrium points are carried out;
- The roles of age heterogeneity, treatment, re-infection, vaccination rates are determined.

The rest of the paper is structured as follows: In section 2, we present the description of the model. The stability analysis of our epidemic model is presented in section 3. Section 4 presents numerical simulations of our model. Finally, section 5 presents the concluding part of the paper.

2 The Model Description

In this section, we present the construction and description of SEQ_3IHR chickenpox multi-group model in a heterogeneous population. The population is made up of group of susceptible individuals, S, exposed individuals, E, quarantined individuals, Q, individuals in incubation stage, I_1, individuals in prodromal stage, I_2, individuals in active stage, I_3, hospitalized individuals, H and recovered individuals, R. We let $N(t)$ be the total population of the individual in the population at time t.

Each of the classes of population is made up of subgroups, defined as follows: $S_i(t)$ the number of susceptible individuals for age group i at time t, $E_i(t)$ the number of exposed individuals for age group i at time t, $Q_i(t)$ the number of quarantined individuals for age group i at time t, $I_{i1}(t)$ the number of individuals in incubation stage of the disease for age group i at time t, $I_{i2}(t)$ the number of individuals in prodromal stage of the disease for age group i at time t, $I_{i3}(t)$ the number of individuals in symptomatic stage of the disease for age group i at time t, $H_i(t)$ the number of individuals undergoing treatment for age group i at time t, $R_i(t)$ the number infected that recovered from the disease for age group i at time t, $N_i(t)$ the total population of the individuals in age group i at time t.

Figure 1 represents the schematic diagram for the SEQ_3IHR model design for chickenpox from which, we arrived the following nonlinear system of differential equations:

$$\begin{aligned}\frac{dS_i(t)}{dt} &= (1-\kappa_i)\Lambda_i - \sum_{j=1}^{n}\frac{\beta_{ij}S_i(t)}{N(t)}\left(\sum_{k=1}^{3}\alpha_{jk}I_{jk}(t)\right.\\ &\left.+\alpha_{j4}H_j(t)+\alpha_{j5}E_j(t)+\alpha_{j6}Q_j(t)\right)\\ &-(\mu+\theta_i)S_i(t), \quad t\geq 0,\end{aligned} \tag{1}$$

$$\begin{aligned}\frac{dE_i(t)}{dt} &= \sum_{j=1}^{n}\frac{\beta_{ij}S_i(t)}{N(t)}\left(\sum_{k=1}^{3}\alpha_{jk}I_{jk}(t)\right.\\ &\left.+\alpha_{j4}H_j(t)+\alpha_{j5}E_j(t)+\alpha_{j6}Q_j(t)\right)\\ &-(\rho_{i1}+\gamma_i+\mu)E_i(t), \quad t\geq 0,\end{aligned} \tag{2}$$

$$\frac{dQ_i(t)}{dt} = \rho_{i1}E_i(t)-(\rho_{i2}+\rho_{i3}+\mu)Q_i(t), \quad t\geq 0, \tag{3}$$

$$\begin{aligned}\frac{dI_{i1}(t)}{dt} &= \gamma_i E_i(t)+\rho_{i2}Q_i(t)+\phi_i R_i(t)\\ &-(\omega_{i1}+\mu_{i1}+\mu)I_{i1}(t), \quad t\geq 0,\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{dI_{i2}(t)}{dt} &= \omega_{i1}I_{i1}(t)+\varepsilon_i R_i(t)\\ &-(\omega_{i2}+\nu_{i1}+\mu_{i2}+\mu)I_{i2}(t), \quad t\geq 0,\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{dI_{i3}(t)}{dt} &= \omega_{i2}I_{i2}(t)+\sigma_i R_i(t)\\ &-(\omega_{i3}+\nu_{i2}+\mu_{i3}+\mu)I_{i3}(t), \quad t\geq 0,\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{dH_i(t)}{dt} &= \rho_{i3}Q_i(t)+\nu_{i1}I_{i2}(t)+\nu_{i2}I_{i3}(t)+\vartheta_i R_i(t)\\ &-(\nu_{i3}+\mu_{i4}+\mu)H_i(t), \quad t\geq 0,\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{dR_i(t)}{dt} &= \kappa_i\Lambda_i+\theta_i S_i(t)+\nu_{i3}H_i(t)+\omega_{i3}I_{i3}\\ &-(\mu+\phi_i+\varepsilon_i+\sigma_i+\vartheta_i)R_i(t), \quad t\geq 0,\end{aligned} \tag{8}$$

subject to the initial values $S_i(0)=S_i^0$, $E_i(0)=E_i^0$, $Q_i(0)=Q_i^0$, $I_{ik}(0)=I_{ik}^0$, $k=1,2,3$, $H_i(0)=H_i^0$, $R_i(0)=R_i^0$, where

$$\lambda_i(t)=\sum_{j=1}^{n}\frac{\beta_{ij}}{N}\left(\sum_{k=1}^{3}\alpha_{jk}I_{jk}+\alpha_{j4}H_j+\alpha_{j5}E_j+\alpha_{j6}Q_j\right),$$
$$i=1,2,\ldots,n,$$

is the force of infection at time t.

3 Stability Analysis of SEQ_3IHR Epidemic Model

In this section, we consider the stability analysis of our SEQ_3IHR chickenpox epidemic model. We first, determine the dynamics of the total subpopulation which is structured according to age groups in the population. From which, we deduce the dynamics of the entire population of all the age groups at time t. From now on, we replace H_i with I_{i4} for simplicity.

Theorem 1. (*Positivity of Solutions*). *Let the initial values* $S_{i0}\geq 0, E_{i0}\geq 0, Q_{i0}\geq 0, I_{i10}\geq 0, I_{i20}\geq 0, I_{i30}\geq 0, I_{i40}\geq 0, R_{i0}\geq 0, i=1,2,\ldots,n$. *Then, the solutions* $(S_1,E_1,Q_1,I_{11},I_{12},I_{13},I_{14},R_1), i=1,2,\ldots,n$ *of model* (1) *are nonnegative for all* $t>0$ *in* $\mathbb{R}_+^8$.

Proof. Since all the parameters in our model are positive, it then follows that the lower bounds can be obtained for each equation in (1) - (8) and for $i=1,2,\ldots,n$ as follows:

$$\begin{aligned}\frac{dS_i(t)}{dt} &> -\sum_{j=1}^{n}\frac{\beta_{ij}S_i(t)}{N(t)}\left(\sum_{k=1}^{3}a_{jk}I_{jk}(t)\right.\\ &\left.+a_{j4}H_j(t)+a_{j5}E_j(t)+a_{j6}Q_j(t)\right)\\ &-(\mu+\theta_i)S_i(t),\end{aligned}$$

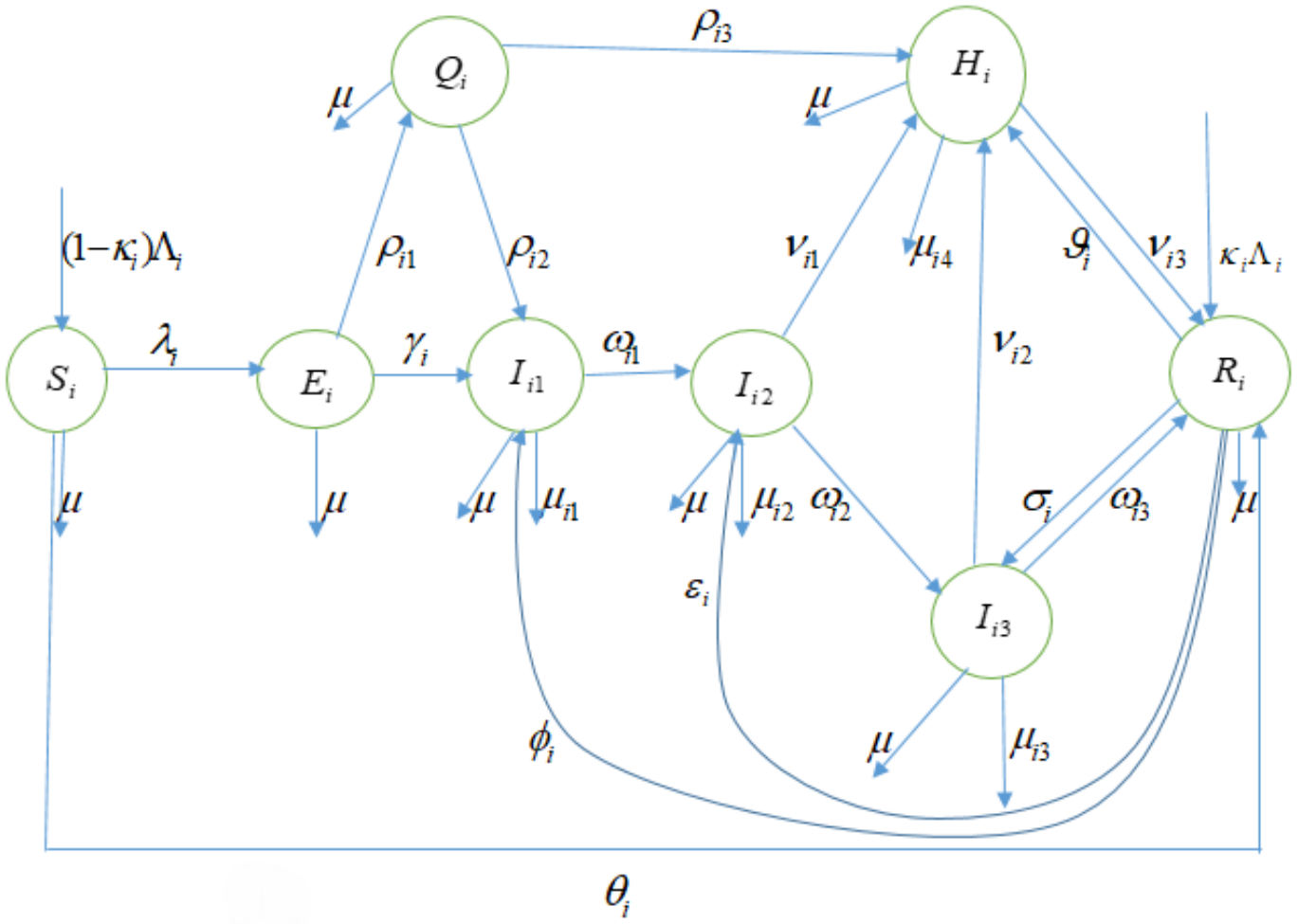

Figure 1. The Schematic Diagram for Multi-Stage SEQ_3IHR Model.

$$\frac{dE_i(t)}{dt} > -(\rho_{i1}+\gamma_i+\mu)E_i(t),$$
$$\frac{dQ_i(t)}{dt} > -(\rho_{i2}+\rho_{i3}+\mu)Q_i(t),$$
$$\frac{dI_{i1}(t)}{dt} > -(\omega_{i1}+\mu_{i1}+\mu)I_{i1}(t),$$
$$\frac{dI_{i2}(t)}{dt} > -(\omega_{i2}+v_{i1}+\mu_{i2}+\mu)I_{i2}(t),$$
$$\frac{dI_{i3}(t)}{dt} > -(\omega_{i3}+v_{i2}+\mu_{i3}+\mu)I_{i3}(t),$$
$$\frac{dI_{i4}(t)}{dt} > -(\nu_{i3}+\mu_{i4}+\mu)I_{i4}(t),$$
$$\frac{dR_i(t)}{dt} > -(\mu+\phi_i+\epsilon_i+\sigma_i+\delta_i)R_i(t).$$

Solving the above resulting inequalities, we have respectively:

$$S_i(t) > S_i(0)\exp\Bigg((\mu+\theta_i)t - \int_{j=1}^{n}\frac{\beta_{ij}S_i(u)}{N(u)}\Big(\sum_{k=1}^{3}a_{jk}I_{jk}(u) + a_{j4}H_j(u) + a_{j5}E_j(u) + a_{j6}Q_j(u)\Big)du\Bigg) \geq 0,$$

$$E_i(t) > E_i(0)\exp((\rho_{i1}+\gamma_i+\mu)t) \geq 0,$$

$$I_{i1}(t) > I_{i1}(0)\exp((\omega_{i1}+\mu_{i1}+\mu)t) \geq 0,$$

$$I_{i3}(t) > I_{i3}(0)\exp((\omega_{i3}+\nu_{i2}+\mu_{i3}+\mu)t) \geq 0,$$

$$R_i(t) > R_i(0)\exp((\mu+\phi_i+\epsilon_i+\sigma_i+\delta_i)t) \geq 0.$$

Hence, for all $t > 0$, the solutions $(S_1, E_1, Q_1, I_{11}, I_{12}, I_{13}, I_{14}, R_1), i = 1, 2, \ldots, n$ of model (1) are nonnegative in $\mathbb{R}^8_+$. □

Theorem 2. (*Boundedness of Solutions*). *The solutions* $(S_i, E_i, Q_i, I_{i1}, I_{i2}, I_{i3}, I_{i4}, R_i)$, $i = 1, 2, \ldots, n$ *of model* (1) – (8) *are bounded.*

Proof. We are to determine the boundedness of solution of model (1) in a positively invariant set D and in an invariant subset D_i, for $i = 1, 2, \ldots, n$. Now, the dynamics of $N_i(t)$ is given as:

$$N_i(t) = S_i(t) + E_i(t) + Q_i(t) + \sum_{k=1}^{4} I_{ik}(t) + R_i(t),$$
$$i = 1, 2, \ldots, n.$$

It then follows that:

$$N(t) = \sum_{i=1}^{n} \left(S_i(t) + E_i(t) + Q_i(t) + \sum_{k=1}^{4} I_{ik}(t) + R_i(t) \right)$$

The derivative of $N_i(t)$ can be obtained as:

$$\dot{N}_i(t) = \Lambda_i - \mu S_i(t) - \mu E_i(t) - \mu Q_i(t) - \sum_{k=1}^{4} (\mu + \mu_{ik}) I_{ik}(t) - \mu R_i(t).$$

Simplifying, we have:

$$\dot{N}_i(t) = \Lambda_i - \mu N_i(t) - \sum_{k=1}^{4} \mu_{ik} I_{ik}(t).$$

Now, for $\sum_{k=1}^{4} \mu_{ik} I_{ik}(t) \leq 0$, we have:

$$\dot{N}_i(t) \leq \Lambda_i - \mu N_i(t). \tag{9}$$

From (9), we have that $\lim_{t\to\infty} \sup N_i(t) \leq \frac{\Lambda_i}{\mu}$, for $i = 1, 2, \ldots, n$ and when $N_i(t) > \frac{\Lambda_i}{\mu}, \dot{N}_i(t) < 0$.

Similarly, it follows from (1) – (8) that

$$\limsup_{t\to\infty} S_i \leq \frac{(1-\kappa_i)\Lambda_i}{\mu + \theta_i}.$$

So the feasible region D_i of (1) – (8) is, for $i = 1, 2, \ldots, n$,

$$D_i = \begin{cases} (S_i, E_i, Q_i, I_{ik}, R_i) \in \mathbb{R}_+^{8n} \mid 0 < S_i(t) + E_i(t) + Q_i(t) \\ + \sum_{k=1}^{4} I_{ik}(t) + R_i(t) \leq N_i(t) \leq \frac{\Lambda_i}{\mu}, \quad S_i^0 = \frac{(1-\kappa_i)\Lambda_i}{\mu+\theta_i}, \\ S_i \geq 0, E_i \geq 0, Q_i \geq 0, I_{ik} \geq 0, k = 1, 2, 3, 4, R_i \geq 0 \end{cases}$$

Similarly, for $\Lambda = \sum_{i=1}^{n} \Lambda_i$, $S = \sum_{i=1}^{n} S_i$, $E = \sum_{i=1}^{n} E_i$, $Q = \sum_{i=1}^{n} Q_i$, $I_k = \sum_{i=1}^{n} I_{ik}$ and $R = \sum_{i=1}^{n} R_i$ we have,

$$D = \begin{cases} (S, E, Q, I_{ik}, R) \in \mathbb{R}_+^{8n} \mid 0 < S(t) + E(t) + Q(t) \\ + \sum_{k=1}^{4} I_{ik}(t) + R(t) \leq N(t) \leq \frac{\Lambda}{\mu}, \quad S^0 = \frac{(1-\kappa)\Lambda}{\mu+\theta}, \\ S \geq 0, E \geq 0, Q \geq 0, I_{ik} \geq 0, k = 1, 2, 3, 4, R \geq 0 \end{cases}$$

Since all solutions of (1) – (8) will remain or allow to tends to the field of D_i and the sum tends to the field D, it is easy to see that the feasible region D_i and D are the positive invariant sets for the system of differential equations (1) – (8).

The progression through the compartments is illustrated in Figure 1. New infections in compartment E_i arise by contacts between susceptible in compartments S_i, and infected individuals in compartments Q_i and I_{1k}, at a rate $\sum_{j=1}^{n} \frac{\beta_{ij} S_i(t)}{N(t)} \left(\sum_{k=1}^{4} a_{jk} I_{jk}(t) + a_{j5} E_j(t) + a_{j6} Q_j(t) \right)$.

The system has a unique disease-free equilibrium for the system of differential equations (1) – (8),

$$P_0 = \big(S_1^0, 0, 0, 0, 0, 0, 0, 0, S_2^0, 0, 0, 0, 0, 0, 0, 0, \ldots, S_n^0, 0, 0, 0, 0, 0, 0, 0\big)$$

where $S_i^0 = \frac{(1-\kappa_i)\Lambda_i}{\mu+\theta_i}$ in D_i, $i = 1, 2, \ldots, n$, is the equilibrium of the S_i population in the absence of disease. An endemic equilibrium for the system of differential equations (1) – (8) is

$$P^* = (S_1^*, E_1^*, Q_1^*, I_{11}^*, I_{12}^*, I_{13}^*, I_{14}^*, R_1^*, S_2^*, E_2^*, Q_2^*, I_{21}^*, I_{22}^*, I_{23}^*, I_{24}^*, R_2^*, \ldots, S_n^*, E_n^*, Q_n^*, I_{n1}^*, I_{n2}^*, I_{n3}^*, I_{n4}^*, R_n^*)$$

in D_i, where $S_i^* > 0, E_i^* > 0, Q_i^* > 0, I_{i1}^* > 0, I_{i2}^* > 0, I_{i3}^* > 0, I_{i4}^* > 0, R_i^* > 0, i = 1, 2, \ldots, n$.

Taking the infected compartments to be E_i, Q_i, and $I_{ik}, i = 1, 2, 3, 4$ gives

$$\sum_{j=1}^{n} \frac{\beta_{ij} S_i(t)}{N(t)} \left[\sum_{k=1}^{3} a_{jk} I_{jk}(t) + a_{j4} H_j(t) + a_{j5} E_j(t) + a_{j6} Q_j(t) \right]$$

$$\mathcal{S}_i = \begin{pmatrix} \sum_{j=1}^{n} & \frac{\beta_{ij} S_i(t)}{N(t)} \left[\sum_{k=1}^{3} a_{jk} I_{jk}(t) + a_{j4} H_j(t) \right. \\ & \left. + a_{j5} E_j(t) + a_{j6} Q_j(t) \right] \\ & 0 \\ & 0 \\ & 0 \\ & 0 \end{pmatrix}$$

and

$$\Xi_i = \begin{pmatrix} (\rho_{i1}+\gamma_i+\mu)E_i(t) - \rho_{i1}E_i(t) + (\rho_{i2} \\ +\rho_{i3}+\mu)Q_i(t) - \gamma_i E_i(t) - \rho_{i2}Q_i(t) \\ -\phi_i R_i(t) + (\omega_{i1}+\mu_{i1}+\mu)I_{i1}(t) - \omega_{i1}I_{i1}(t) \\ -\epsilon_i R_i(t) + (\omega_{i2}+\nu_{i1}+\mu_{i2}+\mu)I_{i2}(t) \\ -\omega_{i2}I_{i2}(t) - \sigma_i R_i(t) \\ +(\omega_{i3}+\nu_{i2}+\mu_{i3}+\mu)I_{i3}(t) \\ -\rho_{i3}Q_i(t) - \nu_{i1}I_{i2}(t) - \nu_{i2}I_{i3}(t) \\ -\delta_i R_i(t) + (\nu_{i3}+\mu_{i4}+\mu)H_i(t) \end{pmatrix}$$

Hence,

$$F_i = \begin{pmatrix} \sum_{j=1}^{n}\frac{\alpha_{j5}\beta_{ij}S_i^0(t)}{N(t)} & \sum_{j=1}^{n}\frac{\alpha_{j6}\beta_{ij}S_i^0(t)}{N(t)} & \sum_{j=1}^{n}\frac{\alpha_{j1}\beta_{ij}S_i^0(t)}{N(t)} & \sum_{j=1}^{n}\frac{\alpha_{j2}\beta_{ij}S_i^0(t)}{N(t)} & \sum_{j=1}^{n}\frac{\alpha_{j3}\beta_{ij}S_i^0(t)}{N(t)} & \sum_{j=1}^{n}\frac{\alpha_{j4}\beta_{ij}S_i^0(t)}{N(t)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V_i = \begin{pmatrix} \rho_{i1}+\gamma_i+\mu & 0 & 0 & 0 & 0 & 0 \\ -\rho_{i1} & \rho_{i2}+\rho_{i3}+\mu & 0 & 0 & 0 & 0 \\ -\gamma_i & -\rho_{i2} & \omega_{i1}+\mu_{i1}+\mu & 0 & 0 & 0 \\ 0 & 0 & -\omega_{i1} & \omega_{i2}+\nu_{i1}+\mu_{i2}+\mu & 0 & 0 \\ 0 & 0 & 0 & -\omega_{i2} & \omega_{i3}+\nu_{i2}+\mu_{i3}+\mu & 0 \\ 0 & -\rho_{i3} & 0 & -\nu_{i1} & -\nu_{i2} & \nu_{i3}+\mu_{i4}+\mu \end{pmatrix},$$

and using the next generation matrix, we have the basic reproduction number for our chickenpox model for the transmission of infection between compartments S_i and the infection classes j, that is, $E_j, Q_j, I_{kj}, k = 1,2,3,4$ is obtained as follows:

$$\begin{aligned} R_o^i = &\frac{\hat{\alpha}_{5i}s_i}{P_{1i}} + \frac{\hat{\alpha}_{6i}\rho_{i1}s_i}{P_{1i}P_{2i}} + \frac{\hat{\alpha}_{1i}(\gamma_i P_{2i}+\rho_{i1}\rho_{i2})s_i}{P_{1i}P_{2i}P_{3i}} \\ &+ \frac{\hat{\alpha}_{2i}\omega_{i1}(\gamma_i P_{2i}+\rho_{i1}\rho_{i2})s_i}{P_{1i}P_{2i}P_{3i}P_{4i}} \\ &+ \frac{\hat{\alpha}_{3i}\omega_{i1}\omega_{i2}(\gamma_i P_{2i}+\rho_{i1}\rho_{i2})s_i}{P_{1i}P_{2i}P_{3i}P_{4i}P_{5i}} \\ &+ \frac{\hat{\alpha}_{4i}\left[\omega_{i1}(\omega_{i2}\nu_{i2}+\nu_{i1}P_{5i})(\gamma_i P_{2i}+\rho_{i1}\rho_{i2})\right]s_i}{P_{1i}P_{2i}P_{3i}P_{4i}P_{5i}P_{6i}} \\ &+ \frac{\hat{\alpha}_{4i}\left[P_{3i}P_{4i}P_{5i}\rho_{i1}\rho_{i3}\right]s_i}{P_{1i}P_{2i}P_{3i}P_{4i}P_{5i}P_{6i}}, \end{aligned} \tag{10}$$

where $s_i = \frac{\mu(1-\kappa_i)\Lambda_i}{(\mu+\theta_i)\Lambda}$, $\hat{\alpha}_{1i} = \sum_{j=1}^{n}\alpha_{j1}\beta_{ij}$, $\hat{\alpha}_{2i} = \sum_{j=1}^{n}\alpha_{j2}\beta_{ij}$, $\hat{\alpha}_{3i} = \sum_{j=1}^{n}\alpha_{j3}\beta_{ij}$, $\hat{\alpha}_{4i} = \sum_{j=1}^{n}\alpha_{j4}\beta_{ij}$,

$\hat{\alpha}_{5i} = \sum_{j=1}^{n}\alpha_{j5}\beta_{ij}$, $\hat{\alpha}_{6i} = \sum_{j=1}^{n}\alpha_{j6}\beta_{ij}$, $P_{1i} = \rho_{i1}+\gamma_i+\mu$,
$P_{2i} = \rho_{i2}+\rho_{i3}+\mu$, $P_{3i} = \omega_{i1}+\mu_{i1}+\mu$,

$P_{4i} = \omega_{i2}+\nu_{i1}+\mu_{i2}+\mu$, $P_{5i} = \omega_{i3}+\nu_{i2}+\mu_{i3}+\mu$,
$P_{6i} = \nu_{i3}+\mu_{i4}+\mu$.

(10) is the basic reproduction number of the sub-groups of the system, and it represents the number of secondary infections produced by an index case within the same age group, as well as between different age groups. The six terms of R_i^0 are the number of secondary infections produced by an index case initially in compartment E_i. To interpret these terms, recall that β_{ij} is the transmission coefficient between compartments S_i and the infection classes j, that is, $E_j, Q_j, I_{kj}, k = 1,2,3,4$ for introducing a single infected individual into a population of S_i, the susceptible individuals in age group i with probability s_i. The ratio $\frac{\hat{\alpha}_{5i}}{P_{1i}}$ in the first term, is the expected duration of the infectious period of individual in compartment E_i that progresses to compartments Q_j and I_{j1} over the course of its infection. The ratio $\frac{\hat{\alpha}_{6i}\rho_{1i}}{P_{1i}P_{2i}}$ in the second term, is the expected duration of the infectious period of individual in compartment Q_i that progress to compartments I_{j1} and I_{j4} over the course of its infection. The ratio $\frac{\hat{\alpha}_{1i}(\gamma_i P_{2i}+\rho_{1i}\rho_{i2})}{P_{1i}P_{2i}P_{3i}P_{4i}}$ in third term, is the expected duration of the infectious period of individual in compartment I_{i1} that progress to compartments I_{j2} over the course of its infection. The ratio $\frac{\hat{\alpha}_{2i}\omega_{1i}(\gamma_i P_{2i}+\rho_{1i}\rho_{i2})}{P_{1i}P_{2i}P_{3i}P_{4i}}$ in the fourth term, is the expected duration of the infectious period of individual in compartment I_{i2} that progress to compartments I_{j3} and I_{j4} over the course of its infection. The ratio $\frac{\hat{\alpha}_{3i}\omega_{1i}\omega_{i2}(\gamma_i P_{2i}+\rho_{1i}\rho_{i2})}{P_{1i}P_{2i}P_{3i}P_{4i}P_{5i}}$ in the fifth term, is the expected duration of the infectious period of individual in compartment I_{i3} that progress to compartment I_{j4} and immunized stage over the course of its infection. The ratio $\frac{\hat{\alpha}_{4i}\left[\omega_{i1}(\omega_{i2}\nu_{i2}+\nu_{i1}P_{5i})(\gamma_i P_{2i}+\rho_{1i}\rho_{i2})+P_{3i}P_{4i}P_{5i}\rho_{i1}\rho_{i3}\right]}{P_{1i}P_{2i}P_{3i}P_{4i}P_{5i}P_{6i}}$ in sixth term, is the expected duration of the infectious period of individual to progress compartment I_{i4} to immunized stage.

It then follows from (10) that the basic reproduction number of the entire system is obtained as:

$$R_0 = \sum_{i=1}^{n} R_i^0.$$

We now determine the equilibrium points

$$\begin{aligned} P^* = &(S_1^*, E_1^*, Q_1^*, I_{11}^*, I_{12}^*, I_{13}^*, I_{14}^*, R_1^*, S_2^*, E_2^*, Q_2^*, \\ &I_{21}^*, I_{22}^*, I_{23}^*, I_{24}^*, R_2^*, \ldots, S_n^*, E_n^*, Q_n^*, I_{n1}^*, I_{n2}^*, \\ &I_{n3}^*, I_{n4}^*, R_n^*) \end{aligned}$$

in D_i, $i = 1,2,\ldots,n$, for endemic case as follows:

$$S_i^* = \frac{(1-\kappa_i)\Lambda_i}{\mu+\theta_i} - \frac{P_{1i}E_i^*}{\mu+\theta_i},$$
$$\psi_1 E_i^* + \psi_2 E_i^* + \psi_3 = 0,$$
$$Q_i^* = \frac{P_{1i}E_i^*}{P_{2i}},$$

$$I_{i1}^* = \frac{\phi_i\kappa_i\Lambda_i}{P_{3i}\Phi_i} + \frac{\phi_i\theta_i(1-\kappa_i)\Lambda_i}{P_{3i}\Phi_i(\mu+\theta_i)} + \left(\gamma_i\frac{P_{2i}+\rho_{1i}\rho_{12}}{P_{2i}P_{3i}} + \frac{\phi_i(f_{1i}+f_{13})}{P_{3i}\Phi_i} - \frac{\phi_i\theta_i P_{1i}}{P_{3i}\Phi_i(\mu+\theta_i)}\right)E_i^*,$$

$$I_{i2}^* = \frac{\Lambda_i(\omega_{1i}\phi_i+\epsilon_i P_{3i})}{P_{3i}P_{4i}\Phi_i}\left(\kappa_i + \frac{\theta_i(1-\kappa_i)}{\mu+\theta_i}\right) + \left(\frac{\omega_{1i}(\gamma_i P_{2i}+\rho_{1i}\rho_{12})}{P_{2i}P_{3i}P_{4i}} + \frac{(\omega_{1i}\phi_i+\epsilon_i P_{3i})}{P_{3i}P_{4i}\Phi_i}(f_{1i}+f_{13} - \frac{\theta_i P_{1i}}{\mu+\theta_i})\right)E_i^*,$$

$$I_{i3}^* = \frac{f_{i4}(1-\kappa_i)\Lambda_i}{\mu+\theta_i} + \left(f_{i3} - \frac{P_{1i}f_{i4}}{\mu+\theta_i}\right)E_i^*,$$
$$I_{4i}^* = \frac{f_{i2}(1-\kappa_i)\Lambda_i}{\mu+\theta_i} + \left(f_{i1} - \frac{P_{1i}f_{i2}}{\mu+\theta_i}\right)E_i^*,$$

$$R_i^* = \frac{\kappa_i\Lambda_i}{\Phi_i} + \frac{\theta_i(1-\kappa_i)\Lambda_i}{\mu+\theta_i} + \left(\frac{f_{1i}+f_{i3}}{\Phi_i} - \frac{P_{1i}\theta_i}{\mu+\theta_i}\right)E_i^*.$$

where:

$$\psi_1 = h_{i1} - \frac{h_{i2}\theta_i P_{1i}}{\mu+\theta_i} - \frac{P_{1i}\Lambda R_0^i}{\mu(1-\kappa_i)\Lambda_i},$$
$$\psi_2 = R_0^i + h_{i1} + h_{i2}\kappa_i\Lambda_i - \frac{h_{i2}\theta_i P_{1i}}{\mu+\theta_i} + \frac{h_{i2}\theta_i(1-\kappa_i)\Lambda_i}{\mu+\theta_i} - 1,$$
$$\psi_3 = h_{2i}\kappa_i\Lambda_i + \frac{h_{2i}\theta_i(1-\kappa_i)\Lambda_i}{\mu+\theta_i},$$
$$M_i = \frac{s_i}{P_{1i}\Phi_i}\left(\frac{\hat\alpha_{1i}\phi_i}{P_{3i}} + \frac{\hat\alpha_{3i}\left[\omega_{i2}(\omega_{i1}\phi_i+\epsilon_i P_{2i}) + \sigma_i P_{3i}P_{4i}\right]}{P_{3i}P_{4i}P_{5i}} + \hat\alpha_{4i}f_{i2}\right),$$
$$h_{1i} = (f_{1i}+f_{13})M_i,\ h_{i2} = M_i + \frac{\hat\alpha_{2i}(\omega_{i1}\phi_i+\epsilon_i P_{3i})s_i}{P_{1i}P_{3i}P_{4i}\Phi_i},$$
$$f_{i1} = \frac{\rho_{i1}\rho_{i3}}{P_{2i}P_{6i}} + \frac{\nu_{1i}\omega_{i1}(\gamma_i P_{2i}+\rho_{i1}\rho_{i2})}{P_{2i}P_{3i}P_{4i}P_{6i}} + \frac{\nu_{2i}f_{i3}}{P_{6i}},$$
$$f_{i2} = \frac{\nu_{1i}(\omega_{1i}\phi_i+\epsilon_i P_{3i})}{P_{3i}P_{4i}P_{6i}} + \frac{\nu_{13}f_{i4}}{P_{6i}} + \frac{g_i}{P_{1i}},$$
$$f_{i3} = \frac{\omega_{1i}\omega_{12}(\gamma_i P_{2i}+\rho_{1i}\rho_{12})}{P_{2i}P_{3i}P_{4i}P_{5i}},$$
$$f_{i4} = \frac{\omega_{12}(\omega_{1i}\phi_i+\epsilon_i P_{3i})}{P_{3i}P_{4i}P_{5i}},$$
$$\Phi_i = P_{7i} - \nu_{13}f_{i2} - \omega_{13}f_{i4},$$
$$P_{7i} = \mu + \phi_i + \epsilon_i + \sigma_i + g_i.$$

□

Theorem 3. *If $R_o^i \le 1$, then the disease-free equilibrium P_0 is globally asymptotically stable in D_i, and if $R_o^i > 1$, P_0 will be unstable in D_i, $i = 1, 2, \ldots, n$.*

Proof. Let $\{\beta_{ij}\}_{i,j=1}^n$ be a matrix and C_{ii} the cofactor of the i-th, $1 \le i \le n$, diagonal entry of $\{\beta_{ij}\}_{i,j=1}^n$. Again, let $c_i = C_{ii}$, $i = 1, 2, \ldots, n$. We now construct a Lyapunov function for our system of differential equations (1)–(8) as follows:

$$V = \sum_{i=1}^n c_i\left(S_i - S_i^*\ln S_i + E_i - E_i^*\ln E_i + Q_i - Q_i^*\ln Q_i + \sum_{k=1}^4 (I_{ik} - I_{ik}^*\ln I_{ik}) + \dot R_i - R_i^*\ln R_i\right).$$

Differentiating with respect to t, we have

$$\dot V = \sum_{i=1}^n c_i\left(\dot S_i - \frac{S_i^*}{S_i}\dot S_i + \dot E_i - \frac{E_i^*}{E_i}\dot E_i + \dot Q_i - \frac{Q_i^*}{Q_i}\dot Q_i + \sum_{k=1}^4\left(\dot I_{ik} - \frac{I_{ik}^*}{I_{ik}}\dot I_{ik}\right) + \dot R_i - \frac{R_i^*}{R_i}\dot R_i\right).$$

Using (1)–(8) and the equilibrium equations, we have

$$(1-\kappa_i)\Lambda_i = \sum_{j=1}^n \frac{\beta_{ij}S_i^*}{N}\left(\sum_{k=1}^4 \alpha_{jk}I_{jk}^* + \alpha_{j5}E_j^* + \alpha_{j6}Q_j^*\right) + (\mu+\theta_i)S_i^*,$$

$$\sum_{j=1}^n \frac{\beta_{ij}S_i^*}{N}\left(\sum_{k=1}^4 \alpha_{jk}I_{jk} + \alpha_{j5}E_j + \alpha_{j6}Q_j\right) = (\rho_{i1}+\gamma_i+\mu)E_i.$$

Using (1), we have

$$\begin{aligned}\dot{V} = \sum_{i=1}^{n} c_i \Bigg\{ & 2\sum_{j=1}^{n} \frac{\beta_{ij} S_i^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^* \right) \\ & - (\mu + \theta_i) S_i^* \left(\frac{S_i^*}{S_i} + \frac{S_i}{S_i^*} - 2 \right) \\ & + \sum_{j=1}^{n} \frac{\beta_{ij} S_i^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk} + \alpha_{j5} E_j + \alpha_{j6} Q_j \right) \\ & - \sum_{j=1}^{n} \frac{\beta_{ij} S_i^{*2}}{S_i N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^* \right) \\ & - (\rho_{i1} + \gamma_i + \mu) E_i \\ & - \sum_{j=1}^{n} \frac{\beta_{ij} E_i^* S_i}{E_i N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk} + \alpha_{j5} E_j + \alpha_{j6} Q_j \right) \\ & + \left(1 - \frac{Q_i^*}{Q_i} \right) \rho_{i1} E_i + (\rho_{i2} + \rho_{i3} + \mu)(Q_i^* - Q_i) \\ & + \left(1 - \frac{I_{i1}^*}{I_{i1}} \right) (\gamma_i E_i + \rho_{i2} Q_i + \phi_i R_i) \\ & + (\omega_{i1} + \mu_{i1} + \mu)(I_{i1}^* - I_{i1}) \\ & + \left(1 - \frac{I_{i2}^*}{I_{i2}} \right) (\omega_{i1} I_{i1} + \varepsilon_i R_i) \\ & + (\omega_{i2} + \nu_{i1} + \mu_{i2} + \mu)(I_{i2}^* - I_{i2}) \\ & + \left(1 - \frac{I_{i3}^*}{I_{i3}} \right) (\omega_{i2} I_{i2} + \sigma_i R_i) \\ & + (\omega_{i3} + \nu_{i2} + \mu_{i3} + \mu)(I_{i3}^* - I_{i3}) \\ & + \left(1 - \frac{I_{i4}^*}{I_{i4}} \right) (\rho_{i3} Q_i + \nu_{i1} I_{i2} + \nu_{i2} I_{i3} + \vartheta_i R_i) \\ & + (\nu_{i3} + \mu_{i4} + \mu)(I_{i4}^* - I_{i4}) \\ & + \left(1 - \frac{R_i^*}{R_i} \right) (\kappa_i \Lambda_i + \theta_i S_i + \nu_{i3} I_{i4} + \omega_{i3} I_{i3}) \\ & + (\mu + \phi_i + \varepsilon_i + \sigma_i + \vartheta_i)(R_i^* - R_i) \Bigg\}. \end{aligned}$$

Setting $E_i^* = E_i$, $Q_i^* = Q_i$, $I_{i1}^* = I_{i1}$, $I_{i2}^* = I_{i2}$, $I_{i3}^* = I_{i3}$, $I_{i4}^* = I_{i4}$, $R_i^* = R_i$, it then follows that

$$\begin{aligned}\dot{V} = -\sum_{i=1}^{n} c_i \Bigg[& S_i^*(\mu + \theta_i) \left(\frac{S_i^*}{S_i} + \frac{S_i}{S_i^*} - 2 \right) \\ & + \left(\frac{S_i^*}{S_i} - 1 \right) \sum_{j=1}^{n} \frac{\beta_{ij} S_i^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^* \right) \Bigg]. \end{aligned}$$

Since $\left(\frac{S_i^*}{S_i} + \frac{S_i}{S_i^*} - 2 \right) \geq 0$ and $\left(\frac{S_i^*}{S_i} - 1 \right) \geq 0$, we have that

$$\begin{aligned} & S_i^*(\mu + \theta_i) \left(\frac{S_i^*}{S_i} + \frac{S_i}{S_i^*} - 2 \right) \\ & + \left(\frac{S_i^*}{S_i} - 1 \right) \sum_{j=1}^{n} \frac{\beta_{ij} S_i^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^* \right) \geq 0. \end{aligned}$$

Hence,

$$\begin{aligned}\dot{V} = -\sum_{i=1}^{n} c_i \Bigg[& S_i^*(\mu + \theta_i) \left(\frac{S_i^*}{S_i} + \frac{S_i}{S_i^*} - 2 \right) \\ & + \left(\frac{S_i^*}{S_i} - 1 \right) \sum_{j=1}^{n} \frac{\beta_{ij} S_i^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^* \right) \Bigg] \leq 0. \end{aligned}$$

If $E_i^* = Q_i^* = I_{i1}^* = I_{i2}^* = I_{i3}^* = I_{i4}^* = 0$, we have that

$$\dot{V} = -\sum_{i=1}^{n} c_i S_i^*(\mu + \theta_i) \left(\frac{S_i^*}{S_i} + \frac{S_i}{S_i^*} - 2 \right) \leq 0.$$

It is observed that $\dot{V} = 0$ if and only if $S_i^* = S_i$. By adopting LaSalle's (1976) extension to Lyapunov's method, we see the limit set of each solution is contained in the largest invariant set D, we obtain that P_0 is globally asymptotically stable in D, that is, $R_o^i \leq 1$ and unstable in D, if $R_o^i > 1$. □

Theorem 4. *If $R_o^i > 1$, then the unique endemic equilibrium point,*

$$\begin{aligned} P^* = & \big(S_1^*, E_1^*, Q_1^*, I_{11}^*, I_{12}^*, I_{13}^*, I_{14}^*, R_1^*, S_2^*, E_2^*, Q_2^*, I_{21}^*, \\ & I_{22}^*, I_{23}^*, I_{24}^*, R_2^*, \ldots, \\ & S_n^*, E_n^*, Q_n^*, I_{n1}^*, I_{n2}^*, I_{n3}^*, I_{n4}^*, R_n^* \big) \end{aligned}$$

in D, where $S_i^* > 0$, $E_i^* > 0$, $Q_i^* > 0$, $I_{i1}^* > 0$, $I_{i2}^* > 0$, $I_{i3}^* > 0$, $I_{i4}^* > 0$, $R_i^* > 0$, $i = 1, 2, \ldots, n$ is globally asymptotically stable in D for our system of differential equations (1)–(8).

Proof. We commence by expressing

$$\begin{aligned}\dot{V} \leq & \sum_{i=1}^{n} c_i \Bigg[S_i^*(\mu + \theta_i) \left(2 - \frac{S_i^*}{S_i} - \frac{S_i}{S_i^*} \right) \\ & + \left(1 - \frac{S_i^*}{S_i} \right) \sum_{j=1}^{n} \frac{\beta_{ij} S_i^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^* \right) \Bigg] \\ \leq & \sum_{i=1}^{n} c_i \left[S_i^*(\mu + \theta_i) \left(2 - \frac{S_i^*}{S_i} - \frac{S_i}{S_i^*} \right) \right] \\ & + \sum_{i=1}^{n} c_i \left(1 - \frac{S_i^*}{S_i} \right) \sum_{j=1}^{n} \frac{\beta_{ij} S_i^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^* \right) \\ \leq & \sum_{i=1}^{n} c_i \left[S_i^*(\mu + \theta_i) \left(2 - \frac{S_i^*}{S_i} - \frac{S_i}{S_i^*} \right) \right] \\ & + \sum_{i,j=1}^{n} c_i \left(1 - \frac{S_i^*}{S_i} \right) \frac{\beta_{ij} S_i^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^* \right). \end{aligned}$$

Denote

$$\begin{aligned}Z_n &= Z_n(S_1, E_1, Q_1, I_{11}, I_{12}, I_{13}, I_{14}, R_1, \ldots, S_n, E_n, Q_n, \\ &\quad I_{n1}, I_{n2}, I_{n3}, I_{n4}, R_n) \\ &= \sum_{i=1}^{n} c_i \left[S_i^*(\mu+\theta_i)\left(2 - \frac{S_i^*}{S_i} - \frac{S_i}{S_i^*}\right)\right] \\ &+ \sum_{i,j=1}^{n} c_i \left(1 - \frac{S_i^*}{S_i}\right) \frac{\beta_{ij} S_i^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^*\right).\end{aligned}$$

We now show that $Z_n \leq 0$ for all $Z_n(S_1, E_1, Q_1, I_{11}, I_{12}, I_{13}, I_{14}, R_1, \ldots, S_n, E_n, Q_n, I_{n1}, I_{n2}, I_{n3}, I_{n4}, R_n) \in D$. Since the proof for the general case is somehow complicated, we consider proofs for $n = 2$ and $n = 3$. Then, we show the major steps of the proof for arbitrary $n \geq 2$ is given.

Case $n = 2$: We have

$$\begin{aligned}Z_2 &= Z_2(S_1, E_1, Q_1, I_{11}, I_{12}, I_{13}, I_{14}, R_1, S_2, E_2, Q_2, I_{21}, \\ &\quad I_{22}, I_{23}, I_{24}, R_2) \\ &= \sum_{i=1}^{2} c_i \left[S_i^*(\mu+\theta_i)\left(2 - \frac{S_i^*}{S_i} - \frac{S_i}{S_i^*}\right)\right] \\ &+ \sum_{i,j=1}^{2} c_i \left(1 - \frac{S_i^*}{S_i}\right) \frac{\beta_{ij} S_i^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^*\right).\end{aligned}$$

Expanding, we have

$$\begin{aligned}Z_2 &= c_1 \left[S_1^*(\mu+\theta_1)\left(2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*}\right)\right] \\ &+ c_1 \left(1 - \frac{S_1^*}{S_1}\right) \frac{\beta_{11} S_1^*}{N} \left(\sum_{k=1}^{4} \alpha_{1k} I_{1k}^* + \alpha_{15} E_1^* + \alpha_{16} Q_1^*\right) \\ &+ c_1 \left(1 - \frac{S_1^*}{S_1}\right) \frac{\beta_{12} S_1^*}{N} \left(\sum_{k=1}^{4} \alpha_{2k} I_{2k}^* + \alpha_{25} E_2^* + \alpha_{26} Q_2^*\right) \\ &+ c_2 \left[S_2^*(\mu+\theta_2)\left(2 - \frac{S_2^*}{S_2} - \frac{S_2}{S_2^*}\right)\right] \\ &+ c_2 \left(1 - \frac{S_2^*}{S_2}\right) \frac{\beta_{21} S_2^*}{N} \left(\sum_{k=1}^{4} \alpha_{1k} I_{1k}^* + \alpha_{15} E_1^* + \alpha_{16} Q_1^*\right) \\ &+ c_2 \left(1 - \frac{S_2^*}{S_2}\right) \frac{\beta_{22} S_2^*}{N} \left(\sum_{k=1}^{4} \alpha_{2k} I_{2k}^* + \alpha_{25} E_2^* + \alpha_{26} Q_2^*\right)\end{aligned}$$

Since

$$c_1 \left[S_1^*(\mu+\theta_1)\left(2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*}\right)\right] \leq 0,$$

$$c_1 \left(1 - \frac{S_1^*}{S_1}\right) \frac{\beta_{11} S_1^*}{N} \left(\textstyle\sum_{k=1}^{4} \alpha_{1k} I_{1k}^* + \alpha_{15} E_1^* + \alpha_{16} Q_1^*\right) \leq 0,$$

$$c_1 \left(1 - \frac{S_1^*}{S_1}\right) \frac{\beta_{12} S_1^*}{N} \left(\textstyle\sum_{k=1}^{4} \alpha_{2k} I_{2k}^* + \alpha_{25} E_2^* + \alpha_{26} Q_2^*\right) \leq 0,$$

$$c_2 \left[S_2^*(\mu+\theta_2)\left(2 - \frac{S_2^*}{S_2} - \frac{S_2}{S_2^*}\right)\right] \leq 0,$$

$$c_2 \left(1 - \frac{S_2^*}{S_2}\right) \frac{\beta_{21} S_2^*}{N} \left(\textstyle\sum_{k=1}^{4} \alpha_{1k} I_{1k}^* + \alpha_{15} E_1^* + \alpha_{16} Q_1^*\right) \leq 0,$$

and

$$c_2 \left(1 - \frac{S_2^*}{S_2}\right) \frac{\beta_{22} S_2^*}{N} \left(\textstyle\sum_{k=1}^{4} \alpha_{2k} I_{2k}^* + \alpha_{25} E_2^* + \alpha_{26} Q_2^*\right) \leq 0,$$

we obtain $Z_2 \leq 0$ for all $Z_2(S_1, E_1, Q_1, I_{11}, I_{12}, I_{13}, I_{14}, R_1, S_2, E_{2n}, Q_2, I_{21}, I_{22}, I_{23}, I_{24}, R_2) \in D$.

Thus, $\dot{V} = 0$ if and only if $S_i^* = S_i$, $i = 1, 2$ and $Z_2 = 0$.

Case $n = 3$: We have

$$\begin{aligned}Z_3 &= Z_3(S_1, E_1, Q_1, I_{11}, I_{12}, I_{13}, I_{14}, R_1, S_2, E_2, Q_2, I_{21}, \\ &\quad I_{22}, I_{23}, I_{24}, R_2, S_3, E_3, Q_3, I_{31}, I_{32}, I_{33}, I_{34}, R_3) \\ &= \sum_{i=1}^{3} c_i \left[S_i^*(\mu+\theta_i)\left(2 - \frac{S_i^*}{S_i} - \frac{S_i}{S_i^*}\right)\right] \\ &+ \sum_{i,j=1}^{3} c_i \left(1 - \frac{S_i^*}{S_i}\right) \frac{\beta_{ij} S_i^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^*\right).\end{aligned}$$

Expanding, we have

$$\begin{aligned}Z_3 &= Z_3(S_1, E_1, Q_1, I_{11}, I_{12}, I_{13}, I_{14}, R_1, S_2, E_2, Q_2, I_{21}, \\ &\quad I_{22}, I_{23}, I_{24}, R_2, S_3, E_3, Q_3, I_{31}, I_{32}, I_{33}, I_{34}, R_3) \\ &= c_1 \left[S_1^*(\mu+\theta_1)\left(2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*}\right)\right] \\ &+ c_1 \left(1 - \frac{S_1^*}{S_1}\right) \sum_{j=1}^{n} \frac{\beta_{1j} S_1^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^*\right) \\ &+ c_2 \left[S_2^*(\mu+\theta_2)\left(2 - \frac{S_2^*}{S_2} - \frac{S_2}{S_2^*}\right)\right] \\ &+ c_2 \left(1 - \frac{S_2^*}{S_2}\right) \sum_{j=1}^{n} \frac{\beta_{2j} S_2^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^*\right) \\ &+ c_3 \left[S_3^*(\mu+\theta_3)\left(2 - \frac{S_3^*}{S_3} - \frac{S_3}{S_3^*}\right)\right] \\ &+ c_3 \left(1 - \frac{S_3^*}{S_3}\right) \sum_{j=1}^{n} \frac{\beta_{3j} S_3^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^*\right).\end{aligned}$$

Again, since

$$c_1 \left[S_1^*(\mu+\theta_1)\left(2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*}\right)\right]$$

$$+ c_1 \left(1 - \frac{S_1^*}{S_1}\right) \sum_{j=1}^{n} \frac{\beta_{1j} S_1^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^*\right) \leq 0,$$

$$c_2 \left[S_2^*(\mu+\theta_2)\left(2 - \frac{S_2^*}{S_2} - \frac{S_2}{S_2^*}\right)\right]$$

$$+ c_2 \left(1 - \frac{S_2^*}{S_2}\right) \sum_{j=1}^{n} \frac{\beta_{2j} S_2^*}{N} \left(\sum_{k=1}^{4} \alpha_{jk} I_{jk}^* + \alpha_{j5} E_j^* + \alpha_{j6} Q_j^*\right) \leq 0,$$

and

$$c_3\left[S_3^*(\mu+\theta_3)\left(2-\frac{S_3^*}{S_3}-\frac{S_3}{S_3^*}\right)\right] + c_3\left(1-\frac{S_3^*}{S_3}\right)\sum_{j=1}^{n}\frac{\beta_{3j}S_3^*}{N}\left(\sum_{k=1}^{4}\alpha_{jk}I_{jk}^*+\alpha_{j5}E_j^*+\alpha_{j6}Q_j^*\right)\leq 0,$$

we obtain $Z_3 \leq 0$ for all $Z_3(S_1, E_1, Q_1, I_{11}, I_{12}, I_{13}, I_{14}, R_1, S_2, E_2, Q_2, I_{21}, I_{22}, I_{23}, I_{24}, R_2, S_3, E_3, Q_3, I_{31}, I_{32}, I_{33}, I_{34}, R_3) \in D$.

Thus, $\dot{V} = 0$ if and only if $S_i^* = S_i$, $i = 1, 2, 3$ and $Z_3 = 0$.

Case $n \geq 2$: We have

$$Z_n = \sum_{i=1}^{n} c_i\left[S_i^*(\mu+\theta_i)\left(2-\frac{S_i^*}{S_i}-\frac{S_i}{S_i^*}\right)\right] + \sum_{i,j=1}^{n} c_i\left(1-\frac{S_i^*}{S_i}\right)\frac{\beta_{ij}S_i^*}{N}\left(\sum_{k=1}^{4}\alpha_{jk}I_{jk}^*+\alpha_{j5}E_j^*+\alpha_{j6}Q_j^*\right).$$

Expanding, we have

$$\begin{aligned}Z_n = {} & c_1\left[S_1^*(\mu+\theta_1)\left(2-\frac{S_1^*}{S_1}-\frac{S_1}{S_1^*}\right)\right] + c_1\left(1-\frac{S_1^*}{S_1}\right)\sum_{j=1}^{n}\frac{\beta_{1j}S_1^*}{N}\left(\sum_{k=1}^{4}\alpha_{jk}I_{jk}^*+\alpha_{j5}E_j^*+\alpha_{j6}Q_j^*\right)\\ & + c_2\left[S_2^*(\mu+\theta_2)\left(2-\frac{S_2^*}{S_2}-\frac{S_2}{S_2^*}\right)\right] + c_2\left(1-\frac{S_2^*}{S_2}\right)\sum_{j=1}^{n}\frac{\beta_{2j}S_2^*}{N}\left(\sum_{k=1}^{4}\alpha_{jk}I_{jk}^*+\alpha_{j5}E_j^*+\alpha_{j6}Q_j^*\right)+\\ & + c_n\left[S_n^*(\mu+\theta_n)\left(2-\frac{S_n^*}{S_n}-\frac{S_n}{S_n^*}\right)\right] + c_n\left(1-\frac{S_n^*}{S_n}\right)\sum_{j=1}^{n}\frac{\beta_{nj}S_n^*}{N}\left(\sum_{k=1}^{4}\alpha_{jk}I_{jk}^*+\alpha_{j5}E_j^*+\alpha_{j6}Q_j^*\right).\end{aligned}$$

In a similar manner, since

$$c_1\left[S_1^*(\mu+\theta_1)\left(2-\frac{S_1^*}{S_1}-\frac{S_1}{S_1^*}\right)\right] + c_1\left(1-\frac{S_1^*}{S_1}\right)\sum_{j=1}^{n}\frac{\beta_{1j}S_1^*}{N}\left(\sum_{k=1}^{4}\alpha_{jk}I_{jk}^*+\alpha_{j5}E_j^*+\alpha_{j6}Q_j^*\right)\leq 0,$$

$$c_2\left[S_2^*(\mu+\theta_2)\left(2-\frac{S_2^*}{S_2}-\frac{S_2}{S_2^*}\right)\right] + c_2\left(1-\frac{S_2^*}{S_2}\right)\sum_{j=1}^{n}\frac{\beta_{2j}S_i^*}{N}\left(\sum_{k=1}^{4}\alpha_{jk}I_{jk}^*+\alpha_{j5}E_j^*+\alpha_{j6}Q_j^*\right)\leq 0,\ldots,$$

$$c_n\left[S_n^*(\mu+\theta_n)\left(2-\frac{S_n^*}{S_n}-\frac{S_n}{S_n^*}\right)\right] + c_n\left(1-\frac{S_n^*}{S_n}\right)\sum_{j=1}^{n}\frac{\beta_{nj}S_i^*}{N}\left(\sum_{k=1}^{4}\alpha_{jk}I_{jk}^*+\alpha_{j5}E_j^*+\alpha_{j6}Q_j^*\right)\leq 0,$$

we have that $Z_n \leq 0$ for all $Z_n \in D$. Thus, $\dot{V} = 0$ if and only if $S_i^* = S_i$, $i = 1, 2, \ldots, n$ and $Z_n = 0$.

Hence, the only compact invariant subset of the set where $\dot{V} = 0$ is the singleton P^*. By LaSalle's Invariance Principle, P^* is global asymptotically stable in D if $R_o^i > 1$.

□

4 Numerical Results

In this section, we present the numerical results of our model. Table 1 presents the baseline values of the parameters for the SEQ_3IHR chickenpox disease model for multi-dimensional age groups. Table 2 presents the initial values for model's state variables for the SEQ_3IHR chickenpox disease model, using Nineveh Province of Iraq as a case study. We have that the total population of Nineveh Province of Iraq as of 2021, according to the United Nations-World is 4,030,006. Hence, in this paper, the total population $N = 4{,}030{,}006$, of which 201,500 is the population infants below one year old, 403,001 is a population of age group 1–4 years, 806,002 is a population of age group 5–14 years, 2,015,003 is a population of age group 15–44 years, and 604,500 is a population of age group 45 years and above. It is imperative to note that most of the parameter values are derived and estimated from Dawood [12]. Again, the age group $i =$ (below 1 year, 1–4 years, 5–14 years, 15–44 years, above 45 years), and $n = 5$.

Using the above data and information, we have that the basic reproduction number for each of the subgroups is obtained as $\mathcal{R}_{01} = 0.0535$, $\mathcal{R}_{02} = 0.0007$, $\mathcal{R}_{03} = 0.0031$, $\mathcal{R}_{04} = 0.0024$, $\mathcal{R}_{05} = 0.0050$. It simply implies that for age group below 1 year, the basic reproduction number is 0.0535, for 1–4 years, the basic reproduction number is 0.0007, for 5–14 years is 0.0031, for 15–44 years is 0.0024 and for 45 years and above is 0.0050. It is obvious from the results that, it is possible to eradicate the chickenpox disease in the subpopulation. Again, the basic reproduction number for the entire population is obtained as the sum of the basic reproduction of the subpopulation, which is $\mathcal{R}_0 = 0.0647$, showing that chickenpox can be eradicated from the entire population.

Table 1. The baseline values of the parameters for the SEQ_3IHR disease model for different age groups.

Parameter	Baseline values for $i = (1, 2, 3, 4, 5)$ and $j = (1, 2, 3, 4, 5, 6)$	References
β_{1i}	(0.00017448, 0.000152, 0.0001, 0.00012, 0.042, 0.4) day^{-1}	Dawood [12]
β_{2i}	(0.0015, 0.00134, 0.001, 0.0001, 0.142, 0.4) day^{-1}	Dawood [12]
β_{3i}	(0.0044, 0.00411, 0.0038, 0.0002, 0.142, 0.4) day^{-1}	Dawood [12]
β_{4i}	(0.00071382, 0.0007, 0.00068, 0.0005, 0.7, 0.2) day^{-1}	Dawood [12]
β_{5i}	(0.00013942, 0.000114, 0.0001, 0.0003, 0.9, 0.2) day^{-1}	Dawood [12]
κ_i	(0.01, 0.45, 0.15, 0.20, 0.19) day^{-1}	Estimated
θ_i	(0.01, 0.50, 0.20, 0.20, 0.05) day^{-1}	Estimated
ϕ_i	(0.05, 0.02, 0.01, 0.01, 0.03) day^{-1}	Estimated
ε_i	(0.09, 0.06, 0.02, 0.04, 0.12) day^{-1}	Estimated
σ_i	(0.1, 0.25, 0.30, 0.12, 0.4) day^{-1}	Estimated
ϑ_i	(0.05, 0.08, 0.10, 0.6, 0.20) day^{-1}	Estimated
μ_{1i}	(0.001, 0.001, 0.001, 0.001, 0.001) day^{-1}	Dawood [12]
μ_{2i}	(0.024, 0.02, 0.02, 0.01, 0.02) day^{-1}	Dawood [12]
μ_{3i}	(0.03, 0.03, 0.03, 0.02, 0.04) day^{-1}	Dawood [12]
μ_{4i}	(0.02, 0.01, 0.01, 0.002, 0.04) day^{-1}	Dawood [12]
α_{1i}	(0.90, 0.95, 0.95, 0.95, 0.85)	[13]
α_{2i}	(0.95, 0.97, 0.97, 0.97, 0.90)	[13]
α_{3i}	(1.0, 1.0, 1.0, 1.0, 1.0)	Estimated
α_{4i}	(0.80, 0.85, 0.85, 0.90, 0.95)	[13]
α_{5i}	(0.75, 0.78, 0.80, 0.85, 0.95)	[13]
α_{6i}	(0.12, 0.16, 0.20, 0.25, 0.30)	Estimated
ω_{1i}	(0.2079, 0.1313, 0.0745, 0.2382, 0.7661) day^{-1}	Dawood [12]
ω_{2i}	(0.12, 0.12, 0.125, 0.5, 0.3) day^{-1}	Dawood [12]
ω_{3i}	(0.15, 0.256, 0.27, 0.4, 0.15) day^{-1}	Dawood [12]
ρ_{1i}	(0.6, 0.6, 0.62, 0.65, 0.7) day^{-1}	Assumed
ρ_{2i}	(0.15, 0.2, 0.1, 0.2, 0.1) day^{-1}	Assumed
ρ_{3i}	(0.10, 0.12, 0.10, 0.08, 0.15) day^{-1}	Assumed
ν_{i1}	(0.041, 0.035, 0.030, 0.025, 0.050) day^{-1}	Dawood [12]
ν_{i2}	(0.050, 0.045, 0.040, 0.035, 0.060) day^{-1}	Dawood [12]
ν_{i3}	(0.080, 0.085, 0.090, 0.095, 0.070) day^{-1}	Dawood [12]
γ_i	(0.041, 0.211, 0.695, 0.041, 0.012) day^{-1}	Dawood [12]
μ	0.004677 day^{-1}	[13]

Figure 2 shows the number of people who will possibly be in incubation stage of chickenpox infection for different age groups: below 1 year, 1-4 years, 5-14 years, 15-44 years and 45 years and above for a period of 0-50 days. It is observed that individuals at age 15-44 years will experience more chickenpox infection for the first 15 days, and gradually declined as time evolves. In the first five days, the number of individuals who are at the age of 5-14 years and number of individuals who are 45 years and above intersect. Also intersect at day 12, is number of individuals who are 1-4 years and number of individuals who are 45 years and above. At day 18, the number of individuals who are 15-44 years and number of individuals who 45 years and above, attained the same number. At day 22, the number of individuals who 5-14 years and number of individuals who are below 1 year, intersect. Similarly, at day 35, number of individuals who are below 1 year and 1-4 years, intersect. At day 42, number of individuals who are below 1 year and 15-44 years, intersect. Also intersect at day 42, are individuals who are age 1-4 years and 5-14 years. In day 47, number of individuals who are 5-14 years and 15-44 years, intersect as well. The result shows that if incubation stage is to last for 50 days, individuals at age 45 years and above, will experience more chickenpox infection, and those who

Table 2. Initial values for model's state variables for the SEQ_3IHR disease model.

Variable	Initial values, for $i = (1, 2, 3, 4, 5)$	References
Λ_i	(6095, 1050, 918, 373, 120)	Estimated
N_i	(201500, 403001, 806002, 2015003, 604500)	[13]
S_i	(200604, 401262, 802535, 2006461, 601927)	Computed
E_i	(752, 1478, 3101, 7513, 2475)	[13]
Q_i	(100, 225, 305, 1002, 80)	[13]
I_{i1}	(8, 5, 10, 2, 1)	Dawood [12]
I_{i2}	(15, 12, 20, 9, 5)	Dawood [12]
I_{i3}	(20, 18, 30, 15, 11)	Dawood [12]
H_i	(1, 1, 1, 1, 1)	Assumed
R_i	(0, 0, 0, 0, 0)	Assumed

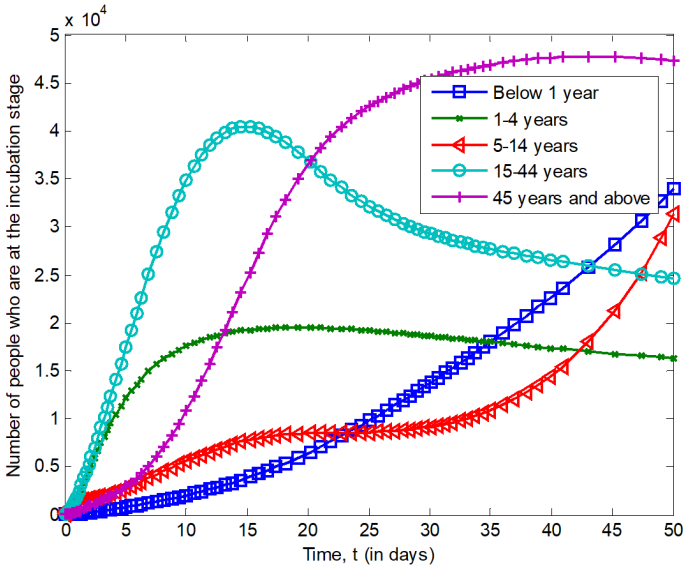

Figure 2. The number of people who are in incubation stage of chickenpox infection for age groups below 1 year, 1-4 years, 5-14 years, 15-44 years and 45 years and above for a period of 0-50 days.

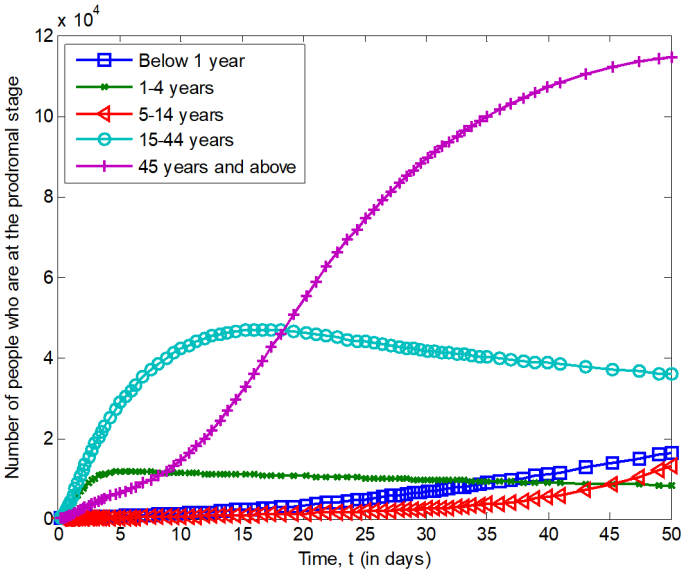

Figure 3. The number of people who are in prodromal stage of chickenpox infection for age groups below 1 year, 1-4 years, 5-14 years, 15-44 years and 45 years and above for a period of 0-50 days.

are 1-4 will have low chickenpox infection. Practically, during incubation stage of chickenpox infection, which last from 7-21 days, groups mainly affected at this period are 45 years and above, follow by 15-44 years of age, then 1-4 years, and then age groups below 1 year and 5-14 years, of which they have almost equal number of infections at this stage.

Figure 3 shows the number of people who are in prodromal stage of chickenpox infection for age groups below 1 year, 1-4 years, 5-14 years, 15-44 years and 45 years and above for a period of 0-50 days. It is observed that individuals who are 45 years and above will be more infected, follow by individual group who are between the ages of 15 and 44 years. It is lower at other age groups, as we can see in Figure 3.

Figure 4 shows the number of people who are in active stage of chickenpox infection for age groups below 1 year, 1-4 years, 5-14 years, 15-44 years and 45 years and above for a period of 0-50 days. It is observed that individuals who are 45 years and above will be more infected at the active stage, follows by individual group who are between the ages of 15 and 44 years, though between 0 and 49 days, since at 49 day upwards, individuals at age range 5-14 happened to be higher, when compare with individuals of age 15-44 years. It is lower at other age groups, as we can see in Figure 4, but at day 33, individuals who are at age range of 5-14 years, experience a sudden increase of chickenpox infection at the active stage of the virus.

Figure 5 shows the number of people who are hospitalized of chickenpox infection for age groups

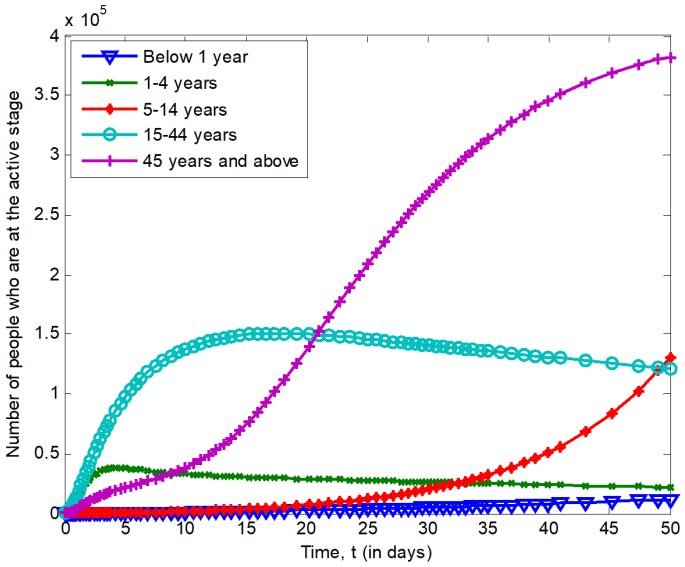

Figure 4. The number of people who are in active stage of chickenpox infection for age groups below 1 year, 1-4 years, 5-14 years, 15-44 years and 45 years and above for a period of 0-50 days.

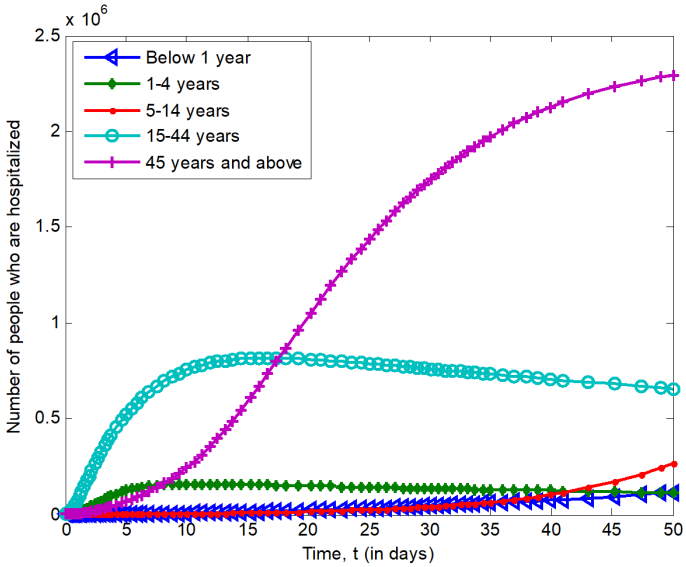

Figure 5. The number of people who are hospitalized of chickenpox infection for age groups below 1 year, 1-4 years, 5-14 years, 15-44 years and 45 years and above for a period of 0-50 days.

below 1 year, 1-4 years, 5-14 years, 15-44 years and 45 years and above for a period of 0-50 days. It is found that individuals who are 45 years and above will be more at the hospitalization stage, also follows by individual group who are between the ages of 15 and 44 years. It is lower for other age groups.

Figure 6 shows the number of people who recovered of chickenpox infection for age groups below 1 year, 1-4 years, 5-14 years, 15-44 years and 45 years and above for a period of 0-50 days. It is observed that higher

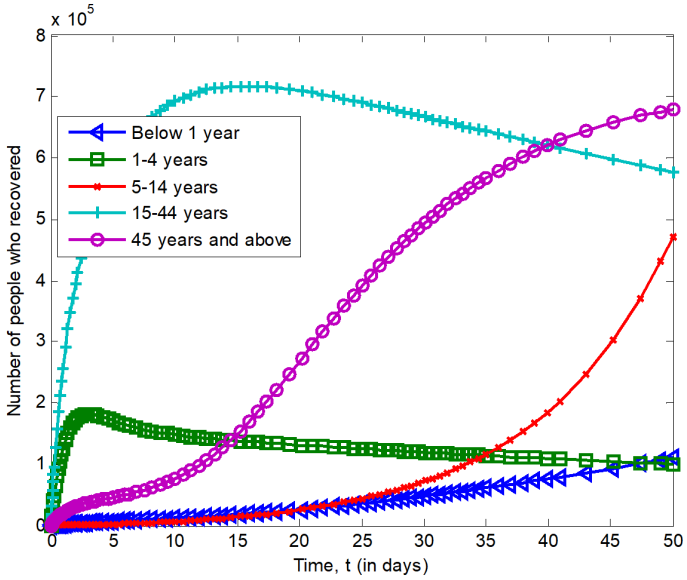

Figure 6. The number of people who recovered of chickenpox infection for age groups below 1 year, 1-4 years, 5-14 years, 15-44 years and 45 years and above for a period of 0-50 days.

recovery are experience by individuals in age group 15-44, between day 0 and 40, and declined thereafter. Individuals of age 45 years and above continue to experience high recovery. Also, individuals who are of age 5-14 years. It is imperative to mention that there is a continuous reduction of number of recovered individuals of age group 1-4 years.

Table 3 presents the basic reproduction numbers for the five different age groups. The table is made up of five compartments. The first compartment comprises of changes in basic reproduction numbers for the five different age groups for varying value of ω_1. The second compartment involves changes in basic reproduction numbers for the five different age groups for varying value of ω_2. The third compartment comprises of change in basic reproduction numbers for the five different age groups for varying value of γ. The fourth compartment comprises of change in basic reproduction numbers for the five different age groups for varying value of ρ_2. And the last compartment comprises of change in basic reproduction numbers for the five different age groups for varying value of θ.

We observed in Table 3 from the first compartment that as the rate of chickenpox progresses from age groups in incubation stage to age group in prodromal stage, ω_1, the basic reproduction number for individuals below 1 year, R_0^1 decreases from 0.0749 to 0.0478 as ω_1 increases over time, from 0.05 to 0.90. For age group of individuals 1–4 years, for the first compartment, R_0^2 decreases from 0.0010 to 0.0005, and thereafter

Table 3. The basic reproduction numbers for the five age groups for varying values of $\omega_1, \omega_2, \gamma, \rho_2$, and θ.

Compartment	Parameter	Value	R_0^1	R_0^2	R_0^3	R_0^4	R_0^5
First	ω_1	0.05	0.0749	0.0010	0.0039	0.0040	0.0063
		0.10	0.0612	0.0007	0.0026	0.0031	0.0056
		0.30	0.0513	0.0005	0.0016	0.0024	0.0051
		0.50	0.0492	0.0005	0.0014	0.0022	0.0050
		0.70	0.0483	0.0005	0.0013	0.0021	0.0050
		0.90	0.0478	0.0005	0.0013	0.0021	0.0050
Second	ω_2	0.10	0.0535	0.0007	0.0031	0.0026	0.0050
		0.20	0.0535	0.0007	0.0030	0.0026	0.0050
		0.60	0.0533	0.0006	0.0030	0.0024	0.0050
Third	γ	0.05	0.0538	0.0006	0.0018	0.0025	0.0050
		0.10	0.0552	0.0006	0.0020	0.0025	0.0050
		0.70	0.0636	0.0007	0.0031	0.0030	0.0050
Fourth	ρ_2	0.05	0.0493	0.0006	0.0030	0.0021	0.0049
		0.10	0.0514	0.0006	0.0031	0.0022	0.0050
		0.15	0.0556	0.0007	0.0032	0.0024	0.0051
		0.20	0.0577	0.0007	0.0033	0.0026	0.0052
		0.25	0.0599	0.0007	0.0034	0.0027	0.0053
Fifth	θ	0.05	0.0144	0.0060	0.0115	0.0092	0.0050
		0.10	0.0075	0.0032	0.0060	0.0048	0.0026
		0.20	0.0038	0.0016	0.0031	0.0024	0.0013
		0.30	0.0026	0.0011	0.0021	0.0016	0.0009
		0.50	0.0016	0.0007	0.0012	0.0010	0.0005

remain constant as ω_1 increases over time, from 0.05 to 0.90. For age group of individuals 5–14 years, for the first compartment, R_0^3 decreases from 0.0039 to 0.0013, and thereafter remain constant as ω_1 increases over time, from 0.05 to 0.90. For age group of individuals 15–44 years, for the first compartment, R_0^4 decreases from 0.0040 to 0.0021, and thereafter remain constant as ω_1 increases over time, from 0.05 to 0.90. For age group of individuals 45 years and above, for the first compartment, R_0^5 decreases from 0.0063 to 0.0050, and thereafter remain constant as ω_1 increases over time, from 0.05 to 0.90. We then conclude that, the rate at which chickenpox move from the incubation stage to the prodromal stage will decrease the spread of chickenpox in the population over time.

Also in Table 3, first compartment, it is observed that for all values of ω_1, individuals who are below a year old have the highest tendency of spreading the chickenpox disease, followed by individuals in age group of 45 years and above, followed by individuals in age group 15–44 years, then individuals in age group of 5–14 years of age, and the least are individuals in age group of 1–4 years.

It is observed in Table 3 from the second compartment that as the rate of chickenpox progresses from age groups in prodromal stage to age group in active stage, ω_2, the basic reproduction number for individuals below 1 year, R_0^1 remains constant at the value 0.053 as ω_2 increases over time, from 0.10 to 0.60. For age group of individuals 1–4 years, for the second compartment, R_0^2 decreases from 0.0007 to 0.0006 as ω_2 increases over time, from 0.10 to 0.60. For age group of individuals

5–14 years, for the second compartment, R_0^3 decreases from 0.0031 to 0.0030 as ω_2 increases over time, from 0.10 to 0.60. For age group of individuals 15–44 years, for the second compartment, R_0^4 decreases from 0.0026 to 0.0024 as ω_2 increases over time, from 0.10 to 0.60. For age group of individuals 45 years and above, for the second compartment, R_0^5 remains constant at 0.0050 as ω_2 increases over time, from 0.10 to 0.60. We therefore conclude that, the rate at which chickenpox moves from the prodromal stage to active stage will trigger the spread of chickenpox in the population over time.

Also in Table 3, second compartment, it is observed that for all values of ω_2, individuals who are below a year old have the highest tendency of spreading the chickenpox disease, followed by individuals in age group of 45 years and above, followed by individuals in age group 5–14 years, then individuals in age group of 15–44 years of age, and the least are individuals in age group of 1–4 years.

We observed in Table 3 from the third compartment that as the rate of chickenpox progresses from age groups in exposed class to age group in incubation stage, γ, the basic reproduction number for individuals below 1 year, R_0^1 increases from 0.0538 to 0.0636 as γ increases over time, from 0.05 to 0.70. For age group of individuals 1–4 years, for the third compartment, R_0^2 increases from 0.0006 to 0.0007 as γ increases over time, from 0.05 to 0.70. For age group of individuals 5–14 years, for the third compartment, R_0^3 increases from 0.0018 to 0.0031 as γ increases over time, from 0.05 to 0.70. For age group of individuals 15–44 years, for the third compartment, R_0^4 increases from 0.0025 to 0.0030 as γ increases over time, from 0.05 to 0.70. For age group of individuals 45 years and above, for the third compartment, R_0^5 remains constant at 0.0050 as γ increases over time, from 0.05 to 0.70. It then follows that, the rate at which chickenpox moves from the exposed class to incubation stage will increase the spread of chickenpox in the population over time.

Furthermore, in Table 3, third compartment, it is observed that for all values of γ, individuals who are below a year old have the highest tendency of spreading the chickenpox disease, followed by individuals in age group of 45 years and above, followed by individuals in age group 15–44 years, then individuals in age group of 5–14 years of age, and the least are individuals in age group of 1–4 years.

We observed in Table 3 from the fourth compartment that as the rate at which quarantined individuals from age groups move to incubation stage, ρ_2, the basic reproduction number for individuals below 1 year, R_0^1 increases from 0.0493 to 0.0599 as ρ_2 increases over time, from 0.05 to 0.25. For age group of individuals 1–4 years, for the fourth compartment, R_0^2 increases from 0.0006 to 0.0007 as ρ_2 increases over time, from 0.05 to 0.25. For age group of individuals 5–14 years, for the fourth compartment, R_0^3 increases from 0.0030 to 0.0034 as ρ_2 increases over time, from 0.05 to 0.25. For age group of individuals 15–44 years, for the fourth compartment, R_0^4 increases from 0.0021 to 0.0027 as ρ_2 increases over time, from 0.05 to 0.25.

For age group of individuals 45 years and above, for the fourth compartment, R_0^5 increases from 0.0049 to 0.0053 as ρ_2 increases over time, from 0.05 to 0.25. Thus, the rate at which quarantined individuals move to incubation stage will increase the spread of chickenpox in the population over time.

Furthermore, in Table 3, fourth compartment, it is observed that for all values of ρ_2, individuals who are below a year old have the highest tendency of spreading the chickenpox disease, followed by individuals in age group of 45 years and above, followed by individuals in age group 5–14 years, then individuals in age group of 15–44 years of age, and the least are individuals in age group of 1–4 years.

We observed in Table 3 from the last compartment that as the rate at which individuals from age groups are vaccinated changes, θ, the basic reproduction number for individuals below 1 year, R_0^1 decreases from 0.0144 to 0.0016 as ϑ increases over time, from 0.05 to 0.50. For age group of individuals 1–4 years, for the last compartment, R_0^2 decreases from 0.0060 to 0.0007 as ϑ increases over time, from 0.05 to 0.50. For age group of individuals 5–14 years, for the last compartment, R_0^3 decreases from 0.0115 to 0.0012 as ϑ increases over time, from 0.05 to 0.50.

For age group of individuals 15–44 years, for the last compartment, R_0^4 decreases from 0.0092 to 0.0010 as ϑ increases over time, from 0.05 to 0.50. For age group of individuals 45 years and above, for the last compartment, R_0^5 decreases from 0.0050 to 0.0005 as ϑ increases over time, from 0.05 to 0.50. Hence, vaccination goes a long way in reducing the spread of chickenpox in the population.

Furthermore, in Table 3, last compartment, it is observed that for all values of θ, individuals who are below a year old have the highest tendency of spreading the chickenpox disease, followed by individuals in age group 5–14 years, then individuals

Table 4. The basic reproduction numbers for the five age groups with varying values of $\alpha_j, j = 1, 2, \ldots, 6$.

Compartment	Parameter 1	Parameter 2	R_0^1	R_0^2	R_0^3	R_0^4	R_0^5
First	$\alpha_1 = 0.20$	$\alpha_2 = 0.95$	0.0478	0.0005	0.0016	0.0021	0.0049
	$\alpha_1 = 0.40$	$\alpha_3 = 1.00$	0.0496	0.0005	0.0020	0.0022	0.0049
	$\alpha_1 = 0.60$	$\alpha_4 = 0.95$	0.0515	0.0006	0.0025	0.0023	0.0049
	$\alpha_1 = 0.80$	$\alpha_5 = 0.95$	0.0533	0.0006	0.0030	0.0024	0.0050
	$\alpha_1 = 0.90$	$\alpha_6 = 0.30$	0.0542	0.0007	0.0032	0.0025	0.0050
Second	$\alpha_2 = 0.20$	$\alpha_1 = 0.95$	0.0532	0.0007	0.0031	0.0024	0.0049
	$\alpha_2 = 0.40$	$\alpha_3 = 1.00$	0.0536	0.0007	0.0032	0.0024	0.0049
	$\alpha_2 = 0.60$	$\alpha_4 = 0.95$	0.0539	0.0007	0.0032	0.0025	0.0050
	$\alpha_2 = 0.80$	$\alpha_5 = 0.95$	0.0543	0.0007	0.0033	0.0025	0.0050
	$\alpha_2 = 0.90$	$\alpha_6 = 0.30$	0.0545	0.0007	0.0033	0.0025	0.0050
Third	$\alpha_3 = 0.20$	$\alpha_1 = 0.95$	0.0545	0.0007	0.0033	0.0024	0.0050
	$\alpha_3 = 0.40$	$\alpha_2 = 0.97$	0.0545	0.0007	0.0033	0.0024	0.0050
	$\alpha_3 = 0.60$	$\alpha_4 = 0.95$	0.0546	0.0007	0.0033	0.0025	0.0050
	$\alpha_3 = 0.80$	$\alpha_5 = 0.95$	0.0546	0.0007	0.0033	0.0025	0.0050
	$\alpha_3 = 0.90$	$\alpha_6 = 0.30$	0.0546	0.0007	0.0033	0.0025	0.0050
Fourth	$\alpha_4 = 0.20$	$\alpha_1 = 0.95$	0.0331	0.0005	0.0028	0.0019	0.0024
	$\alpha_4 = 0.40$	$\alpha_2 = 0.97$	0.0388	0.0005	0.0029	0.0020	0.0031
	$\alpha_4 = 0.60$	$\alpha_3 = 1.00$	0.0446	0.0006	0.0031	0.0022	0.0038
	$\alpha_4 = 0.80$	$\alpha_5 = 0.95$	0.0503	0.0006	0.0032	0.0024	0.0045
	$\alpha_4 = 0.90$	$\alpha_6 = 0.30$	0.0532	0.0007	0.0033	0.0025	0.0048
Fifth	$\alpha_5 = 0.20$	$\alpha_1 = 0.95$	0.0435	0.0006	0.0031	0.0019	0.0041
	$\alpha_5 = 0.40$	$\alpha_2 = 0.97$	0.0464	0.0006	0.0032	0.0020	0.0043
	$\alpha_5 = 0.60$	$\alpha_3 = 1.00$	0.0494	0.0006	0.0032	0.0022	0.0046
	$\alpha_5 = 0.80$	$\alpha_4 = 0.95$	0.0524	0.0007	0.0033	0.0024	0.0048
	$\alpha_5 = 0.90$	$\alpha_6 = 0.30$	0.0539	0.0007	0.0033	0.0025	0.0049
Sixth	$\alpha_6 = 0.20$	$\alpha_1 = 0.95$	0.0537	0.0007	0.0033	0.0024	0.0049
	$\alpha_6 = 0.40$	$\alpha_2 = 0.97$	0.0555	0.0007	0.0033	0.0025	0.0051
	$\alpha_6 = 0.60$	$\alpha_3 = 1.00$	0.0573	0.0007	0.0033	0.0026	0.0052
	$\alpha_6 = 0.80$	$\alpha_4 = 0.95$	0.0590	0.0007	0.0034	0.0028	0.0054
	$\alpha_6 = 0.90$	$\alpha_5 = 0.30$	0.0599	0.0007	0.0034	0.0028	0.0055

in age group of 15–44 years of age, then individuals in age group of 1–4 years of age, and the least are individuals in age group of 45 years and above.

Table 4 shows the basic reproduction numbers for the five age groups with varying values of $\alpha_j, j = 1, 2, \ldots, 6$. The table is made up of six compartments, each for varying values of α_j.

In the first compartment of Table 4, for a fixed value of the proportion of infectious individuals in infectious classes, $\alpha_j, j = 2, \ldots, 6$, and varying values of α_1, we have that individuals below a year have the highest tendency of spreading the disease, while individuals

Table 5. Sensitivity analysis of re-infections on the incubation, prodromal, active and treatment stages of chickenpox disease.

Parameter	Value	$I_1 \times 10^4$	$I_2 \times 10^4$	$I_3 \times 10^4$	$I_4 \times 10^5$
	0.10	5.0321	1.8469	1.0569	1.1447
ϕ	0.30	8.7735	2.3088	0.80797	1.1541
	0.50	10.639	2.5454	0.69051	1.1648
	0.10	3.3554	1.7246	1.5650	1.1574
ε	0.15	3.2261	2.1033	1.1224	1.2363
	0.20	3.1208	2.4308	1.0942	1.3059
	0.10	3.0211	1.4904	1.0659	1.0200
σ	0.30	2.8113	1.2709	2.3163	1.3032
	0.50	2.7284	1.1379	3.1791	1.5126
	0.20	2.9975	1.3884	9.5721	1.3820
ϑ	0.40	2.7543	1.2005	7.9546	1.6431
	0.60	2.5954	1.0716	6.8323	1.8314

in age group 1–4 years have the least tendency of spreading the disease over time. We observed that as α_1 increases, $R_0^i, i = 1, 2, \ldots, 5$ increases alongside.

In the second compartment of Table 4, for a fixed value of the proportion of infectious individuals in infectious classes, $\alpha_j, j = 1, 3, 4, 5, 6$, and varying values of α_2, we have that individuals below a year have the highest tendency of spreading the disease, while individuals in age group 1–4 years have the least and constant tendency of spreading the disease over time. We observed that as α_2 increases, $R_0^i, i = 1, 3, 4, 5$ increases slightly, and R_0^2 remains constant over time.

In the third compartment of Table 4, for a fixed value of the proportion of infectious individuals in infectious classes, $\alpha_j, j = 1, 2, 4, 5, 6$, and varying values of α_3, we have that individuals below a year have the highest tendency of spreading the disease, while individuals in age group 1–4 years have the least and constant tendency of spreading the disease over time. We observed that as α_3 increases, $R_0^i, i = 1, 4$ increases slightly, and $R_0^3, i = 2, 3, 5$ remain constant over time.

In the fourth compartment of Table 4, for a fixed value of the proportion of infectious individuals in infectious classes, $\alpha_j, j = 1, 2, 3, 5, 6$, and varying values of α_4, we have that individuals below a year have the highest tendency of spreading the disease, while individuals in age group 1–4 years have the least tendency of spreading the disease over time. We observed that as α_4 increases, $R_0^i, i = 1, 2, \ldots, 5$ increases alongside.

In the fifth compartment of Table 4, for a fixed value of the proportion of infectious individuals in infectious classes, $\alpha_j, j = 1, 2, 3, 4, 6$, and varying values of α_5, we have that individuals below a year have the highest tendency of spreading the disease, while individuals in age group 1–4 years have the least tendency of spreading the disease over time. We observed that as α_5 increases, $R_i^0, i = 1, 2, \ldots, 5$ increases alongside.

In the sixth compartment of Table 4, for a fixed value of the proportion of infectious individuals in infectious classes, $\alpha_j, j = 1, 2, 3, 4, 5$, and varying values of α_6, we have that individuals below a year have the highest tendency of spreading the disease, while individuals in age group 1–4 years have the least and constant tendency of spreading the disease over time. We observed that as α_6 increases, R_i^1 increases sharply while $R_i^0, i = 3, 4, 5$ increase slightly, and R_i^2 remains constant over time.

Table 5 shows the sensitivity analysis of effect of re-infection of chickenpox disease on the number of individuals in incubation stage, prodromal stage, active stage and treatment compartment for a period of 50 days. It is observed that as the rate at which re-infected individuals move to incubation stage, ϕ increases, the number of individuals who

are in incubation, prodromal stages, and treatment compartment will increase, but decreases the number of individuals in the active stage. Hence, re-infection into the incubation stage has the tendency of reducing the number of individuals in the active stage, and increasing the number of individuals in the incubation stage, prodromal stage as well as the number of individuals in the treatment compartment over time.

Also observed in Table 5 is that as the rate at which re-infected individuals move to prodromal stage, ϵ increases, the number of individuals who are in prodromal stage and treatment compartment will increase, while the number of individuals in incubation stage and active stage decreases. Thus, re-infection into the prodromal stage has the tendency of reducing the number of individuals in incubation and active stages of chickenpox, and increasing the number of individuals in the prodromal stage and treatment compartment.

Again, it is observed in Table 5 that as the rate at which re-infected individuals move to the active stage, σ increases, the number of individuals who are in active stage and treatment compartment increases, while the number of individuals in incubation stage and prodromal stage decreases. We therefore conclude that re-infection into the active stage of chickenpox infection has the tendency of reducing the number of individuals in the incubation and prodromal stages, and increasing the number of individuals in active stage and treatment compartment of chickenpox infection.

It is observed in Table 5 that as the rate at which re-infected individuals move to the treatment compartment, ϑ increases, the number of individuals who are in incubation stage, prodromal stage, and active stage decreases, whereas the number of individuals in the treatment compartment increases. Hence, re-infection into the treatment compartment has the tendency of reducing the number of individuals in incubation, prodromal, and active stages of chickenpox infection, and increasing the number of individuals in the treatment compartment over time.

5 Conclusion

This paper constructed a class of multi-group chickenpox epidemic models that put into consideration the effectiveness of vaccination in an age-classified subpopulation. The basic reproduction numbers of the subpopulation are obtained. Also determined are the disease-free and endemic equilibrium states. We found that vaccination rate plays a prominent role in reducing the spread of chickenpox disease in the population. We found that the basic reproduction number is higher in the age group below 1 year and lower in the age group 1–4 years. It was also found that the rate at which chickenpox moves from the incubation stage to the prodromal stage will decrease the spread of chickenpox in the population over time. Also found is that the rate at which chickenpox moves from the prodromal stage to the active stage will trigger the spread of chickenpox in the population over time. We found that the rate at which chickenpox moves from the exposed class to the incubation stage will increase the spread of chickenpox in the population over time. It was further found that the rate at which quarantined individuals move to the incubation stage will increase the spread of chickenpox in the population over time. Again, we found in this paper that vaccination goes a long way in reducing the spread of chickenpox in the population. Also, we found that chickenpox re-infections have little or no effect on the chickenpox reproduction number; that is, chickenpox re-infection does not contribute to the spread of the disease, but re-infection into the:

(i) incubation stage has the tendency of reducing the number of individuals in the active stage, and increasing the number of individuals in the incubation stage, prodromal stage, as well as the number of individuals in the treatment compartment over time.

(ii) prodromal stage has the tendency of reducing the number of individuals in incubation and active stages of chickenpox, and increasing the number of individuals in the prodromal stage and treatment compartment.

(iii) active stage of chickenpox infection has the tendency of reducing the number of individuals in the incubation and prodromal stages, and increasing the number of individuals in the active stage and treatment compartment of chickenpox infection.

(iv) treatment compartment has the tendency of reducing the number of individuals in incubation, prodromal, and active stages of chickenpox infection, and increasing the number of individuals in the treatment compartment over time.

Notation

Parameter	=	Description
Λ_i	=	influx of people into age group i
κ_i	=	the fraction of the influx of immune people into age group i
θ_i	=	the fraction of the susceptible persons in age group i who are vaccinated
ϵ_i	=	the rate at which recovered individuals in age group i who are re-infected of chickenpox disease and are moved to prodromal stage
σ_i	=	the rate at which recovered individuals in age group i who are re-infected of chickenpox disease and are moved to active stage
ϕ_i	=	the rate at which recovered individuals in age group i who are re-infected of chickenpox disease and are moved to incubation stage
ϑ_i	=	fraction of the recovered individuals in age group i who are hospitalized of chickenpox disease
μ	=	the natural death rate of the population
β_{ij}	=	the transmission coefficient between compartments S_i and the subclasses of the infectious classes E_j, Q_j, I_{1j}, I_{2j}, I_{3j}, H_j
ρ_{i1}	=	the rate at which an exposed individual in age group i move to quarantined class
ρ_{i2}	=	the rate at which a quarantined individual in age group i move to incubation class
ρ_{i3}	=	the rate at which a quarantined individual in age group i move to hospitalization or treatment class
γ_i	=	the rate at which an exposed individual in age group i move to incubation class
$\beta_{ij}\alpha_{j1}$	=	the transmission rates for compartment I_{j1}, $0 \leq \alpha_{j1} \leq 1$, where α_{j1} is the proportion of infectious individuals in incubation class
$\beta_{ij}\alpha_{j2}$	=	the transmission rates for compartment I_{j2}, $0 \leq \alpha_{j2} \leq 1$, where α_{j2} is the proportion of infectious individuals in prodromal class
$\beta_{ij}\alpha_{j3}$	=	the transmission rates for compartment I_{j3}, $0 \leq \alpha_{j3} \leq 1$, where α_{j3} is the proportion of infectious individuals in active class
$\beta_{ij}\alpha_{j4}$	=	the transmission rates for compartment H_j, $0 \leq \alpha_{j4} < 1$, where α_{j4} is the proportion of infectious individuals in treatment class
$\beta_{ij}\alpha_{j5}$	=	the transmission rates for compartment E_j, $0 \leq \alpha_{j5} < 1$, where α_{j5} is the proportion of infectious individuals in exposed class
$\beta_{ij}\alpha_{j6}$	=	the transmission rates for compartment Q_j, $0 \leq \alpha_{j6} < 1$, where α_{j6} is the proportion of infectious individuals in quarantined class
μ_{ik}	=	the disease-induced relative death rate for age group i in infectious compartment (i.e., I_{ik}), $k = 1, 2, 3$
ω_{ik}	=	the relative rate of disease progression for a person in age group i and infection compartment $k = 1, 2, 3$
ν_{i1}	=	the relative rate of progression of a person in age group i from compartment I_{i2} to compartment H_i
ν_{i2}	=	the relative rate of progression of a person in age group i from compartment I_{i3} to compartment H_i
ν_{i3}	=	the relative rate of progression of a person in age group i from compartment H_i to compartment R_i
λ_i	=	$\sum_{j=1}^{n} \frac{\beta_{ij}}{N} \left(\sum_{k=1}^{3} \alpha_{jk} I_{jk} + \alpha_{j4} H_j + \alpha_{j5} E_j + \alpha_{j6} Q_j \right), i = 1, 2, \ldots, n$

Data Availability Statement

Data will be made available on request.

Funding

This work was supported without any funding.

Conflicts of Interest

The authors declare no conflicts of interest.

AI Use Statement

The authors declare that no generative AI was used in the preparation of this manuscript.

Ethical Approval and Consent to Participate

Not applicable.

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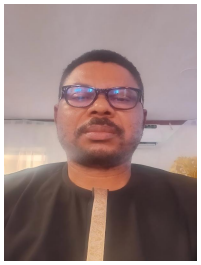

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