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RESEARCH ARTICLE

Dynamic Behavior of a Population Model Based on Second-order Fuzzy Difference Equation

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Abstract

This article examines the dynamic behavior of a second-order fuzzy difference equation that models the quantitative changes in a specific biological population:

$$E_{n+1} = \frac{S}{C + E_n + E_{n-1}},\ n \in \mathbb{Z}\ and\ n \geq 0,$$

Here, parameter S represents the carrying capacity of the environment, while C signifies the minimum resources required for population survival. The initial values E_0, E_{-1}, and parameters S , C are all positive fuzzy numbers. By employing the generalized division (g-division) with respect to fuzzy numbers, we establish the existence, uniqueness, persistence, and boundedness of positive fuzzy solutions to the equation under specified conditions. Furthermore, we derive the local and global asymptotic stability of these fuzzy solutions. Finally, two examples are provided to substantiate the conclusions drawn.

Keywords: fuzzy difference equation, persistence and boundedness, local and global behavior, g-division.

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1 Introduction

In the twenty-first century, substantial advancements have been achieved in the realm of discrete dynamical systems and difference equation (DE) theory, both pivotal constituents of mathematics. Discrete dynamical systems find extensive application across diverse domains, while difference equation theory assumes a vital role in applied analysis. High-order linear difference equations assume paramount significance in practical applications, as they can model various phenomena in numerous disciplines, including biology, ecology, engineering, economics, genetics, and resource management (see [1–6]). Further exploration of the features exhibited by the solutions of higher-order rational DE is a fascinating and promising area of research [7]. This has led numerous scholars to devote themselves to the study of rational DE, especially attracted by the states of solutions of such equations. When confronted with numerous unresolved real-world challenges, traditional difference equation models frequently encounter difficulties in managing complex uncertainties and ambiguities. To address these challenges, we have incorporated fuzzy set theory, which provides a robust framework for addressing uncertainty and accommodating vague results within mathematical models. Within the realm of fuzzy difference equations (FDE), initial values

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and parameters are commonly denoted as fuzzy numbers (FN). When examining the dynamic behavior of FDE, results can be derived by analyzing the corresponding family of difference equations, and fuzzy modeling techniques can be employed to generalize these findings to fuzzy environments. The essence of this approach resides in the incorporation of fuzzy numbers, which can more effectively depict the uncertainty inherent in real-world systems. Concurrently, through fuzzy modeling techniques, we can generalize traditional deterministic models to fuzzy models, thereby more comprehensively characterizing the dynamic behavior of intricate systems [8]. This approach offers substantial advantages in addressing ambiguity and uncertainty, particularly in scenarios where conventional methods prove ineffective (see [5, 10, 11]). The application of fuzzy methods to decision-making problems under uncertainty, such as facility location optimization with fuzzy demands, further demonstrates the broad utility of fuzzy mathematical frameworks (see [18, 19]).

In 1996, Deeba et al. [9] investigated first-order FDE in the following form

$$w_{n+1} = Qw_n + P,\ n \in \mathbb{Z}\ and\ n \geq 0,$$

where w_n represents a fuzzy sequence, and $w_0, P, Q \in \mathbb{R}_F^+$ (positive fuzzy numbers). This research was applied to the field of genetics.

In 2002, Papaschinopoulos and Papadopoulos [12] employed the Zadeh extension principle to investigate fuzzy differencing, represented by the equation

$$D_{n+1} = K + \frac{L}{D_n},\ n \in \mathbb{Z}\ and\ n \geq 0,$$

where D_0, K, and L denote positive fuzzy numbers (PFN), and D_n represents a fuzzy sequence. In the same year, they [13] conducted an analysis of the global asymptotic behavior of high-order fuzzy differences, governed by the equation

$$D_{n+1} = \varUpsilon + \frac{D_n}{D_{n-m}},$$

where $n, m \in (0, 1, \cdots)$, and $\varUpsilon$ represents a fuzzy number.

In 2006, Stefanidou and Papaschinopoulos [14] performed a periodic analysis on the positive solutions of a nonlinear fuzzy maximum DE expressed as

$$z_{n+1} = \max\left\{\frac{P_0}{z_{n-i}}, \frac{P_1}{z_{n-j}}\right\},$$

where $n, j, i \in \mathbb{N}$, and P_0, P_1 represent fuzzy numbers. The initial values z_m, for $m \in \{-s, -s+1, \cdot\cdot\cdot, -1\}$ with $s = \max\{i, j\}$, are positive FN.

In 2017, Wang et al. [15] conducted a series of analyses on the positive solutions of fifth-order FDE

$$z_{n+1} = \frac{Pz_{n-1}z_{n-2}}{\varUpsilon + Qz_{n-3} + Lz_{n-4}},\ n \in \mathbb{Z}\ and\ n \geq 0.$$

Here, z_n represents a fuzzy sequence with initial conditions z_i being PFN where $i = -4, -3, -2, -1, 0$, and the parameters $P, Q, L, \varUpsilon \in \mathbb{R}_F^+$.

In 2015, Zhang et al. [16] employed g-division to investigate the positive solutions of third-order rational FDE as

$$z_{n+1} = Q + \frac{z_n}{z_{n-1}z_{n-2}},\ n \in \mathbb{Z}\ and\ n \geq 0,$$

where Q and $z_k (k = -2, -1, 0)$ are PFN. In 2023, Zhang et al. [4] again utilized g-division to explore the dynamic behavior of a second-order fuzzy discrete population model defined by

$$D_{n+1} = \frac{PD_{n-1}}{1 + D_{n-1} + QD_{n-2}},\ n \in \mathbb{Z}\ and\ n \geq 0.$$

In this context, both P and Q are characterized as positive fuzzy numbers, whereas D_t signifies a PFN that corresponds to the population size at the specific observation time point t .

Under the framework of fuzzy mathematics, modeling dynamic systems frequently encounters the dual challenges of information uncertainty and linguistic ambiguity. The study of rational difference equations and their dynamical properties provides the deterministic foundation upon which fuzzy generalizations are built [17]. Consider the second-order fuzzy DE

$$E_{n+1} = \frac{S}{C + E_n + E_{n-1}},\ n \in \mathbb{Z}\ and\ n \geq 0. \tag{1}$$

In practical scenarios, relying solely on precise parameters and initial conditions is often impractical. Parameters such as S (environmental carrying capacity) and C (minimum required resources for survival) are difficult to quantify precisely and often necessitate the use of fuzzy numbers or membership functions to capture their inherent uncertainty.

This article investigates the application of extending the equilibrium point formula to fuzzy contexts. By introducing fuzzy difference equations and g-division, classical equilibrium points are transformed into fuzzy equilibrium sets, thereby characterizing the "stable possibility distribution" of the system under fuzzy parameters. This approach not only provides a quantitative tool for fuzzy decision-making in fields such as ecological regulation and economic forecasting but also unveils the dynamic trade-off between parameter sensitivity and robustness through η-cut set analysis, thus opening a new avenue for addressing incomplete information problems in complex systems (see [5, 6]).

2 Some Definitions

This section presents definitions that will be utilized in subsequent chapters.

Definition 1. [1] *A function $X : \mathbb{R} \to \mathbb{I}$ is termed a FN if it adheres to the following four conditions (i)-(iv):*

(i) *Normality: There $\exists s \in \mathbb{R}$ such that $X(s) = 1$.*

(ii) *Fuzzy Convexity: Specifically, for every $\lambda \in [0,1]$ and any $s_1, s_2 \in \mathbb{R}$, the inequality*

$$X(\lambda s_1 + (1-\lambda)s_2) \geq \min\{X(s_1), X(s_2)\}$$

holds true.

(iii) *Upper Semicontinuity: X with definition domain $\mathbb{R}$ is upper semicontinuous.*

(iv) *Compact Support: We write supp$X = \overline{\bigcup_{\eta\in(0,1]}[X]_\eta} = \overline{\{s \in \mathbb{R} : X(s) > 0\}}$ as the support set of X and it is compact.*

The η-cut of the FN X is defined as $[X]_\eta = \{s \in \mathbb{R} : X(s) \geq \eta\}$ for $\eta \in (0,1]$. The fuzzy number X is deemed positive (or negative) if its support supp X is a subset of $(0, +\infty)$ (or supp $X \subset (-\infty, 0)$).

Given two fuzzy numbers X and U with their respective η-cut represented as $[X]_\eta = [X_{l,\eta}, X_{r,\eta}]$ and $[U]_\eta = [U_{l,\eta}, U_{r,\eta}]$ for $\eta \in (0,1]$. The addition and number multiplication operations defined by the η-cut set over FN are denoted

$$[X+U]_\eta = [X_{l,\eta} + U_{l,\eta}, X_{r,\eta} + U_{r,\eta}], \tag{2}$$

$$[kX]_\eta = [kX_{l,\eta}, kX_{r,\eta}], k > 0. \tag{3}$$

Define $\mathbb{R}_F = \left\{\theta \middle| [\theta]_\eta = [\theta_l(\eta), \theta_r(\eta)] \ for\ all\ \eta \in [0,1]\right\}$ as the space of all FN.

The image (see Figure 1) below shows the trapezoidal fuzzy number along with its η-cut set as well as the support set, etc [20].

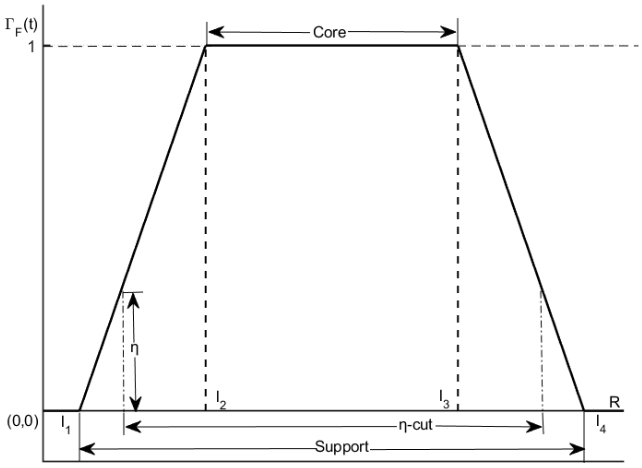

Figure 1. Trapezoidal FN display.

Definition 2. [21] *Let l_1, l_2, l_3 belong to $\mathbb{R}$ and satisfy $l_1 < l_2 < l_3$, then we say that the ordered ternary array $F = (l_1, l_2, l_3)$ is a triangular FN over $\mathbb{R}$. Its affiliation function $\Gamma_F(t)$ is defined by*

$$\Gamma_F(t) = \begin{cases} 0 & for\ t \in (-\infty, l_1), \\ \frac{t-l_1}{l_2-l_1} & for\ t \in [l_1, l_2), \\ 1 & for\ t = l_2, \\ \frac{l_3-t}{l_3-l_2} & for\ t \in (l_2, l_3], \\ 0 & for\ t \in (l_3, +\infty). \end{cases}$$

Definition 3. [1] *Suppose D is a metric which denotes the distance between two FN X and U. It is easy to see that $(\mathbb{R}_F, D)$ is a complete metric space. Its concrete form is*

$$D(X,U) = \underset{\eta\in[0,1]}{sup} \max\{|X_{l,\eta} - U_{l,\eta}|, |X_{r,\eta} - U_{r,\eta}|\}.$$

Definition 4. [22] *Let $S, U \in \mathbb{R}_F$ have η-cuts, $[S]_\eta = [S_{l,\eta}, S_{r,\eta}]$, $[U]_\eta = [U_{l,\eta}, U_{r,\eta}]$, with $0 \notin [U]_\eta$ for $\eta \in [0,1]$. The g-division$(\div g)$ is denoted by $H = S \div_g U$ having η-cut, $[H]_\eta = [H_{l,\eta}, H_{r,\eta}]$, where $[S]_\eta^{-1} = [1/S_{r,\eta}, 1/S_{l,\eta}]$*

$$[H]_\eta = [S]_\eta \div_g [U]_\eta \Longleftrightarrow \begin{cases} (i)\ [S]_\eta = [U]_\eta [H]_\eta, \\ or \\ (ii)\ [U]_\eta = [S]_\eta [H]_\eta^{-1}. \end{cases}$$

Provided that H is FN, $H_{l,\eta}$ is nondecreasing, $H_{r,\eta}$ is nonincreasing, respectively, with $H_{l,1} \leq H_{r,1}$.

Comparing the g-division method utilized in this study with the Zadeh [22] extension principle, it is clear that g-division has significant advantages. Specifically, it can effectively reduce the support set interval of the fuzzy positive solution. As a result, the uncertainty inherent in fuzzy solutions (FS) is greatly alleviated. The support set of a fuzzy solution represents the range of values whose membership degree is greater than zero, and the wider the support set interval, the greater the uncertainty of the solution. The Zadeh extension principle is a fundamental and widely used method in fuzzy theory for extending a parsimonious function to the fuzzy case, but it may sometimes lead to relatively large support set intervals of a fuzzy solution, especially in complex cases involving positive solutions. This may pose a challenge in situations where accurate and reliable fuzzy solutions are required. g-division method takes a unique approach to the partitioning operation within the fuzzy domain, which enables better restriction of the support set intervals (see [23, 24]).

Remark 1. *As stated in* [22], *if the PFN* $H \div_g P = W \in \mathbb{R}_F$ *exists, exactly one of the following two scenarios will occur.*

Case I if $H_{l,\eta}P_{r,\eta} \leq H_{r,\eta}P_{l,\eta}, \forall \eta \in [0,1]$, *then* $W_{l,\eta} = \frac{H_{l,\eta}}{P_{l,\eta}}, W_{r,\eta} = \frac{H_{r,\eta}}{P_{r,\eta}}$.

Case II if $H_{l,\eta}P_{r,\eta} > H_{r,\eta}P_{l,\eta}, \forall \eta \in [0,1]$, *then* $W_{l,\eta} = \frac{H_{r,\eta}}{P_{r,\eta}}, W_{r,\eta} = \frac{H_{l,\eta}}{P_{l,\eta}}$.

Definition 5. *Let* E_n *be a sequence of PFN, if there* $\exists \Omega_1 > 0$, *resp.* $\exists \Omega_2 > 0$, *satisfying*

$$\begin{gathered} supp\ E_n \subset [\Omega_1, \infty), n \in \mathbb{N}_0, \\ resp.\ supp\ E_n \subset (0, \Omega_2], n \in \mathbb{N}_0, \end{gathered}$$

then E_n *is persistent, resp. bounded(see* [12, 13]*).*

Further, if there exist Ω_1 *and* Ω_2 *satisfying* $0 < \Omega_1 < \Omega_2$ *such that*

$$supp\ E_n \subset [\Omega_1, \Omega_2], n \in \mathbb{N}_0,$$

then E_n *is said to be boundedly persistent.*

Conversely, if none of the above holds then the sequence E_n *is unbounded.*

Lemma 1. [25] *Let* $\tau \in \mathbb{R}_F^+$ *with* $[\tau]_\eta = [\tau_l(\eta), \tau_r(\eta)]$ *for* $\eta \in (0,1]$. *The functions* $\tau_l(\eta)$ *and* $\tau_r(\eta)$ *can defined on* $(0,1]$ *can satisfy the conditions* $\tau_l(1) \leq \tau_r(1)$ *and* $\tau_r(\eta)$ *is nonincreasing and* $\tau_l(\eta)$ *is nondecreasing, but both are left continuous.*

3 Main Results

This subsection consists of two parts, one is to analyze the existence and uniqueness of the FS of the Eq.(1) and give the proof procedure. The second is to discuss the global behavior of the Eq.(1) for each of the two cases after using g-division.

Lemma 2. [1] *Let* $T_i \in \mathbb{R}_F^+, i = 1,2,3, g : (0,+\infty)^4 \to (0,+\infty)$ *be continuous, then*

$$[g(T_1, T_2, T_3)]_\eta = g\left([T_1]_\eta, [T_2]_\eta, [T_3]_\eta\right), 0 < \eta \leq 1.$$

Theorem 1. *Consider Eq.*(1), *which has a unique positive fuzzy solution* E_n *if the unknowns* S, C *and* E_i $(i = -1, 0)$ *are positive FN in* $\mathbb{R}_F^+$.

Proof. Assume a sequence of FN, denoted as $\{E_n\}$, adheres to the condition expressed by Eq.(1), given the initial conditions $E_i \in \mathbb{R}_F^+$ for $i = -1, 0$. Furthermore, let η be a value within the interval $(0,1]$.

$$\begin{aligned} [E_n]_\eta = [E_{n,l,\eta}, E_{n,r,\eta}], [S_n]_\eta = [S_{n,l,\eta}, S_{n,r,\eta}], \\ [C_n]_\eta = [C_{n,l,\eta}, C_{n,r,\eta}], n = 0,1,2,\cdots. \end{aligned} \tag{4}$$

Based on (4) and Lemma 2, when considering the η-cuts, it can be deduced from Eq.(1) that

$$\begin{aligned} [E_{n+1}]_\eta &= \left[\frac{S}{C + E_n + E_{n-1}}\right]_\eta \\ &= \frac{[S]_\eta}{[C]_\eta + [E_n]_\eta + [E_{n-1}]_\eta} \\ &= \frac{[S_{l,\eta}, S_{r,\eta}]}{[C_{l,\eta} + E_{n,l,\eta} + E_{n-1,l,\eta}, C_{r,\eta} + E_{n,r,\eta} + E_{n-1,r,\eta}]}. \end{aligned} \tag{5}$$

According to Remark 1, we have two cases

Case I

$$[E_{n+1}]_\eta = \left[\frac{S_{l,\eta}}{C_{l,\eta} + E_{n,l,\eta} + E_{n-1,l,\eta}}, \frac{S_{r,\eta}}{C_{r,\eta} + E_{n,r,\eta} + E_{n-1,r,\eta}}\right]. \tag{6}$$

Case II

$$[E_{n+1}]_\eta = \left[\frac{S_{r,\eta}}{C_{r,\eta} + E_{n,r,\eta} + E_{n-1,r,\eta}}, \frac{S_{l,\eta}}{C_{l,\eta} + E_{n,l,\eta} + E_{n-1,l,\eta}}\right]. \tag{7}$$

If Case I occurs, for $n \in \mathbb{N}$, $0 < \eta \leq 1$, based on equation (6), we can derive the following relationships

$$E_{n+1,l,\eta} = \frac{S_{l,\eta}}{C_{l,\eta} + E_{n,l,\eta} + E_{n-1,l,\eta}},$$
$$E_{n+1,r,\eta} = \frac{S_{r,\eta}}{C_{r,\eta} + E_{n,r,\eta} + E_{n-1,r,\eta}}.$$

Then, for every set of initial values represented as $(E_{j,l,\eta}, E_{j,r,\eta})$, where j takes the values -1 and 0, and η belongs to the interval $(0,1]$, there exists a unique solution denoted as $E_{n,\eta}$.

All that follows is to show that at an initial value of E_j for $j = -1, 0$, the solution $E_{n,\eta}$ of the above-mentioned Eq.(1), where η belongs to the interval $(0,1]$, satisfies

$$[E_n]_\eta = [E_{n,l,\eta}, E_{n,r,\eta}]\,,\ n \in \mathbb{N}\ and\ 0 < \eta \leq 1. \quad (8)$$

From $E_j \in \mathbb{R}_F^+, j = -1, 0$, based on the literature [25], one obtains

$$0 < E_{j,l,\eta_1} \leq E_{j,l,\eta_2} \leq E_{j,r,\eta_2} \leq E_{j,r,\eta_1}, j = -1, 0, \quad (9)$$

where $\eta_1, \eta_2 \in (0,1]$, $\eta_1 \leq \eta_2$.

Now suppose that for $n \in \mathbb{Z},\ n \geq 0$,

$$E_{n,l,\eta_1} \leq E_{n,l,\eta_2} \leq E_{n,r,\eta_2} \leq E_{n,r,\eta_1}. \quad (10)$$

Next we use mathematical induction. From the fact that (9) and (10) are right, we assume that $n = m(m \in \mathbb{Z},\ m \geq 0)$. According to (6) there is

$$\begin{aligned} E_{m+1,l,\eta_1} &= \frac{S_{l,\eta_1}}{C_{l,\eta_1} + E_{m,l,\eta_1} + E_{m-1,l,\eta_1}} \\ &\leq \frac{S_{l,\eta_2}}{C_{l,\eta_2} + E_{m,l,\eta_2} + E_{m-1,l,\eta_2}} \\ &= E_{m+1,l,\eta_2} \\ &= \frac{S_{l,\eta_2}}{C_{l,\eta_2} + E_{m,l,\eta_2} + E_{m-1,l,\eta_2}} \\ &\leq \frac{S_{r,\eta_2}}{C_{r,\eta_2} + E_{m,r,\eta_2} + E_{m-1,r,\eta_2}} \\ &= E_{m+1,r,\eta_2} \\ &= \frac{S_{r,\eta_2}}{C_{r,\eta_2} + E_{m,r,\eta_2} + E_{m-1,r,\eta_2}} \\ &\leq \frac{S_{r,\eta_1}}{C_{r,\eta_1} + E_{m,r,\eta_1} + E_{m-1,r,\eta_1}} \\ &= E_{m+1,r,\eta_1} \end{aligned}$$

so we can get that (9) holds.

From (6), we have

$$\begin{aligned} E_{1,l,\eta} &= \frac{S_{l,\eta}}{C_{l,\eta} + E_{0,l,\eta} + E_{-1,l,\eta}}, \\ E_{1,r,\eta} &= \frac{S_{r,\eta}}{C_{r,\eta} + E_{0,r,\eta} + E_{-1,r,\eta}}, \eta \in (0,1]\,. \end{aligned} \quad (11)$$

Since $E_t \in \mathbb{R}_F^+$, $t = -1, 0$, and both S and C are elements of $\mathbb{R}_F^+$, we can deduce that both $E_{1,l,\eta}$ and $E_{1,r,\eta}$ exhibit left continuous. Therefore, it is natural to deduce that $E_{n,l,\eta}$ and $E_{n,r,\eta}$ is left continuous using mathematical induction.

In the following we should prove that $\cup_{\eta\in(0,1]}[E_{n,l,\eta}, E_{n,r,\eta}]$ is bounded, since in combination with the previous proof it follows that $suppE_n$ is tight under $\cup_{\eta\in(0,1]}[E_{n,l,\eta}, E_{n,r,\eta}]$ is bounded.

Given that $S, C \in \mathbb{R}_F^+$ and $E_t \in \mathbb{R}_F^+$ $(t = -1, 0)$, for every $\eta \in (0,1]$, there exist +ve real numbers K_S, D_S, K_C, D_C as well as $P_t, Q_t (t = -1, 0)$ such that the following inclusions hold

$$\begin{aligned} [S_{l,\eta}, S_{r,\eta}] \subset [K_S, D_S]\,, [C_{l,\eta}, C_{r,\eta}] \\ \subset [K_C, D_C]\,, [E_{t,l,\eta}, E_{t,r,\eta}] \\ \subset [P_t, Q_t]\,. \end{aligned} \quad (12)$$

From (11) and (12), we can derive

$$[E_{1,l,\eta}, E_{1,r,\eta}] \subset \left[\frac{K_S}{D_C + Q_0 + Q_{-1}}, \frac{D_S}{K_C + P_0 + P_{-1}}\right], \quad \eta \in (0,1]\,.$$

Further derivation gives us obviously

$$\bigcup_{\eta\in(0,1]} [E_{1,l,\eta}, E_{1,r,\eta}] \subset \left[\frac{K_S}{D_C + Q_0 + Q_{-1}}, \frac{D_S}{K_C + P_0 + P_{-1}}\right]. \quad (13)$$

Then $\overline{\bigcup_{\eta\in(0,1]}[E_{1,l,\eta}, E_{1,r,\eta}]} \subset (0, \infty)$ is compact.

Applying induction, we can easily get $\overline{\bigcup_{\eta\in(0,1]}[E_{n,l,\eta}, E_{n,r,\eta}]}$ is compact,

$$\overline{\bigcup_{\eta\in(0,1]} [E_{n,l,\eta}, E_{n,r,\eta}]} \subset (0, \infty)\,,\ n \in \mathbb{Z},\ n \geq 0. \quad (14)$$

Since (10) and (14) hold, $E_{n,l,\eta}$ and $E_{n,r,\eta}$ are left continuous. Thus it follows that the PFN E_n satisfying (8) is determined by $[E_{n,l,\eta}, E_{n,r,\eta}]$.

Next, we will demonstrate that E_n is a positive fuzzy solution of the Eq.(1) with the initial value set to $E_t \in \mathbb{R}_F^+, t = -1, 0$, where $\eta \in (0,1]$,

$$[E_{n+1}]_\eta = [E_{n+1,l,\eta}, E_{n+1,r,\eta}] = \left[\frac{S}{C + E_n + E_{n-1}}\right]_\eta.$$

Hence, with the initial value of $E_t, t=-1,0$, we can conclude that E_n is a positive fuzzy solution of the Eq.(1).

Suppose there is another positive fuzzy solution to the Eq.(1) under the initial condition of $E_t, t=-1,0$, denoted as $\bar{E}_n$. By a simple derivation, we can derive

$$\left[\bar{E}_n\right]_\eta = \left[E_{n,l,\eta}, E_{n,r,\eta}\right], \; n \geq 0 \; and \; 0 < \eta \leq 1. \quad (15)$$

According to (8) and (15), it can be concluded that $[E_n]_\eta = \left[\bar{E}_n\right]_\eta$, thus $E_n = \bar{E}_n$.

If Case II occurs, for $n \in \mathbb{N}$, $0 < \eta \leq 1$, based on equation (7), we can derive the following relationships

$$\begin{aligned} E_{n,l,\eta} &= \frac{S_{r,\eta}}{C_{r,\eta} + E_{n,r,\eta} + E_{n-1,r,\eta}}, \\ E_{n,r,\eta} &= \frac{S_{l,\eta}}{C_{l,\eta} + E_{n,l,\eta} + E_{n-1,l,\eta}}. \end{aligned} \quad (16)$$

The proof of Case II is analogous to that of Case I, with the reasoning and steps closely mirroring each other. To avoid redundancy and given the space constraints, the detailed demonstration for Case II is intentionally omitted. We hereby complete the proof of Theorem 1.

□

Case I

Lemma 3. *Consider the DE*

$$u_{n+1} = \frac{p}{q + u_n + u_{n-1}}, \; n \in \mathbb{N}, \quad (17)$$

where initial value $u_i > 0, i = 0, -1$, *if*

$$q > 0, p > 0, \quad (18)$$

then (i) *and* (ii) *below holds.*

(i)*Every positive solution* u_n *derived from* (17) *satisfies*

$$\frac{qp}{q^2 + 2p} < u_n < \frac{p}{q}. \quad (19)$$

(ii)*The* (17) *is globally asymptotically stable with its equilibrium point at* $\bar{u} = \frac{\sqrt{q^2+8p}-q}{4}$.

Proof. (i)From (17), it is easy to see that for $n > 0$, the result is $u_n < \frac{p}{q}$. One has for $n > 0$

$$u_n > \frac{p}{q + \frac{p}{q} + \frac{p}{q}} = \frac{qp}{q^2 + 2p}.$$

(ii)From (17), it can be readily deduced that the equation sole positive equilibrium point is

$$\bar{u} = \frac{\sqrt{q^2 + 8p} - q}{4}.$$

The linear equation given by (17) evaluated at $\bar{u}$ is expressed as

$$u_{n+1} - \omega_1 u_n - \omega_2 u_{n-1} = 0. \quad (20)$$

where

$$\omega_1 = \frac{-2p}{q^2 + 4p + q\sqrt{q^2 + 8p}},$$

$$\omega_2 = \frac{-2p}{q^2 + 4p + q\sqrt{q^2 + 8p}}.$$

From reference [26] and since $q > 0, p > 0$, it is to see that $|\omega_1| < 1 - \omega_2 < 2$. Thus, it can be concluded that $\bar{u}$ is locally asymptotically stable.

Set

$$\lim_{n\to\infty} sup \; u_n = \Lambda, \; \lim_{n\to\infty} inf \; u_n = \lambda, \quad (21)$$

where $\lambda, \Lambda \in (0, \infty)$. It follows from (17) that

$$\Lambda \leq \frac{p}{q + 2\lambda}, \lambda \geq \frac{p}{q + 2\Lambda}.$$

From which,

$$q\left(\Lambda - \lambda\right) \leq 0.$$

Since $\lambda \leq \Lambda$, $q > 0$, $p > 0$, so $\Lambda = \lambda$. Combined with (21) we can infer $\lim_{n\to\infty} u_n = \bar{u}$. Finally, from the above proof process we can infer that $\bar{u}$ is globally asymptotically stable. The above is the entire proof of Lemma 2, which is now complete.

□

Theorem 2. *Consider Eq.*(1), *where* $S, C, E_i \in \mathbb{R}_F^+$, $i = -1, 0$. *Suppose* K_S, D_S, K_C, *and* D_C *all belong to* $\mathbb{R}$, *which satisfies*

$$0 < K_S \leq S_{l,\eta} \leq D_S, 0 < K_C \leq C_{l,\eta} \leq D_C, \eta \in (0, 1], \quad (22)$$

and for $0 < \eta \leq 1, n \in \mathbb{N}$,

$$\begin{aligned} &S_{l,\eta}\left(C_{r,\eta} + E_{n,r,\eta} + E_{n-1,r,\eta}\right) \\ &\quad \leq S_{r,\eta}\left(C_{l,\eta} + E_{n,l,\eta} + E_{n-1,l,\eta}\right). \end{aligned} \quad (23)$$

then (i)-(iii) *below holds.*

(i) *Every positive FS* E_n *to Eq.*(1) *is both bounded and persistent.*

(ii) There exists a unique positive equilibrium $\bar{E}$ for Eq.(1).

(iii) For any positive FS E_n of Eq.(1), it converges to the $\bar{E}$ as $n \to \infty$. Furthermore, this $\bar{E}$ is globally asymptotically stable.

Proof. (i)Assume that E_n represents a positive FS of Eq.(1). Given that (12) and (23) are valid, by invoking Lemma 2 and referring to (6), we can deduce the following

$$\frac{S_{l,\eta}C_{l,\eta}}{C_{l,\eta}^2 + 2S_{l,\eta}} \leq E_{n,l,\eta} \leq \frac{S_{l,\eta}}{C_{l,\eta}}, \tag{24}$$

$$\frac{S_{r,\eta}C_{r,\eta}}{C_{r,\eta}^2 + 2S_{r,\eta}} \leq E_{n,r,\eta} \leq \frac{S_{r,\eta}}{C_{r,\eta}}. \tag{25}$$

From (12), (24), (25), and (10), we can deduce that, for $n = 1, 2, \cdots$, there is

$$[E_{n,l,\eta}, E_{n,r,\eta}] \subset [P, Q], \tag{26}$$

where

$$P = \frac{D_S K_C}{K_S D_C + 2D_S}, Q = \frac{D_S}{K_C}.$$

From (26) we can easily conclude that for $n \in \mathbb{N}_0$,

$$\bigcup_{\eta\in(0,1]} [E_{n,l,\eta}, E_{n,r,\eta}] \subset [P, Q],$$

It can be further obtained that

$$\overline{\bigcup_{\eta\in(0,1]} [E_{n,l,\eta}, E_{n,r,\eta}]} \subset [P, Q].$$

In summary, the proof of part (i) is complete.

(ii) Let $(E_{l,\eta}, E_{r,\eta})$ represent the following system

$$\begin{cases} E_{l,\eta} = \frac{S_{l,\eta}}{C_{l,\eta}+2E_{l,\eta}}, \\ E_{r,\eta} = \frac{S_{r,\eta}}{C_{r,\eta}+2E_{r,\eta}}. \end{cases} \tag{27}$$

Consequently, $(E_{l,\eta}, E_{r,\eta})$ can be written

$$\begin{cases} E_{l,\eta} = \frac{1}{4}\left(\sqrt{C_{l,\eta}^2 + 8S_{l,\eta}} - C_{l,\eta}\right), \\ E_{r,\eta} = \frac{1}{4}\left(\sqrt{C_{r,\eta}^2 + 8S_{r,\eta}} - C_{r,\eta}\right). \end{cases} \tag{28}$$

Suppose that E_n is a positive fuzzy solution to Eq.(1) and that it satisfies (12). Then applying Lemma 3 we can easily get

$$\lim_{n\to\infty} E_{n,l,\eta} = E_{l,\eta}, \lim_{n\to\infty} E_{n,r,\eta} = E_{r,\eta}. \tag{29}$$

where one has to pay attention to (6).

From (26) and (29), we can deduce that, for $0 < \eta_1 \leq \eta_2 \leq 1$, there is

$$0 < E_{l,\eta_1} \leq E_{l,\eta_2} \leq E_{r,\eta_2} \leq E_{r,\eta_1}. \tag{30}$$

Since $S_{l,\eta}$, $S_{r,\eta}$, $C_{l,\eta}$, and $C_{r,\eta}$ are all left continuous, combining (28) infers that $E_{l,\eta}$ and $E_{r,\eta}$ is likewise left continuous. From (12) and (28) it follows that

$$\begin{cases} E_{l,\eta} \geq m = \frac{1}{4}\left(\sqrt{K_C^2 + 8K_S} - D_C\right), \\ 0 < E_{r,\eta} \leq M = \frac{1}{4}\left(\sqrt{D_C^2 + 8D_S} - K_C\right). \end{cases} \tag{31}$$

Therefore $[E_{l,\eta}, E_{r,\eta}] \subset [m, M]$, and so

$$\bigcup_{\eta\in(0,1]} [E_{l,\eta}, E_{r,\eta}] \subset [m, M], \tag{32}$$

Thus $\bigcup_{\eta\in(0,1]} [E_{l,\eta}, E_{r,\eta}]$ is compact,

$$\bigcup_{\eta\in(0,1]} [E_{l,\eta}, E_{r,\eta}] \subset (0, \infty). \tag{33}$$

Since $E_{l,\eta}, E_{r,\eta},\ 0 < \eta \leq 1$ ascertains a FN $\bar{E}$ such that

$$\bar{E} = \frac{S}{C + 2\bar{E}}, \left[\bar{E}\right]_\eta = [E_{l,\eta}, E_{r,\eta}], \eta \in (0, 1]. \tag{34}$$

Therefore $\bar{E}$ is a positive equilibrium of Eq.(1).

(iii)From Definition 3 and (29), we can conclude that

$$\lim_{n\to\infty} D\left(E_n, \bar{E}\right) = \lim_{n\to\infty} \sup_{\eta\in[0,1]} \{\max\{|E_{n,l,\eta} - E_{l,\eta}|, |E_{n,r,\eta} - E_{r,\eta}|\}\}. \tag{35}$$

That is, in the sense of D, each positive FS E_n of Eq.(1) converges to the unique equilibrium point $\bar{E}$ when $n \to \infty$.

Let any $\varepsilon > 0$ be assumed to have

$$0 < \delta < \min\left\{\varepsilon, \frac{1}{2}(K_C + 2m - 2M)\right\}, \tag{36}$$

where m, M comes from (31).

Further, assume that E_n is a positive fuzzy solution to Eq.(1) which satisfies $D\left(E_i, \bar{E}\right) \leq \delta < \varepsilon, i = -1, 0$. It follows that when $\eta \in (0, 1]$ we have

$$|E_{i,l,\eta} - E_{l,\eta}| \leq \delta, |E_{i,r,\eta} - E_{r,\eta}| \leq \delta. \tag{37}$$

According to (6), (12), (27) and (37), it can be obtained that

$$\begin{aligned} E_{1,l,\eta} - E_{l,\eta} &= \frac{S_{l,\eta}}{C_{l,\eta} + E_{0,l,\eta} + E_{-1,l,\eta}} - E_{l,\eta} \\ &\leq \frac{S_{l,\eta}}{C_{l,\eta} + 2E_{l,\eta} - 2\delta} - E_{l,\eta} \\ &= 2\delta \frac{E_{l,\eta}}{C_{l,\eta} + 2E_{l,\eta} - 2\delta} \\ &\leq 2\delta \frac{M}{K_C + 2m - 2\delta}, \end{aligned} \tag{38}$$

$$\begin{aligned} E_{1,r,\eta} - E_{r,\eta} &= \frac{S_{r,\eta}}{C_{r,\eta} + E_{0,r,\eta} + E_{-1,r,\eta}} - E_{r,\eta} \\ &\leq \frac{S_{r,\eta}}{C_{r,\eta} + 2E_{r,\eta} - 2\delta} - E_{r,\eta} \\ &= 2\delta \frac{E_{r,\eta}}{C_{r,\eta} + 2E_{r,\eta} - 2\delta} \\ &\leq 2\delta \frac{M}{K_C + 2m - 2\delta}. \end{aligned} \tag{39}$$

From (36), (38) and (39), one has that

$$|E_{1,l,\eta} - E_{l,\eta}| < \delta < \varepsilon, |E_{1,r,\eta} - E_{r,\eta}| < \delta < \varepsilon. \tag{40}$$

We can easily obtain, using mathematical induction, that when $n \geq 1, \eta \in (0,1]$, there are

$$|E_{n,l,\eta} - E_{l,\eta}| < \varepsilon, |E_{n,r,\eta} - E_{r,\eta}| < \varepsilon. \tag{41}$$

This yields $D(E_n, \bar{E}) < \varepsilon$. Combining (35) yields that the equilibrium point $\bar{E}$ is globally asymptotically stable. At this point the Theorem 2 is fully proved.

□

Case II

Lemma 4. *The DE system is analyzed for* $n \in \mathbb{N}$,

$$u_{n+1} = \frac{b}{a + v_n + v_{n-1}}, v_{n+1} = \frac{p}{q + u_n + u_{n-1}}, \tag{42}$$

where $p, b, q, a, u_i, v_i \in (0, +\infty), i = -1, 0$. *Then* (i) *and* (ii) *below holds.*

(i)*Every positive solution* (u_n, v_n) *of Eq.*(1) *is persistence and bounded.*

(ii)*The system has a unique positive equilibrium*

$$\begin{cases} \bar{u} = \dfrac{2b - 2p - aq}{4a} + \dfrac{\sqrt{(aq + 2p - 2b)^2 + 8abq}}{4a}, \\ \bar{v} = \dfrac{2p - 2b - aq}{4q} + \dfrac{\sqrt{(aq + 2b - 2p)^2 + 8apq}}{4q}. \end{cases} \tag{43}$$

Furthermore, if system (42) *satisfies the condition*

$$\begin{cases} 2ap > \left(\sqrt{2b} - 2a\right)\left(q\sqrt{2b} + 2b - 2p\right), \\ 2bq > \left(\sqrt{2p} - 2a\right)\left(a\sqrt{2p} + 2p - 2b\right). \end{cases}$$

It is globally asymptotically stable.

Proof. (i)According to (42), it can be easily obtained

$$u_n < \frac{b}{a}, v_n < \frac{p}{q}.$$

Which can be further obtained as

$$u_n > \frac{b}{a + 2\frac{p}{q}} = \frac{bq}{aq + 2p}, v_n > \frac{p}{q + 2\frac{b}{a}} = \frac{ap}{aq + 2b},$$

and so the positive solution (u_n, v_n) of (42) is persistence and bounded.

(ii)Its unique equilibrium point $(\bar{u}, \bar{v})$ is obtained from system (42) as in (43). From (42), it's linear equation at the equilibrium point is given by

$$\Psi_{n+1} = J\Psi_n. \tag{44}$$

The Ψ_n, J in (44) is denoted as

$$\Psi_n = \begin{pmatrix} u_n \\ u_{n-1} \\ v_n \\ v_{n-1} \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & 0 & \frac{-b}{(a+2\bar{v})^2} & \frac{-b}{(a+2\bar{v})^2} \\ 1 & 0 & 0 & 0 \\ \frac{-p}{(q+2\bar{u})^2} & \frac{-p}{(q+2\bar{u})^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\Omega = \operatorname{diag}(k_1, k_2, k_3, k_4)$ be a diagonal matrix, where $k_1 = k_3 = 1$ and $k_i = 1 - i\varepsilon\,(i = 2, 4)$. Additionally, the parameter ε satisfies

$$0 < \varepsilon < \min\left\{\frac{1}{4}\left(\frac{(a + 2\bar{v})^2 - 2b}{(a + 2\bar{v})^2 - b}\right), \frac{1}{2}\left(\frac{q + 2\bar{u})^2 - 2p}{(q + 2\bar{u})^2 - p}\right)\right\}. \tag{45}$$

Clearly, $\varOmega$ is reversible. Computing $\varOmega J\varOmega^{-1}$, one has

$$\varOmega J\varOmega^{-1}=\begin{pmatrix} 0 & 0 & \frac{-bk_1k_3^{-1}}{(a+2\bar{v})^2} & \frac{-bk_1k_4^{-1}}{(a+2\bar{v})^2} \\ k_2k_1^{-1} & 0 & 0 & 0 \\ \frac{-pk_3k_1^{-1}}{(q+2\bar{u})^2} & \frac{-pk_3k_2^{-1}}{(q+2\bar{u})^2} & 0 & 0 \\ 0 & 0 & k_4k_3^{-1} & 0 \end{pmatrix}. \tag{46}$$

Since J has the same eigenvalue as $\varOmega J\varOmega^{-1}$, from (45), (46) we have

$$\max_{1\le i\le 4}|\lambda_i|\le\|\varOmega G\varOmega^{-1}\|_\infty=\max\left\{k_2k_1^{-1},k_4k_3^{-1},\right.$$
$$\left.\frac{b\left(1+k_4^{-1}\right)}{(a+2\bar{v})^2},\frac{p\left(1+k_2^{-1}\right)}{(q+2\bar{u})^2}\right\}<1.$$

It follows that (43) is locally stable in it's positive equilibrium $(\bar{u},\bar{v})$.

From proposition (i), we can assume that

$$\lim_{n\to\infty} sup\; u_n=\varLambda_1,\ \lim_{n\to\infty} inf\; u_n=\lambda_1,$$
$$\lim_{n\to\infty} sup\; v_n=\varLambda_2,\ \lim_{n\to\infty} inf\; v_n=\lambda_2.$$

In which $\varLambda_i,\lambda_i\in(0,\infty),i=1,2.$

Then from (42), one has

$$\begin{cases}\varLambda_1\le\frac{b}{a+2\lambda_2}, & \varLambda_2\le\frac{p}{q+2\lambda_1},\\ \\ \lambda_1\ge\frac{b}{a+2\varLambda_2}, & \lambda_2\ge\frac{p}{q+2\varLambda_1}.\end{cases} \tag{47}$$

If follows from (47) that

$$a\left(\varLambda_1-\lambda_1\right)+q\left(\varLambda_2-\lambda_2\right)\le 0.$$

Since the initial value $a,q\in(0,+\infty)$, $\varLambda_i=\lambda_i\,(i=1,2)$ is obtained. That means

$$\lim_{n\to\infty}u_n=\bar{u},\ \lim_{n\to\infty}v_n=\bar{v}.$$

In summary, the system (42) is globally asymptotically stable at its positive equilibrium point $(\bar{u},\bar{v})$. ☐

Theorem 3. *The Eq.(1) is analyzed, where* $S,C,E_i\in\mathbb{R}_F^+$, $i=-1,0$. *Suppose that for any* $\eta\in(0,1]$, *there are*

$$\begin{cases} S_{r,\eta}\left(C_{l,\eta}+E_{n,l,\eta}+E_{n-1,l,\eta}\right)\le S_{l,\eta}\left(C_{r,\eta}+E_{n,r,\eta}+E_{n-1,r,\eta}\right),\\ \\ 2C_{r,\eta}S_{l,\eta}>\left(\sqrt{2S_{r,\eta}}-2C_{r,\eta}\right)\left(C_{l,\eta}\sqrt{2S_{r,\eta}}+2S_{r,\eta}-2S_{l,\eta}\right),\\ \\ 2C_{l,\eta}S_{r,\eta}>\left(\sqrt{2S_{l,\eta}}-2C_{l,\eta}\right)\left(C_{r,\eta}\sqrt{2S_{l,\eta}}+2S_{l,\eta}-2S_{r,\eta}\right).\end{cases} \tag{48}$$

In addition to this, if (12) *and* (22) *are true, then (i)-(iii) below holds..*

(i) Every positive FS E_n *to Eq.(1) is both bounded and persistent.*

(ii) There exists a unique positive equilibrium $\bar{E}$ *for Eq.(1).*

(iii) For any positive FS E_n *of Eq.(1), it converges to the* $\bar{E}$ *as* $n\to\infty$. *Furthermore, this* $\bar{E}$ *is globally asymptotically stable.*

Proof. (i)Assume that E_n represents a positive FS of Eq.(1). Given that (12) and (48) are valid, we invoke Lemma 4 to derive

$$\frac{S_{r,\eta}C_{l,\eta}}{C_{l,\eta}C_{r,\eta}+2S_{l,\eta}}\le E_{n,l,\eta}\le\frac{S_{r,\eta}}{C_{r,\eta}}, \tag{49}$$

$$\frac{S_{l,\eta}C_{r,\eta}}{C_{l,\eta}C_{r,\eta}+2S_{r,\eta}}\le E_{n,r,\eta}\le\frac{S_{l,\eta}}{C_{l,\eta}}. \tag{50}$$

From (12), (49) and (50), we have

$$\left[E_{n,l,\eta},E_{n,r,\eta}\right]\subset[H,G], \tag{51}$$

where

$$H=\frac{K_SK_C}{D_C^2+2D_S},$$
$$G=\frac{D_S}{K_C}.$$

Thus, by induction $\overline{\bigcup_{\eta\in(0,1]}\left[E_{n,l,\eta},E_{n,r,\eta}\right]}\subseteq[H,G]$. This shows that the positive FS of Eq.(1) are bounded and persistent.

(ii) Let PFN $\bar{E}$ satisfy (16). One has for $\eta\in(0,1]$,

$$E_{l,\eta}=\frac{S_{r,\eta}}{C_{r,\eta}+2E_{r,\eta}},E_{r,\eta}=\frac{S_{l,\eta}}{C_{l,\eta}+2E_{l,\eta}}, \tag{52}$$

From (52), we have

$$E_{l,\eta}=\frac{-u+\sqrt{u^2+8v}}{4C_{r,\eta}},E_{r,\eta}=\frac{-\beta+\sqrt{\beta^2+8\alpha}}{4C_{l,\eta}}.$$

where

$$u=2S_{l,\eta}-2S_{r,\eta}+C_{l,\eta}C_{r,\eta},$$
$$v=S_{r,\eta}C_{l,\eta}C_{r,\eta},$$
$$\beta=2S_{r,\eta}-2S_{l,\eta}+C_{l,\eta}C_{r,\eta},$$
$$\alpha=S_{l,\eta}C_{l,\eta}C_{r,\eta}.$$

The further proof is similar to part (ii) of Theorem 2. Thus, it can be clearly obtained that $\bar{E}$ is a positive equilibrium point of Eq.(1).

(iii) Since (22), (29) hold, applying Lemma 4 to the system (7) yields

$$\lim_{n\to\infty}D\left(E_n,\bar{E}\right)=\lim_{n\to\infty}\sup_{\eta\in[0,1]}\left\{\max\left\{\left|E_{n,l,\eta}-E_{l,\eta}\right|,\right.\right.$$
$$\left.\left.\left|E_{n,r,\eta}-E_{r,\eta}\right|\right\}\right\}=0,$$

where E_n is a positive fuzzy solution to Eq.(1) and satisfies the system (16). At this point the proof of all three propositions of Theorem 3 is complete.

$\square$

4 Numerical Simulation

In this section, the correctness and validity of the theoretical findings presented in the previous sections will be further verified through the detailed analysis results of two specific examples.

Example 1. *Consider Eq.*(1)*, the membership function is described as*

$$S(t)=\begin{cases}0.25t-0.5, 2\le t\le 6\\ 4-0.5t, 6\le t\le 8\end{cases}, C(t)=\begin{cases}0.5t-1.5, 3\le t\le 5\\ 6-t, 5\le t\le 6\end{cases}, \tag{53}$$

$$E_0(t)=\begin{cases}4t-2, 0.5\le t\le 0.75\\ 4-4t, 0.75\le t\le 1\end{cases}, E_{-1}(t)=\begin{cases}5t-1, 0.2\le t\le 0.4\\ 5-10t, 0.4\le t\le 0.5\end{cases}. \tag{54}$$

From (53)*, one has*

$$[S]_\eta=[2+4\eta,8-2\eta],[C]_\eta=[3+2\eta,6-\eta],\eta\in(0,1]. \tag{55}$$

From (54)*, one has*

$$\begin{aligned}[E_0]_\eta&=[0.5+0.25\eta,1-0.25\eta],\\ [E_{-1}]_\eta&=[0.2+0.2\eta,0.5-0.1\eta],\eta\in(0,1].\end{aligned} \tag{56}$$

Therefore, it follows that

$$\overline{\bigcup_{\eta\in(0,1]}[S]_\eta}=[2,8],\ \overline{\bigcup_{\eta\in(0,1]}[C]_\eta}=[3,6], \tag{57}$$

$$\overline{\bigcup_{\eta\in(0,1]}[E_0]_\eta}=[0.5,1],\ \overline{\bigcup_{\eta\in(0,1]}[E_{-1}]_\eta}=[0.2,0.5]. \tag{58}$$

When case I occurs, using parameter $\eta\in(0,1]$*, Eq.*(1) *can be transformed into a system of DE as follows*

$$\begin{cases}E_{n+1,l,\eta}=\frac{S_{l,\eta}}{C_{l,\eta}+E_{n,l,\eta}+E_{n-1,l,\eta}},\\ E_{n+1,r,\eta}=\frac{S_{r,\eta}}{C_{r,\eta}+E_{n,r,\eta}+E_{n-1,r,\eta}}.\end{cases} \tag{59}$$

According to the theorem 2, and $0<K_S\le S_{l,\eta}\le D_S$*,* $0<K_C\le C_{l,\eta}\le D_C,\eta\in(0,1]$*,* $E_t(t=-1,0)$ *is a positive fuzzy numbers, all solutions* E_n *of Eq.*(1) *is persistently bounded.*

Furthermore, according to the theorem $2(ii)$*, a unique positive equilibrium point* $\bar{E}=(0.5,0.886,1)$ *can be obtained, and all fuzzy solutions* E_n *converge to* $\bar{E}$ *with respect* D *as* $n\to\infty$*(see Figures 2, 3, 4).*

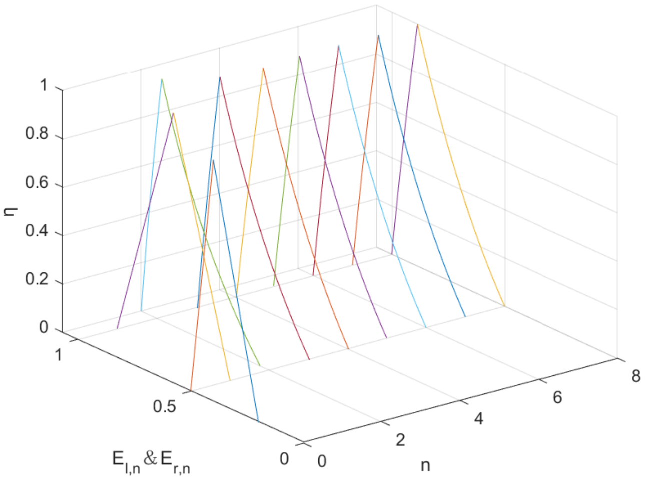

Figure 2. The evolutionary behavior of system (59).

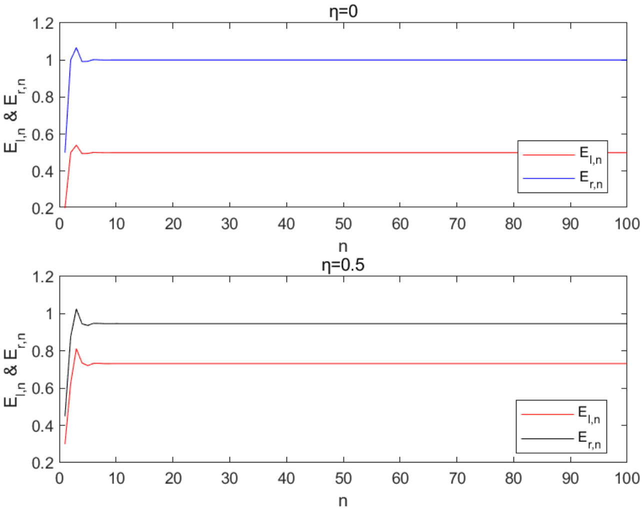

Figure 3. Transition in asymptotic behavior of system (59) for $\eta=0$ and $\eta=0.5$.

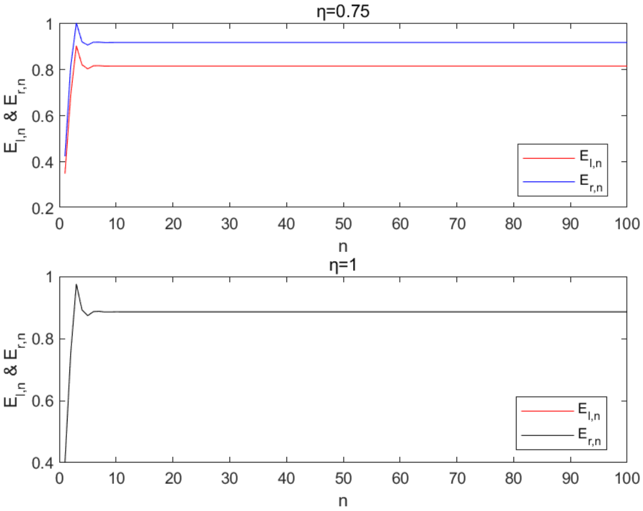

Figure 4. Transition in asymptotic behavior of system (59) for $\eta=0.75$ and $\eta=1$.

Example 2. *Consider Eq.*(1)*, the membership function is* *described as*

$$S\left(t\right)=\begin{cases} t-4, 4\leq t\leq 5 \\ 6-t, 5\leq t\leq 6\end{cases}, C\left(t\right)=\begin{cases} t-2, 2\leq t\leq 3 \\ 2.5-0.5t, 3\leq t\leq 5\end{cases}, \tag{60}$$

$$E_0\left(t\right)=\begin{cases} 4t-2, 0.5\leq t\leq 0.75 \\ 4-4t, 0.75\leq t\leq 1\end{cases}, E_{-1}\left(t\right)=\begin{cases} 5t-1, 0.2\leq t\leq 0.4 \\ 5-10t, 0.4\leq t\leq 0.5\end{cases}. \tag{61}$$

From (60), (61), *it follows that*

$$\begin{aligned} \left[S\right]_\eta &= \left[4+\eta, 6-\eta\right], \\ \left[C\right]_\eta &= \left[2+\eta, 5-2\eta\right], \\ \left[E_0\right]_\eta &= \left[0.5+0.25\eta, 1-0.25\eta\right], \\ \left[E_{-1}\right]_\eta &= \left[0.2+0.2\eta, 0.5-0.1\eta\right], \\ \eta &\in \left(0,1\right]. \end{aligned} \tag{62}$$

Therefore, it follows that

$$\overline{\bigcup_{\eta\in(0,1]}\left[S\right]_\eta}=\left[4,6\right], \overline{\bigcup_{\eta\in(0,1]}\left[C\right]_\eta}=\left[2,7\right], \tag{63}$$

$$\overline{\bigcup_{\eta\in(0,1]}\left[E_0\right]_\eta}=\left[0.5,1\right], \overline{\bigcup_{\eta\in(0,1]}\left[E_{-1}\right]_\eta}=\left[0.2,0.5\right]. \tag{64}$$

When case II occurs, using parameter $\eta\in(0,1]$, *Eq.*(1) *can be transformed into a system of DE as follows*

$$\begin{cases} E_{n,l,\eta}=\frac{S_{r,\eta}}{C_{r,\eta}+E_{n,r,\eta}+E_{n-1,r,\eta}}, \\ E_{n,r,\eta}=\frac{S_{l,\eta}}{C_{l,\eta}+E_{n,l,\eta}+E_{n-1,l,\eta}}. \end{cases} \tag{65}$$

According to the theorem 3, a unique positive equilibrium point $\bar{E}=(0.8358,1,1.0895)$ *can be obtained, so using theorem 3(iii), it can be obtained that all fuzzy solutions* E_n *converge to* $\bar{E}$, *where D is* $n\to\infty$(*see Figures* 5, 6, 7).

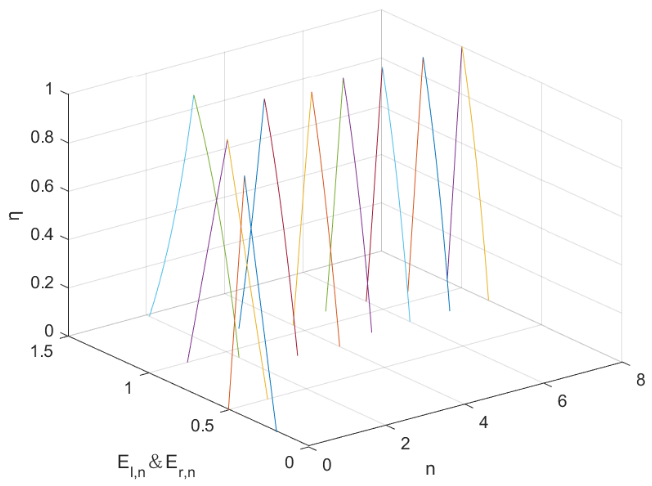

Figure 5. The evolutionary behavior of system (65).

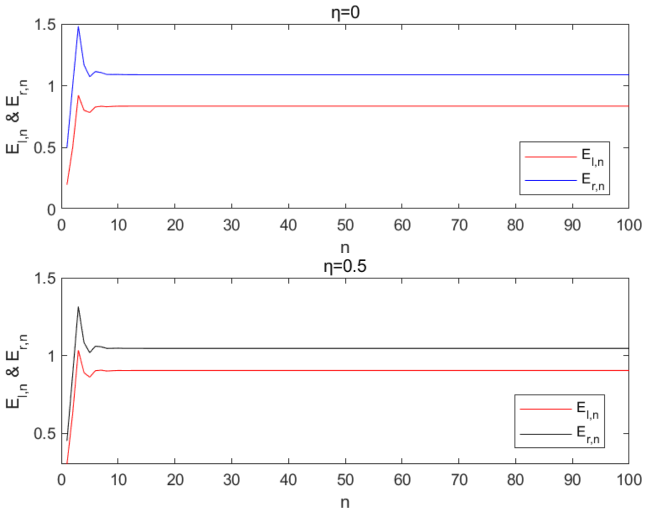

Figure 6. Transition in asymptotic behavior of system (65) for $\eta=0$ and $\eta=0.25$.

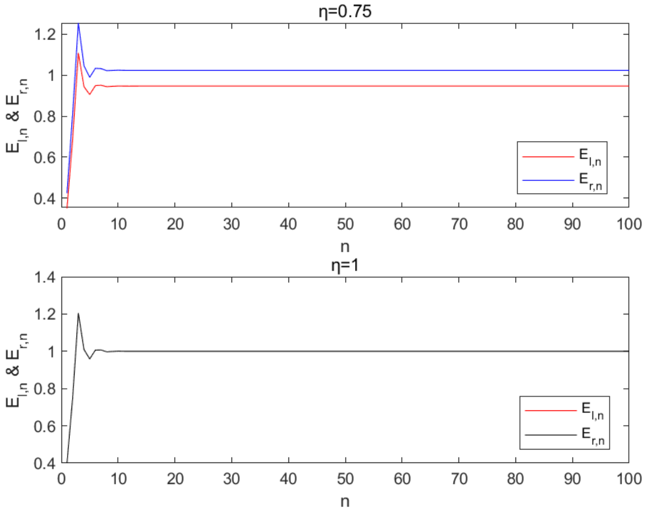

Figure 7. Transition in asymptotic behavior of system (65) for $\eta=0.75$ and $\eta=1$.

5 Conclusion

This article uses the uncertainty of FN to analyze some species and organisms in uncertain environments, and studies the dynamic behavior of second-order FDE. The FDE $E_{n+1}=\frac{S}{C+E_n+E_{n-1}}$ is studied using the generalized division (g-division) method proposed by Stefanini [22]. The uniqueness of the existence of positive solutions to the equation and some properties at its equilibrium are explored and verified.

Under g-division, the two results (i), (ii) are obtained after exploration.

(i) When the Case I occurs, if (59) satisfies $0<K_S\leq S_{l,\eta}\leq D_S, 0<K_C\leq C_{l,\eta}\leq D_C, \eta\in(0,1]$, then the positive FS E_n are boundedly persistent and they all converge to a unique positive equilibrium point $\bar{E}$ as $n\to\infty$.

(ii)When the Case II occurs, if (65) satisfies

$$\begin{cases} S_{r,\eta}\left(C_{l,\eta}+E_{n,l,\eta}+E_{n-1,l,\eta}\right) \leq S_{l,\eta}\left(C_{r,\eta}+E_{n,r,\eta}+E_{n-1,r,\eta}\right), \\ 2C_{r,\eta}S_{l,\eta} > \left(\sqrt{2S_{r,\eta}}-2C_{r,\eta}\right)\left(C_{l,\eta}\sqrt{2S_{r,\eta}}+2S_{r,\eta}-2S_{l,\eta}\right), \\ 2C_{l,\eta}S_{r,\eta} > \left(\sqrt{2S_{l,\eta}}-2C_{l,\eta}\right)\left(C_{r,\eta}\sqrt{2S_{l,\eta}}+2S_{l,\eta}-2S_{r,\eta}\right). \end{cases}$$

each solution E_n tends towards a unique equilibrium point $\bar{E}$ as $n \to \infty$ and remains persistently bounded.

As far as the current research status of fuzzy discrete models is concerned, the few theories available at this stage make it difficult to carry out research on some of their complex dynamic behaviors, such as chaos, fractals, and the existence of periodic solutions. These complex dynamic behaviors are very worthy of further in-depth study.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

AI Use Statement

The authors declare that no generative AI was used in the preparation of this manuscript.

Ethical Approval and Consent to Participate

Not applicable.

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