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RESEARCH ARTICLE

A Formal Analytic Program for the Three-Dimensional Navier–Stokes Equations

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Abstract

This paper develops a deliberately expanded analytic program around the three-dimensional incompressible Navier–Stokes equations. The exposition is intentionally heavy in algebra, index notation, multilinear expansions, differentiated hierarchies, and page-filling calculations. The central idea is to organize the continuation problem around a mixed pressure–vorticity–anisotropic functional, while keeping visible every major nonlinear channel: energy, Hessian growth, high-order commutators, vorticity stretching, elliptic pressure reconstruction, dyadic localization, paradifferential splitting, kernel representation, cylindrical-coordinate geometry, and dense symbolic closure systems. The rigorous portions are clearly distinguished from the formal continuation layer. Thus the text should be read as a strengthened and heavily expanded attempt toward the global smoothness, singularity exclusion, existence, and uniqueness program in dimension three, rather than as an unconditional claim of having resolved the problem.

Keywords: Navier-Stokes equations, incompressible flow, global regularity, finite-time singularity, vorticity, anisotropy, pressure reconstruction, formal continuation.

Introductory declaration

This article is written in a maximal analytic style. The purpose is not to hide the nonlinear structure of the three-dimensional Navier–Stokes equations behind compressed notation, but to display it. For that reason, the paper repeatedly opens derivatives, expands summations, writes full tensor blocks, and includes long differential identities. The ambition of the program is classical and extremely high: global smoothness, exclusion of singularity, persistence of existence, and uniqueness in three dimensions. The paper maintains a classical standard of mathematical honesty: the precise bottlenecks that remain formal are explicitly identified. A substantial body of classical and recent work on regularity criteria, global existence for special classes, and conditional uniqueness informs the broader context of this investigation.

Introduction

We consider the three-dimensional incompressible Navier–Stokes system

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p - \nu \Delta u = 0, \qquad \operatorname{div} u = 0, \quad (1)$$

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or, equivalently,

$$\partial_t u + (u \cdot \nabla)u + \nabla p - \nu\Delta u = 0, \qquad \operatorname{div} u = 0, \quad (2)$$

for a velocity field $u = u(x,t) \in \mathbb{R}^3$, pressure $p = p(x,t) \in \mathbb{R}$, and viscosity $\nu > 0$; see [1–3]. Weak finite-energy solutions are known in the Leray–Hopf class; the unrestricted global regularity of smooth solutions in three dimensions is one of the central open problems in nonlinear PDE [3–6]. Significant progress on global regularity for large classes of initial data under special structural assumptions has been achieved, for example, in the works of [7].

The continuation problem in the strong class is reduced to the growth of the Lipschitz norm

$$\int_0^T \|\nabla u(t)\|_{L^\infty}\, dt. \quad (3)$$

Hence a genuine analytic attack on the three-dimensional problem must isolate the mechanisms that can force $\|\nabla u\|_{L^\infty}$ to diverge. The present manuscript advances a mixed continuation viewpoint based on three visible channels:

$$\begin{gathered}\text{vorticity}, \qquad \text{pressure gradient}, \\ \text{anisotropic vertical derivatives}.\end{gathered} \quad (4)$$

The formal target estimate is

$$\|\nabla u(t)\|_{L^\infty} \lesssim 1 + \|\omega(t)\|_{L^\infty} + \|\nabla p(t)\|_{L^q} + \|\partial_3 u(t)\|_{H^{m-1}}, \quad (5)$$

for suitable $q > 3$ and $m > \frac{5}{2}$.

The philosophy is the following. The vorticity channel isolates local rotation and the vortex-stretching instability. The pressure channel encodes the nonlocal elliptic back-reaction generated by the quadratic transport tensor. The anisotropic channel records the possibility that one direction may carry the true continuation burden more transparently than a fully isotropic norm. This is not presented as a final theorem. It is presented as an explicit analytic route whose precise closure point is kept visible.

1 Functional setting and local strong class

Let $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3$, and let $m > \frac{5}{2}$. We explicitly recall that

$$\begin{aligned} H^m(\Omega) = \{ u \in L^2(\Omega) \ : \ & \partial^\alpha u \in L^2(\Omega), \\ & \forall\, \alpha \in \mathbb{N}^3, \quad |\alpha| \le m \}. \end{aligned} \quad (6)$$

In fully expanded form,

$$\begin{aligned} \|u\|_{H^m}^2 &= \sum_{|\alpha|\le m} \int_\Omega \left|\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3} u(x)\right|^2\, dx \\ &= \int_\Omega |u|^2 dx + \sum_{i=1}^3 \int_\Omega |\partial_i u|^2 dx \\ &\quad + \sum_{i,j=1}^3 \int_\Omega |\partial_{ij} u|^2 dx + \cdots . \end{aligned} \quad (7)$$

For divergence-free initial data $u_0 \in H^m(\Omega)$ satisfying

$$\partial_1 u_{0,1} + \partial_2 u_{0,2} + \partial_3 u_{0,3} = 0, \quad (8)$$

we consider the strong class

$$\begin{gathered} u \in C([0,T]; H^m(\Omega)) \cap L^2_{\mathrm{loc}}([0,T]; H^{m+1}(\Omega)), \\ \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0. \end{gathered} \quad (9)$$

In expanded derivative form, this means that for every multi-index α with $|\alpha| \le m$,

$$\partial^\alpha u(t,\cdot) \in L^2(\Omega), \quad \forall t \in [0,T], \quad (10)$$

and additionally

$$\int_0^T \sum_{|\alpha|\le m+1} \|\partial^\alpha u(s)\|_{L^2}^2\, ds < \infty. \quad (11)$$

By the classical local theory [1, 2, 9–11], there exists a maximal strong solution on $[0, T^*)$ satisfying

$$T^* < \infty \quad \Longrightarrow \quad \limsup_{t\uparrow T^*} \sum_{|\alpha|\le m} \|\partial^\alpha u(t)\|_{L^2} = +\infty. \quad (12)$$

Now we expand the differential inequality in full multi-index form:

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_{H^m}^2 &= \frac{d}{dt} \sum_{|\alpha|\le m} \int_\Omega |\partial^\alpha u|^2 dx \\ &= \sum_{|\alpha|\le m} \frac{d}{dt} \int_\Omega |\partial^\alpha u|^2 dx \\ &= 2 \sum_{|\alpha|\le m} \int_\Omega \partial^\alpha u \cdot \partial^\alpha(\partial_t u)\, dx. \end{aligned} \quad (13)$$

Thus an inequality of the form

$$\frac{d}{dt}\|u(t)\|_{H^m}^2 \le G(t)\,\|u(t)\|_{H^m}^2, \quad (14)$$

corresponds explicitly to

$$\sum_{|\alpha|\le m} \int_\Omega \partial^\alpha u \cdot \partial^\alpha(\partial_t u) \le G(t) \sum_{|\alpha|\le m} \int_\Omega |\partial^\alpha u|^2. \quad (15)$$

2 The full componentwise Navier–Stokes system

We now expand the nonlinear term fully before writing components.

$$(u \cdot \nabla)u = \left(\sum_{j=1}^{3} u_j \partial_j u_1,\ \sum_{j=1}^{3} u_j \partial_j u_2,\ \sum_{j=1}^{3} u_j \partial_j u_3 \right). \tag{16}$$

Thus the system reads

$$\partial_t u_i + \sum_{j=1}^{3} u_j \partial_j u_i + \partial_i p - \nu \sum_{j=1}^{3} \partial_{jj} u_i = 0. \tag{17}$$

Now fully expanded:

$$\begin{aligned} &\partial_t u_1 + u_1 \partial_1 u_1 + u_2 \partial_2 u_1 + u_3 \partial_3 u_1 \\ &\qquad + \partial_1 p - \nu \partial_{11} u_1 - \nu \partial_{22} u_1 - \nu \partial_{33} u_1 = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} &\partial_t u_2 + u_1 \partial_1 u_2 + u_2 \partial_2 u_2 + u_3 \partial_3 u_2 \\ &\qquad + \partial_2 p - \nu \partial_{11} u_2 - \nu \partial_{22} u_2 - \nu \partial_{33} u_2 = 0, \end{aligned} \tag{19}$$

$$\begin{aligned} &\partial_t u_3 + u_1 \partial_1 u_3 + u_2 \partial_2 u_3 + u_3 \partial_3 u_3 \\ &\qquad + \partial_3 p - \nu \partial_{11} u_3 - \nu \partial_{22} u_3 - \nu \partial_{33} u_3 = 0. \end{aligned} \tag{20}$$

The divergence constraint expanded is

$$\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0. \tag{21}$$

We now expand the gradient matrix entry by entry:

$$\nabla u = \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 & \partial_3 u_1 \\ \partial_1 u_2 & \partial_2 u_2 & \partial_3 u_2 \\ \partial_1 u_3 & \partial_2 u_3 & \partial_3 u_3 \end{pmatrix}. \tag{22}$$

The symmetric part:

$$S(u) = \frac{1}{2} \begin{pmatrix} 2\partial_1 u_1 & \partial_2 u_1 + \partial_1 u_2 & \partial_3 u_1 + \partial_1 u_3 \\ \partial_1 u_2 + \partial_2 u_1 & 2\partial_2 u_2 & \partial_3 u_2 + \partial_2 u_3 \\ \partial_1 u_3 + \partial_3 u_1 & \partial_2 u_3 + \partial_3 u_2 & 2\partial_3 u_3 \end{pmatrix}. \tag{23}$$

The antisymmetric part:

$$A(u) = \frac{1}{2} \begin{pmatrix} 0 & \partial_2 u_1 - \partial_1 u_2 & \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 & 0 & \partial_3 u_2 - \partial_2 u_3 \\ \partial_1 u_3 - \partial_3 u_1 & \partial_2 u_3 - \partial_3 u_2 & 0 \end{pmatrix}. \tag{24}$$

Special algebraic structures in the nonlinearity, including self-dual reductions that may admit particular classes of solutions, have been investigated in [8].

3 L^2 energy identity fully expanded

We now expand every contraction index explicitly.

$$\begin{aligned} \int_\Omega (u \cdot \nabla) u \cdot u \, dx &= \sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Omega u_j (\partial_j u_i) u_i \, dx \\ &= \sum_{j=1}^{3} \int_\Omega u_j \left(\sum_{i=1}^{3} u_i \partial_j u_i \right) dx \\ &= \sum_{j=1}^{3} \int_\Omega u_j \partial_j \left(\frac{1}{2} \sum_{i=1}^{3} u_i^2 \right) dx \\ &= \frac{1}{2} \sum_{j=1}^{3} \int_\Omega u_j \partial_j |u|^2 dx. \end{aligned} \tag{25}$$

Now integrate by parts term by term:

$$\int_\Omega u_j \partial_j |u|^2 dx = - \int_\Omega (\partial_j u_j) |u|^2 dx, \qquad j = 1, 2, 3. \tag{26}$$

Summing:

$$\begin{aligned} \sum_{j=1}^{3} \int_\Omega u_j \partial_j |u|^2 dx &= - \int_\Omega (\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3) |u|^2 dx \\ &= - \int_\Omega (\operatorname{div} u) |u|^2 dx = 0. \end{aligned} \tag{27}$$

Thus the nonlinear term cancels exactly.

Finally we obtain, in fully expanded dissipation form:

$$\begin{aligned} \nu \|\nabla u\|_{L^2}^2 &= \nu \sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Omega |\partial_j u_i|^2 dx \\ &= \nu \Bigg(\int |\partial_1 u_1|^2 + \int |\partial_2 u_1|^2 + \int |\partial_3 u_1|^2 \\ &\quad + \int |\partial_1 u_2|^2 + \int |\partial_2 u_2|^2 + \int |\partial_3 u_2|^2 \\ &\quad + \int |\partial_1 u_3|^2 + \int |\partial_2 u_3|^2 + \int |\partial_3 u_3|^2 \Bigg). \end{aligned} \tag{28}$$

4 First-order differentiated energy chain

We begin by differentiating the incompressible Navier–Stokes system

$$\partial_t u_i + u_j \partial_j u_i + \partial_i p - \nu \partial_{jj} u_i = 0, \qquad \partial_i u_i = 0, \tag{29}$$

with respect to the spatial variable x_k, where $k \in \{1, 2, 3\}$ is fixed for the moment.

Applying ∂_k to each term, we obtain

$$\partial_k(\partial_t u_i)+\partial_k(u_j\partial_j u_i)+\partial_k(\partial_i p)-\nu\partial_k(\partial_{jj}u_i)=0. \quad (30)$$

Since partial derivatives commute, this becomes

$$\partial_t(\partial_k u_i)+\partial_k(u_j\partial_j u_i)+\partial_i(\partial_k p)-\nu\partial_{jj}(\partial_k u_i)=0. \quad (31)$$

Reassembling the vector notation,

$$\partial_t(\partial_k u)+\partial_k\big((u\cdot\nabla)u\big)+\nabla(\partial_k p)-\nu\Delta(\partial_k u)=0. \quad (32)$$

We now test this differentiated equation against $\partial_k u$ and integrate over Ω:

$$\begin{aligned}0&=\int_\Omega \partial_t(\partial_k u)\cdot\partial_k u\,dx+\int_\Omega \partial_k\big((u\cdot\nabla)u\big)\cdot\partial_k u\,dx\\&\quad+\int_\Omega \nabla(\partial_k p)\cdot\partial_k u\,dx-\nu\int_\Omega \Delta(\partial_k u)\cdot\partial_k u\,dx.\end{aligned} \quad (33)$$

We now expand each term separately.

Expansion of the time term

First,

$$\begin{aligned}\int_\Omega \partial_t(\partial_k u)\cdot\partial_k u\,dx&=\sum_{i=1}^3\int_\Omega \partial_t(\partial_k u_i)\,\partial_k u_i\,dx\\&=\sum_{i=1}^3\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_k u_i|^2\,dx\\&=\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3\int_\Omega |\partial_k u_i|^2\,dx\\&=\frac{1}{2}\frac{d}{dt}\,\|\partial_k u\|_{L^2}^2\,.\end{aligned} \quad (34)$$

Expansion of the differentiated nonlinear term

We now expand

$$\partial_k\big((u\cdot\nabla)u\big). \quad (35)$$

Recall that

$$(u\cdot\nabla)u_i=u_1\partial_1 u_i+u_2\partial_2 u_i+u_3\partial_3 u_i=\sum_{j=1}^3 u_j\partial_j u_i. \quad (36)$$

Therefore,

$$\begin{aligned}\partial_k\big((u\cdot\nabla)u_i\big)&=\partial_k\Big(u_1\partial_1 u_i+u_2\partial_2 u_i+u_3\partial_3 u_i\Big)\\&=\partial_k(u_1\partial_1 u_i)+\partial_k(u_2\partial_2 u_i)+\partial_k(u_3\partial_3 u_i).\end{aligned}$$

Applying the product rule term by term,

$$\partial_k(u_1\partial_1 u_i)=(\partial_k u_1)(\partial_1 u_i)+u_1\partial_k\partial_1 u_i, \quad (37)$$

$$\partial_k(u_2\partial_2 u_i)=(\partial_k u_2)(\partial_2 u_i)+u_2\partial_k\partial_2 u_i, \quad (38)$$

$$\partial_k(u_3\partial_3 u_i)=(\partial_k u_3)(\partial_3 u_i)+u_3\partial_k\partial_3 u_i. \quad (39)$$

Summing these three identities,

$$\begin{aligned}\partial_k\big((u\cdot\nabla)u_i\big)&=(\partial_k u_1)(\partial_1 u_i)+(\partial_k u_2)(\partial_2 u_i)\\&\quad+(\partial_k u_3)(\partial_3 u_i)+u_1\partial_1(\partial_k u_i)\\&\quad+u_2\partial_2(\partial_k u_i)+u_3\partial_3(\partial_k u_i).\end{aligned} \quad (40)$$

In summation notation,

$$\partial_k\big((u\cdot\nabla)u_i\big)=(\partial_k u_j)\partial_j u_i+u_j\partial_j(\partial_k u_i). \quad (41)$$

Returning to vector notation, this is

$$\partial_k\big((u\cdot\nabla)u\big)=(\partial_k u\cdot\nabla)u+(u\cdot\nabla)(\partial_k u). \quad (42)$$

Now test against $\partial_k u$:

$$\begin{aligned}\int_\Omega \partial_k\big((u\cdot\nabla)u\big)\cdot\partial_k u\,dx&=\int_\Omega \big((\partial_k u\cdot\nabla)u\big)\cdot\partial_k u\,dx\\&\quad+\int_\Omega \big((u\cdot\nabla)(\partial_k u)\big)\cdot\partial_k u\,dx.\end{aligned} \quad (43)$$

We now expand the second integral:

$$\begin{aligned}&\int_\Omega \big((u\cdot\nabla)(\partial_k u)\big)\cdot\partial_k u\,dx\\&\quad=\sum_{i=1}^3\int_\Omega \big((u\cdot\nabla)(\partial_k u_i)\big)\,\partial_k u_i\,dx\\&\quad=\sum_{i=1}^3\int_\Omega \Big(\sum_{j=1}^3 u_j\partial_j(\partial_k u_i)\Big)\partial_k u_i\,dx\\&\quad=\sum_{i=1}^3\sum_{j=1}^3\int_\Omega u_j\,\partial_j(\partial_k u_i)\,\partial_k u_i\,dx.\end{aligned} \quad (44)$$

Using the identity

$$\partial_j\Big(\frac{1}{2}|\partial_k u_i|^2\Big)=\partial_j(\partial_k u_i)\,\partial_k u_i, \quad (45)$$

we get

$$\begin{aligned}&\sum_{i=1}^3\sum_{j=1}^3\int_\Omega u_j\,\partial_j(\partial_k u_i)\,\partial_k u_i\,dx\\&\quad=\frac{1}{2}\sum_{i=1}^3\sum_{j=1}^3\int_\Omega u_j\,\partial_j\big(|\partial_k u_i|^2\big)\,dx\\&\quad=\frac{1}{2}\sum_{j=1}^3\int_\Omega u_j\,\partial_j\Big(\sum_{i=1}^3|\partial_k u_i|^2\Big)\,dx\\&\quad=\frac{1}{2}\int_\Omega u\cdot\nabla|\partial_k u|^2\,dx.\end{aligned} \quad (46)$$

Integrating by parts,

$$\frac{1}{2}\int_\Omega u\cdot\nabla|\partial_k u|^2\,dx = -\frac{1}{2}\int_\Omega (\operatorname{div} u)\,|\partial_k u|^2\,dx = 0, \tag{47}$$

because $\operatorname{div} u = 0$. Therefore,

$$\int_\Omega \big((u\cdot\nabla)(\partial_k u)\big)\cdot\partial_k u\,dx = 0. \tag{48}$$

Thus

$$\int_\Omega \partial_k\big((u\cdot\nabla)u\big)\cdot\partial_k u\,dx = \int_\Omega \big((\partial_k u\cdot\nabla)u\big)\cdot\partial_k u\,dx. \tag{49}$$

Expansion of the pressure term

Now consider

$$\int_\Omega \nabla(\partial_k p)\cdot\partial_k u\,dx. \tag{50}$$

Expanding componentwise,

$$\int_\Omega \nabla(\partial_k p)\cdot\partial_k u\,dx = \sum_{i=1}^{3}\int_\Omega \partial_i(\partial_k p)\,\partial_k u_i\,dx. \tag{51}$$

Integrating by parts in x_i,

$$\begin{aligned}\sum_{i=1}^{3}\int_\Omega \partial_i(\partial_k p)\,\partial_k u_i\,dx &= -\sum_{i=1}^{3}\int_\Omega \partial_k p\,\partial_i(\partial_k u_i)\,dx\\ &= -\int_\Omega \partial_k p\sum_{i=1}^{3}\partial_i\partial_k u_i\,dx\\ &= -\int_\Omega \partial_k p\,\partial_k\Big(\sum_{i=1}^{3}\partial_i u_i\Big)\,dx\\ &= -\int_\Omega \partial_k p\,\partial_k(\operatorname{div} u)\,dx = 0.\end{aligned} \tag{52}$$

Expansion of the diffusion term

Now expand

$$-\,\nu\int_\Omega \Delta(\partial_k u)\cdot\partial_k u\,dx. \tag{53}$$

Componentwise,

$$\begin{aligned}&-\,\nu\int_\Omega \Delta(\partial_k u)\cdot\partial_k u\,dx\\ &= -\nu\sum_{i=1}^{3}\int_\Omega \Delta(\partial_k u_i)\,\partial_k u_i\,dx\\ &= -\nu\sum_{i=1}^{3}\int_\Omega \big(\partial_{11}\partial_k u_i + \partial_{22}\partial_k u_i + \partial_{33}\partial_k u_i\big)\partial_k u_i\,dx.\end{aligned} \tag{54}$$

Integrating by parts term by term,

$$-\nu\sum_{i=1}^{3}\int_\Omega \partial_{11}\partial_k u_i\,\partial_k u_i\,dx = \nu\sum_{i=1}^{3}\int_\Omega |\partial_1\partial_k u_i|^2\,dx, \tag{55}$$

$$-\nu\sum_{i=1}^{3}\int_\Omega \partial_{22}\partial_k u_i\,\partial_k u_i\,dx = \nu\sum_{i=1}^{3}\int_\Omega |\partial_2\partial_k u_i|^2\,dx, \tag{56}$$

$$-\nu\sum_{i=1}^{3}\int_\Omega \partial_{33}\partial_k u_i\,\partial_k u_i\,dx = \nu\sum_{i=1}^{3}\int_\Omega |\partial_3\partial_k u_i|^2\,dx. \tag{57}$$

Summing,

$$\begin{aligned}-\nu\int_\Omega \Delta(\partial_k u)\cdot\partial_k u\,dx &= \nu\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega |\partial_j\partial_k u_i|^2\,dx\\ &= \nu\,\|\nabla\partial_k u\|_{L^2}^2.\end{aligned} \tag{58}$$

Conclusion of the first-order estimate

Substituting (34), (49), (52), and (58) into (33), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_k u\|_{L^2}^2+\nu\,\|\nabla\partial_k u\|_{L^2}^2 = -\int_\Omega \big((\partial_k u\cdot\nabla)u\big)\cdot\partial_k u\,dx. \tag{59}$$

Summing over $k = 1, 2, 3$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{k=1}^{3}\|\partial_k u\|_{L^2}^2 + \nu\sum_{k=1}^{3}\|\nabla\partial_k u\|_{L^2}^2\\ &\qquad = -\sum_{k=1}^{3}\int_\Omega \big((\partial_k u\cdot\nabla)u\big)\cdot\partial_k u\,dx.\end{aligned} \tag{60}$$

That is,

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\nu\,\|\Delta u\|_{L^2}^2 = -\sum_{k=1}^{3}\int_\Omega \big((\partial_k u\cdot\nabla)u\big)\cdot\partial_k u\,dx. \tag{61}$$

Finally, estimate the right-hand side:

$$\begin{aligned}&\left|\sum_{k=1}^{3}\int_\Omega \big((\partial_k u\cdot\nabla)u\big)\cdot\partial_k u\,dx\right|\\ &\le \sum_{k=1}^{3}\int_\Omega |\partial_k u|\,|\nabla u|\,|\partial_k u|\,dx\\ &\le \sum_{k=1}^{3}\|\nabla u\|_{L^\infty}\int_\Omega |\partial_k u|^2\,dx\\ &= \|\nabla u\|_{L^\infty}\sum_{k=1}^{3}\|\partial_k u\|_{L^2}^2\\ &= \|\nabla u\|_{L^\infty}\,\|\nabla u\|_{L^2}^2.\end{aligned} \tag{62}$$

Therefore,

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + 2\nu\|\Delta u\|_{L^2}^2 \le 2\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}^2. \quad (63)$$

5 Second-order differentiated chain and Hessian matrix expansion

The transition from first-order to second-order estimates is not merely a mechanical differentiation of the energy inequality. The new terms that appear are the Hessian of the velocity field, which is controlled by the L^2 norm of second derivatives, and the commutator between the transport operator and the second-order derivative. These interactions are made explicit below.

We now pass to second-order derivatives and deliberately expand the algebra more heavily.

Define the Hessian block

$$D^2u = \begin{pmatrix} \partial_{11}u_1 & \partial_{12}u_1 & \partial_{13}u_1 \\ \partial_{21}u_1 & \partial_{22}u_1 & \partial_{23}u_1 \\ \partial_{31}u_1 & \partial_{32}u_1 & \partial_{33}u_1 \end{pmatrix} \oplus \begin{pmatrix} \partial_{11}u_2 & \partial_{12}u_2 & \partial_{13}u_2 \\ \partial_{21}u_2 & \partial_{22}u_2 & \partial_{23}u_2 \\ \partial_{31}u_2 & \partial_{32}u_2 & \partial_{33}u_2 \end{pmatrix} \oplus \begin{pmatrix} \partial_{11}u_3 & \partial_{12}u_3 & \partial_{13}u_3 \\ \partial_{21}u_3 & \partial_{22}u_3 & \partial_{23}u_3 \\ \partial_{31}u_3 & \partial_{32}u_3 & \partial_{33}u_3 \end{pmatrix}. \quad (64)$$

Fix two differentiation indices $k, \ell \in \{1,2,3\}$. Apply $\partial_{\ell k}$ to

$$\partial_t u_i + u_j\partial_j u_i + \partial_i p - \nu\partial_{mm}u_i = 0. \quad (65)$$

Then

$$\partial_t(\partial_{\ell k}u_i) + \partial_{\ell k}(u_j\partial_j u_i) + \partial_i\partial_{\ell k}p - \nu\partial_{mm}\partial_{\ell k}u_i = 0. \quad (66)$$

We now expand the nonlinear term carefully:

$$\begin{aligned} \partial_{\ell k}(u_j\partial_j u_i) &= \partial_\ell\Big(\partial_k u_j\,\partial_j u_i + u_j\,\partial_k\partial_j u_i\Big) \\ &= \partial_\ell(\partial_k u_j)\,\partial_j u_i + (\partial_k u_j)\,\partial_\ell\partial_j u_i \\ &\quad + (\partial_\ell u_j)\,\partial_k\partial_j u_i + u_j\,\partial_\ell\partial_k\partial_j u_i \\ &= (\partial_{\ell k}u_j)\partial_j u_i + (\partial_k u_j)\partial_{\ell j}u_i \\ &\quad + (\partial_\ell u_j)\partial_{kj}u_i + u_j\partial_{\ell kj}u_i. \end{aligned} \quad (67)$$

Thus the twice-differentiated equation is

$$\begin{aligned} &\partial_t(\partial_{\ell k}u_i) + u_j\partial_j(\partial_{\ell k}u_i) + \partial_i\partial_{\ell k}p - \nu\partial_{mm}\partial_{\ell k}u_i \\ &\quad = -(\partial_{\ell k}u_j)\partial_j u_i - (\partial_k u_j)\partial_{\ell j}u_i - (\partial_\ell u_j)\partial_{kj}u_i. \end{aligned} \quad (68)$$

Now multiply by $\partial_{\ell k}u_i$ and integrate:

$$\begin{aligned} &\int_\Omega \partial_t(\partial_{\ell k}u_i)\,\partial_{\ell k}u_i\,dx + \int_\Omega u_j\partial_j(\partial_{\ell k}u_i)\,\partial_{\ell k}u_i\,dx \\ &+ \int_\Omega \partial_i\partial_{\ell k}p\,\partial_{\ell k}u_i\,dx - \nu\int_\Omega \partial_{mm}\partial_{\ell k}u_i\,\partial_{\ell k}u_i\,dx \\ &= -\int_\Omega(\partial_{\ell k}u_j)\partial_j u_i\,\partial_{\ell k}u_i\,dx \\ &\quad - \int_\Omega(\partial_k u_j)\partial_{\ell j}u_i\,\partial_{\ell k}u_i\,dx - \int_\Omega(\partial_\ell u_j)\partial_{kj}u_i\,\partial_{\ell k}u_i\,dx. \end{aligned} \quad (69)$$

We now expand the left-hand side term by term.

For the time derivative,

$$\int_\Omega \partial_t(\partial_{\ell k}u_i)\,\partial_{\ell k}u_i\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_{\ell k}u_i|^2\,dx. \quad (70)$$

For the transport term,

$$\begin{aligned} \int_\Omega u_j\partial_j(\partial_{\ell k}u_i)\,\partial_{\ell k}u_i\,dx &= \frac{1}{2}\int_\Omega u_j\partial_j\left(|\partial_{\ell k}u_i|^2\right)dx \\ &= -\frac{1}{2}\int_\Omega(\partial_j u_j)|\partial_{\ell k}u_i|^2\,dx = 0. \end{aligned} \quad (71)$$

For the pressure term,

$$\begin{aligned} \int_\Omega \partial_i\partial_{\ell k}p\,\partial_{\ell k}u_i\,dx &= -\int_\Omega \partial_{\ell k}p\,\partial_i\partial_{\ell k}u_i\,dx \\ &= -\int_\Omega \partial_{\ell k}p\,\partial_{\ell k}(\partial_i u_i)\,dx = 0. \end{aligned} \quad (72)$$

For the diffusion term,

$$\begin{aligned} -\nu\int_\Omega \partial_{mm}\partial_{\ell k}u_i\,\partial_{\ell k}u_i\,dx &= \nu\int_\Omega |\nabla\partial_{\ell k}u_i|^2\,dx \\ &= \nu\sum_{m=1}^{3}\int_\Omega |\partial_m\partial_{\ell k}u_i|^2\,dx. \end{aligned} \quad (73)$$

Hence

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_{\ell k}u_i|^2\,dx + \nu\int_\Omega |\nabla\partial_{\ell k}u_i|^2\,dx \\ &= -\int_\Omega(\partial_{\ell k}u_j)\partial_j u_i\,\partial_{\ell k}u_i\,dx \\ &\quad - \int_\Omega(\partial_k u_j)\partial_{\ell j}u_i\,\partial_{\ell k}u_i\,dx \\ &\quad - \int_\Omega(\partial_\ell u_j)\partial_{kj}u_i\,\partial_{\ell k}u_i\,dx. \end{aligned} \quad (74)$$

Now sum in i, ℓ, k:

$$\frac{1}{2}\frac{d}{dt}\sum_{i,\ell,k=1}^{3}\int_\Omega |\partial_{\ell k}u_i|^2\,dx + \nu\sum_{i,\ell,k=1}^{3}\int_\Omega |\nabla\partial_{\ell k}u_i|^2\,dx$$
$$= -\sum_{i,\ell,k=1}^{3}\int_\Omega (\partial_{\ell k}u_j)\partial_j u_i\,\partial_{\ell k}u_i\,dx$$
$$-\sum_{i,\ell,k=1}^{3}\int_\Omega (\partial_{k}u_j)\partial_{\ell j} u_i\,\partial_{\ell k}u_i\,dx$$
$$-\sum_{i,\ell,k=1}^{3}\int_\Omega (\partial_{\ell}u_j)\partial_{k j} u_i\,\partial_{\ell k}u_i\,dx. \quad (75)$$

Therefore,

$$\frac{d}{dt}\left\|D^2u\right\|_{L^2}^2 + 2\nu\left\|\nabla D^2u\right\|_{L^2}^2 \le C\left\|\nabla u\right\|_{L^\infty}\left\|D^2u\right\|_{L^2}^2. \quad (76)$$

6 Third-order differentiated chain

Now let us expand one order further. Apply ∂_{abc} to the componentwise equation:

$$\partial_t(\partial_{abc}u_i) + \partial_{abc}(u_j\partial_j u_i) + \partial_i\partial_{abc}p - \nu\partial_{mm}\partial_{abc}u_i = 0. \quad (77)$$

We now open the cubic Leibniz rule one layer at a time:

$$\begin{aligned}\partial_{abc}(u_j\partial_j u_i) &= \partial_a\Big(\partial_{bc}(u_j\partial_j u_i)\Big)\\ &= \partial_a\Big((\partial_{bc}u_j)\partial_j u_i + (\partial_b u_j)\partial_{cj}u_i\\ &\quad + (\partial_c u_j)\partial_{bj}u_i + u_j\partial_{bcj}u_i\Big)\\ &= (\partial_{abc}u_j)\partial_j u_i + (\partial_{bc}u_j)\partial_{aj}u_i\\ &\quad + (\partial_{ab}u_j)\partial_{cj}u_i + (\partial_{b}u_j)\partial_{acj}u_i\\ &\quad + (\partial_{ac}u_j)\partial_{bj}u_i + (\partial_{c}u_j)\partial_{abj}u_i\\ &\quad + (\partial_{a}u_j)\partial_{bcj}u_i + u_j\partial_{abcj}u_i. \end{aligned} \quad (78)$$

Therefore,

$$\begin{aligned}&\partial_t(\partial_{abc}u_i) + u_j\partial_j(\partial_{abc}u_i) + \partial_i\partial_{abc}p - \nu\partial_{mm}\partial_{abc}u_i\\ &\quad= -(\partial_{abc}u_j)\partial_j u_i - (\partial_{bc}u_j)\partial_{aj}u_i\\ &\qquad - (\partial_{ab}u_j)\partial_{cj}u_i - (\partial_{b}u_j)\partial_{acj}u_i\\ &\qquad - (\partial_{ac}u_j)\partial_{bj}u_i - (\partial_{c}u_j)\partial_{abj}u_i\\ &\qquad - (\partial_{a}u_j)\partial_{bcj}u_i. \end{aligned} \quad (79)$$

Testing against $\partial_{abc}u_i$ and summing gives

$$\frac{1}{2}\frac{d}{dt}\sum_{i,a,b,c=1}^{3}\int_\Omega |\partial_{abc}u_i|^2\,dx + \nu\sum_{i,a,b,c=1}^{3}\int_\Omega |\nabla\partial_{abc}u_i|^2\,dx$$
$$\le C\left\|\nabla u\right\|_{L^\infty}\left\|D^3u\right\|_{L^2}^2 + C\left\|D^2u\right\|_{L^4}^2\left\|D^3u\right\|_{L^2}. \quad (80)$$

7 High-order differentiated system and multi-index commutators

We finally write the high-order differentiated equation in a more expanded way.

Start from

$$\partial_t u_i + u_j\partial_j u_i + \partial_i p - \nu\partial_{jj}u_i = 0. \quad (81)$$

Apply ∂^α:

$$\partial^\alpha(\partial_t u_i) + \partial^\alpha(u_j\partial_j u_i) + \partial^\alpha(\partial_i p) - \nu\partial^\alpha(\partial_{jj}u_i) = 0. \quad (82)$$

Using commutation of derivatives,

$$\partial_t\partial^\alpha u_i + \partial^\alpha(u_j\partial_j u_i) + \partial_i\partial^\alpha p - \nu\partial_{jj}\partial^\alpha u_i = 0. \quad (83)$$

Now expand the nonlinear term:

$$\begin{aligned}\partial^\alpha(u_j\partial_j u_i) &= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}(\partial^\beta u_j)\partial^{\alpha-\beta}\partial_j u_i\\ &= u_j\partial_j\partial^\alpha u_i + \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}(\partial^\beta u_j)\partial_j\partial^{\alpha-\beta}u_i. \end{aligned} \quad (84)$$

Thus

$$\begin{aligned}&\partial_t\partial^\alpha u_i + u_j\partial_j\partial^\alpha u_i + \partial_i\partial^\alpha p - \nu\partial_{jj}\partial^\alpha u_i\\ &\quad= -\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}(\partial^\beta u_j)\partial_j\partial^{\alpha-\beta}u_i. \end{aligned} \quad (85)$$

Multiply by $\partial^\alpha u_i$ and integrate:

$$\begin{aligned}&\int_\Omega \partial_t\partial^\alpha u_i\,\partial^\alpha u_i\,dx + \int_\Omega u_j\partial_j\partial^\alpha u_i\,\partial^\alpha u_i\,dx\\ &\quad+ \int_\Omega \partial_i\partial^\alpha p\,\partial^\alpha u_i\,dx - \nu\int_\Omega \partial_{jj}\partial^\alpha u_i\,\partial^\alpha u_i\,dx\\ &\quad= -\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega(\partial^\beta u_j)\partial_j\partial^{\alpha-\beta}u_i\,\partial^\alpha u_i\,dx. \end{aligned} \quad (86)$$

Summing over $i = 1, 2, 3$ gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\partial^\alpha u\right\|_{L^2}^2 + \nu\left\|\nabla\partial^\alpha u\right\|_{L^2}^2\\ &\quad= -\sum_{i=1}^{3}\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega(\partial^\beta u_j)\partial_j\partial^{\alpha-\beta}u_i\,\partial^\alpha u_i\,dx. \end{aligned} \quad (87)$$

Finally sum over all $|\alpha| \le m$:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le m}\left\|\partial^\alpha u\right\|_{L^2}^2 + \nu\sum_{|\alpha|\le m}\left\|\nabla\partial^\alpha u\right\|_{L^2}^2\\ &\quad= -\sum_{|\alpha|\le m}\sum_{i=1}^{3}\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega(\partial^\beta u_j)\partial_j\partial^{\alpha-\beta}u_i\,\partial^\alpha u_i\,dx. \end{aligned} \quad (88)$$

That is,

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^m}^2 + \nu\|\nabla u\|_{H^m}^2 = -\sum_{|\alpha|\le m}\sum_{i=1}^{3}\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega (\partial^\beta u_j)\partial_j\partial^{\alpha-\beta}u_i\,\partial^\alpha u_i\,dx. \quad (89)$$

We denote this multilinear quantity by

$$\Sigma_\alpha = \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega (\partial^\beta u_j)\partial_j\partial^{\alpha-\beta}u_i\,\partial^\alpha u_i\,dx. \quad (90)$$

Split it as

$$\Sigma_\alpha = \Sigma_\alpha^{(1)} + \Sigma_\alpha^{(2)}, \quad (91)$$

where

$$\Sigma_\alpha^{(1)} = \sum_{1\le|\beta|\le\lfloor m/2\rfloor}\binom{\alpha}{\beta}\int_\Omega (\partial^\beta u_j)\partial_j\partial^{\alpha-\beta}u_i\,\partial^\alpha u_i\,dx, \quad (92)$$

and

$$\Sigma_\alpha^{(2)} = \sum_{\lfloor m/2\rfloor+1\le|\beta|\le|\alpha|}\binom{\alpha}{\beta}\int_\Omega (\partial^\beta u_j)\partial_j\partial^{\alpha-\beta}u_i\,\partial^\alpha u_i\,dx. \quad (93)$$

Therefore,

$$|\Sigma_\alpha| \le C_m\|\nabla u\|_{L^\infty}\|u\|_{H^m}^2. \quad (94)$$

Hence

$$\frac{d}{dt}\|u\|_{H^m}^2 + 2\nu\|\nabla u\|_{H^m}^2 \le C_m\|\nabla u\|_{L^\infty}\|u\|_{H^m}^2. \quad (95)$$

8 Vorticity channel and stretching hierarchy

Define the vorticity by

$$\omega = \operatorname{curl} u = \nabla\times u. \quad (96)$$

Writing all components explicitly, we have

$$\omega_1 = \partial_2 u_3 - \partial_3 u_2, \quad (97)$$
$$\omega_2 = \partial_3 u_1 - \partial_1 u_3, \quad (98)$$
$$\omega_3 = \partial_1 u_2 - \partial_2 u_1. \quad (99)$$

Hence

$$\omega = \begin{pmatrix}\partial_2 u_3 - \partial_3 u_2\\ \partial_3 u_1 - \partial_1 u_3\\ \partial_1 u_2 - \partial_2 u_1\end{pmatrix}. \quad (100)$$

We now derive the vorticity equation from the componentwise Navier–Stokes system

$$\partial_t u_i + u_j\partial_j u_i + \partial_i p - \nu\partial_{jj}u_i = 0. \quad (101)$$

Take the curl of the velocity equation. Since

$$\operatorname{curl}(\nabla p) = 0, \quad (102)$$

and since

$$\operatorname{curl}(\Delta u) = \Delta(\operatorname{curl} u) = \Delta\omega, \quad (103)$$

we obtain

$$\partial_t\omega + \operatorname{curl}\big((u\cdot\nabla)u\big) - \nu\Delta\omega = 0. \quad (104)$$

Using the vector identity

$$\operatorname{curl}\big((u\cdot\nabla)u\big) = (u\cdot\nabla)\omega - (\omega\cdot\nabla)u, \quad (105)$$

we arrive at

$$\partial_t\omega + (u\cdot\nabla)\omega - \nu\Delta\omega = (\omega\cdot\nabla)u. \quad (106)$$

We now expand the transport term:

$$(u\cdot\nabla)\omega_i = u_1\partial_1\omega_i + u_2\partial_2\omega_i + u_3\partial_3\omega_i. \quad (107)$$

We also expand the stretching term:

$$(\omega\cdot\nabla)u_i = \omega_1\partial_1 u_i + \omega_2\partial_2 u_i + \omega_3\partial_3 u_i. \quad (108)$$

Therefore the three scalar vorticity equations are

$$\begin{aligned}&\partial_t\omega_1 + u_1\partial_1\omega_1 + u_2\partial_2\omega_1 + u_3\partial_3\omega_1 - \nu\partial_{11}\omega_1\\ &\quad - \nu\partial_{22}\omega_1 - \nu\partial_{33}\omega_1\\ &\quad = \omega_1\partial_1 u_1 + \omega_2\partial_2 u_1 + \omega_3\partial_3 u_1, \quad (109)\end{aligned}$$

$$\begin{aligned}&\partial_t\omega_2 + u_1\partial_1\omega_2 + u_2\partial_2\omega_2 + u_3\partial_3\omega_2 - \nu\partial_{11}\omega_2\\ &\quad - \nu\partial_{22}\omega_2 - \nu\partial_{33}\omega_2\\ &\quad = \omega_1\partial_1 u_2 + \omega_2\partial_2 u_2 + \omega_3\partial_3 u_2, \quad (110)\end{aligned}$$

$$\begin{aligned}&\partial_t\omega_3 + u_1\partial_1\omega_3 + u_2\partial_2\omega_3 + u_3\partial_3\omega_3 - \nu\partial_{11}\omega_3\\ &\quad - \nu\partial_{22}\omega_3 - \nu\partial_{33}\omega_3\\ &\quad = \omega_1\partial_1 u_3 + \omega_2\partial_2 u_3 + \omega_3\partial_3 u_3. \quad (111)\end{aligned}$$

We now test (106) against ω and integrate over Ω:

$$\begin{aligned}\int_\Omega \partial_t\omega\cdot\omega\,dx + \int_\Omega \big((u\cdot\nabla)\omega\big)\cdot\omega\,dx - \nu\int_\Omega \Delta\omega\cdot\omega\,dx\\ = \int_\Omega \big((\omega\cdot\nabla)u\big)\cdot\omega\,dx. \quad (112)\end{aligned}$$

We expand each term one by one.

For the time derivative:

$$\begin{aligned}\int_\Omega \partial_t\omega\cdot\omega\,dx &= \sum_{i=1}^{3}\int_\Omega \partial_t\omega_i\,\omega_i\,dx\\ &= \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}\int_\Omega |\omega_i|^2\,dx\\ &= \frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2. \quad (113)\end{aligned}$$

For the transport term:

$$
\begin{aligned}
\int_\Omega ((u\cdot\nabla)\omega)\cdot\omega\,dx
&= \sum_{i=1}^{3}\int_\Omega ((u\cdot\nabla)\omega_i)\omega_i\,dx \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega u_j\,\partial_j\omega_i\,\omega_i\,dx \\
&= \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega u_j\,\partial_j\left(|\omega_i|^2\right)dx \\
&= \frac{1}{2}\sum_{j=1}^{3}\int_\Omega u_j\,\partial_j\left(\sum_{i=1}^{3}|\omega_i|^2\right)dx \\
&= \frac{1}{2}\int_\Omega u\cdot\nabla|\omega|^2\,dx \\
&= -\frac{1}{2}\int_\Omega (\operatorname{div} u)\,|\omega|^2\,dx = 0. \qquad (114)
\end{aligned}
$$

For the diffusion term:

$$
\begin{aligned}
-\nu\int_\Omega \Delta\omega\cdot\omega\,dx
&= -\nu\sum_{i=1}^{3}\int_\Omega \Delta\omega_i\,\omega_i\,dx \\
&= -\nu\sum_{i=1}^{3}\int_\Omega (\partial_{11}\omega_i+\partial_{22}\omega_i+\partial_{33}\omega_i)\omega_i\,dx \\
&= \nu\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega |\partial_j\omega_i|^2\,dx \\
&= \nu\,\|\nabla\omega\|_{L^2}^2. \qquad (115)
\end{aligned}
$$

For the stretching term:

$$
\begin{aligned}
\int_\Omega ((\omega\cdot\nabla)u)\cdot\omega\,dx &= \sum_{i=1}^{3}\int_\Omega ((\omega\cdot\nabla)u_i)\omega_i\,dx \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega \omega_j\,\partial_j u_i\,\omega_i\,dx. \qquad (116)
\end{aligned}
$$

Thus

$$
\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2+\nu\,\|\nabla\omega\|_{L^2}^2=\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega \omega_j\,\partial_j u_i\,\omega_i\,dx. \qquad (117)
$$

Estimating the right-hand side:

$$
\begin{aligned}
&\left|\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega \omega_j\,\partial_j u_i\,\omega_i\,dx\right| \\
&\le \sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega |\omega_j|\,|\partial_j u_i|\,|\omega_i|\,dx \\
&\le \|\nabla u\|_{L^\infty}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega |\omega_j|\,|\omega_i|\,dx \\
&\le C\,\|\nabla u\|_{L^\infty}\,\|\omega\|_{L^2}^2. \qquad (118)
\end{aligned}
$$

Hence

$$
\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2+\nu\,\|\nabla\omega\|_{L^2}^2\le\|\nabla u\|_{L^\infty}\,\|\omega\|_{L^2}^2. \qquad (119)
$$

High-order differentiated vorticity equation

We now apply ∂^α, with $|\alpha|\le m-1$, to the scalar vorticity equation

$$
\partial_t\omega_i+u_j\partial_j\omega_i-\nu\partial_{jj}\omega_i=\omega_j\partial_j u_i. \qquad (120)
$$

Then

$$
\partial_t\partial^\alpha\omega_i+\partial^\alpha(u_j\partial_j\omega_i)-\nu\partial_{jj}\partial^\alpha\omega_i=\partial^\alpha(\omega_j\partial_j u_i). \qquad (121)
$$

Expand the transport term:

$$
\partial^\alpha(u_j\partial_j\omega_i)=u_j\partial_j\partial^\alpha\omega_i+\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}(\partial^\beta u_j)\,\partial_j\partial^{\alpha-\beta}\omega_i. \qquad (122)
$$

Expand the stretching term:

$$
\partial^\alpha(\omega_j\partial_j u_i)=\omega_j\partial_j\partial^\alpha u_i+\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}(\partial^\beta\omega_j)\,\partial_j\partial^{\alpha-\beta}u_i. \qquad (123)
$$

Therefore,

$$
\begin{aligned}
&\partial_t\partial^\alpha\omega_i+u_j\partial_j\partial^\alpha\omega_i-\nu\partial_{jj}\partial^\alpha\omega_i \\
&\quad=\omega_j\partial_j\partial^\alpha u_i-\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}(\partial^\beta u_j)\,\partial_j\partial^{\alpha-\beta}\omega_i \\
&\qquad+\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}(\partial^\beta\omega_j)\,\partial_j\partial^{\alpha-\beta}u_i. \qquad (124)
\end{aligned}
$$

Now multiply by $\partial^\alpha \omega_i$ and integrate:

$$\begin{aligned}
&\int_\Omega \partial_t \partial^\alpha \omega_i \, \partial^\alpha \omega_i \, dx + \int_\Omega u_j \partial_j \partial^\alpha \omega_i \, \partial^\alpha \omega_i \, dx \\
&\quad - \nu \int_\Omega \partial_{jj} \partial^\alpha \omega_i \, \partial^\alpha \omega_i \, dx \\
&\quad = \int_\Omega \omega_j \partial_j \partial^\alpha u_i \, \partial^\alpha \omega_i \, dx \\
&\qquad - \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \int_\Omega (\partial^\beta u_j) \, \partial_j \partial^{\alpha-\beta} \omega_i \, \partial^\alpha \omega_i \, dx \\
&\qquad + \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \int_\Omega (\partial^\beta \omega_j) \, \partial_j \partial^{\alpha-\beta} u_i \, \partial^\alpha \omega_i \, dx.
\end{aligned} \tag{125}$$

Summing in i and then in $|\alpha| \le m-1$, and using the same transport cancellation as before, we arrive at

$$\begin{aligned}
&\frac{d}{dt} \|\omega\|_{H^{m-1}}^2 + \nu \|\nabla \omega\|_{H^{m-1}}^2 \\
&\quad \le C_m \|\omega\|_{L^\infty} \|u\|_{H^m}^2 + C_m \|\nabla u\|_{L^\infty} \|\omega\|_{H^{m-1}}^2 \\
&\quad \le C_m \big(\|\nabla u\|_{L^\infty} + \|\omega\|_{L^\infty} \big) \|u\|_{H^m}^2 .
\end{aligned} \tag{126}$$

Refined regularity criteria based on vorticity and related quantities have been studied extensively; see, for example, [12] for critical regularity conditions and [13] for recent Green's function estimates in the vorticity formulation.

9 Pressure as a nonlocal quadratic channel

We begin with the incompressible Navier–Stokes system written componentwise:

$$\partial_t u_i + u_j \partial_j u_i + \partial_i p - \nu \partial_{jj} u_i = 0, \qquad \partial_i u_i = 0. \tag{127}$$

We now take the divergence of the momentum equation. Applying ∂_i to the i-th equation and summing over i, we obtain

$$\partial_i(\partial_t u_i) + \partial_i(u_j \partial_j u_i) + \partial_i \partial_i p - \nu \partial_i \partial_{jj} u_i = 0. \tag{128}$$

Since $\partial_i u_i = 0$, we have

$$\partial_t(\partial_i u_i) = 0, \tag{129}$$

and also

$$\partial_i \partial_{jj} u_i = \partial_{jj} \partial_i u_i = \partial_{jj}(0) = 0. \tag{130}$$

Therefore

$$\partial_i(u_j \partial_j u_i) + \Delta p = 0. \tag{131}$$

Equivalently,

$$-\Delta p = \partial_i(u_j \partial_j u_i). \tag{132}$$

We now expand the nonlinear term in detail:

$$\partial_i(u_j \partial_j u_i) = (\partial_i u_j)(\partial_j u_i) + u_j \partial_i \partial_j u_i. \tag{133}$$

But since $\partial_i u_i = 0$, we also have

$$\partial_j(\partial_i u_i) = 0, \qquad u_j \partial_j(\partial_i u_i) = 0. \tag{134}$$

Thus

$$u_j \partial_i \partial_j u_i = u_j \partial_j \partial_i u_i = u_j \partial_j(0) = 0. \tag{135}$$

Hence

$$-\Delta p = (\partial_i u_j)(\partial_j u_i). \tag{136}$$

Another classical equivalent form is

$$-\Delta p = \partial_i \partial_j (u_i u_j). \tag{137}$$

Indeed,

$$\begin{aligned}
\partial_i \partial_j (u_i u_j) &= \partial_i \Big((\partial_j u_i) u_j + u_i (\partial_j u_j) \Big) \\
&= \partial_i \Big((\partial_j u_i) u_j \Big) + \partial_i \Big(u_i (\partial_j u_j) \Big) \\
&= \partial_i \Big((\partial_j u_i) u_j \Big),
\end{aligned} \tag{138}$$

because $\partial_j u_j = 0$. Then

$$\begin{aligned}
\partial_i \Big((\partial_j u_i) u_j \Big) &= (\partial_i \partial_j u_i) u_j + (\partial_j u_i)(\partial_i u_j) \\
&= u_j \partial_j (\partial_i u_i) + (\partial_j u_i)(\partial_i u_j) \\
&= (\partial_j u_i)(\partial_i u_j),
\end{aligned} \tag{139}$$

which matches (136).

We now expand (137) completely in Cartesian coordinates:

$$\begin{aligned}
-\Delta p = {}& \partial_{11}(u_1 u_1) + \partial_{12}(u_1 u_2) + \partial_{13}(u_1 u_3) \\
&+ \partial_{21}(u_2 u_1) + \partial_{22}(u_2 u_2) + \partial_{23}(u_2 u_3) \\
&+ \partial_{31}(u_3 u_1) + \partial_{32}(u_3 u_2) + \partial_{33}(u_3 u_3).
\end{aligned} \tag{140}$$

Since mixed derivatives commute and $u_i u_j = u_j u_i$, this becomes

$$\begin{aligned}
-\Delta p = {}& \partial_{11}(u_1^2) + \partial_{22}(u_2^2) + \partial_{33}(u_3^2) \\
&+ 2\partial_{12}(u_1 u_2) + 2\partial_{13}(u_1 u_3) + 2\partial_{23}(u_2 u_3).
\end{aligned} \tag{141}$$

We now open each one of these six blocks as slowly as possible.

For the first diagonal block:

$$\begin{aligned}
\partial_{11}(u_1^2) &= \partial_1 \big(\partial_1 (u_1^2) \big) \\
&= \partial_1 \big(2 u_1 \partial_1 u_1 \big) \\
&= 2(\partial_1 u_1)(\partial_1 u_1) + 2 u_1 \partial_{11} u_1 \\
&= 2(\partial_1 u_1)^2 + 2 u_1 \partial_{11} u_1.
\end{aligned} \tag{142}$$

For the second diagonal block:

$$\begin{aligned}\partial_{22}(u_2^2) &= \partial_2\big(\partial_2(u_2^2)\big)\\ &= \partial_2\big(2u_2\partial_2 u_2\big)\\ &= 2(\partial_2 u_2)^2 + 2u_2\partial_{22}u_2. \end{aligned} \tag{143}$$

For the third diagonal block:

$$\begin{aligned}\partial_{33}(u_3^2) &= \partial_3\big(\partial_3(u_3^2)\big)\\ &= \partial_3\big(2u_3\partial_3 u_3\big)\\ &= 2(\partial_3 u_3)^2 + 2u_3\partial_{33}u_3. \end{aligned} \tag{144}$$

Now the first mixed block:

$$\begin{aligned}\partial_{12}(u_1u_2) &= \partial_1\big(\partial_2(u_1u_2)\big)\\ &= \partial_1\big((\partial_2 u_1)u_2 + u_1(\partial_2 u_2)\big)\\ &= (\partial_{12}u_1)u_2 + (\partial_2 u_1)(\partial_1 u_2)\\ &\quad + (\partial_1 u_1)(\partial_2 u_2) + u_1\partial_{12}u_2. \end{aligned} \tag{145}$$

The second mixed block:

$$\begin{aligned}\partial_{13}(u_1u_3) &= \partial_1\big(\partial_3(u_1u_3)\big)\\ &= \partial_1\big((\partial_3 u_1)u_3 + u_1(\partial_3 u_3)\big)\\ &= (\partial_{13}u_1)u_3 + (\partial_3 u_1)(\partial_1 u_3)\\ &\quad + (\partial_1 u_1)(\partial_3 u_3) + u_1\partial_{13}u_3. \end{aligned} \tag{146}$$

The third mixed block:

$$\begin{aligned}\partial_{23}(u_2u_3) &= \partial_2\big(\partial_3(u_2u_3)\big)\\ &= \partial_2\big((\partial_3 u_2)u_3 + u_2(\partial_3 u_3)\big)\\ &= (\partial_{23}u_2)u_3 + (\partial_3 u_2)(\partial_2 u_3)\\ &\quad + (\partial_2 u_2)(\partial_3 u_3) + u_2\partial_{23}u_3. \end{aligned} \tag{147}$$

Collecting (142)–(147), we obtain

$$\begin{aligned}-\Delta p = {}& 2(\partial_1 u_1)^2 + 2u_1\partial_{11}u_1 + 2(\partial_2 u_2)^2 + 2u_2\partial_{22}u_2\\ &+ 2(\partial_3 u_3)^2 + 2u_3\partial_{33}u_3\\ &+ 2(\partial_{12}u_1)u_2 + 2(\partial_2 u_1)(\partial_1 u_2) + 2(\partial_1 u_1)(\partial_2 u_2)\\ &+ 2u_1\partial_{12}u_2 + 2(\partial_{13}u_1)u_3 + 2(\partial_3 u_1)(\partial_1 u_3)\\ &+ 2(\partial_1 u_1)(\partial_3 u_3) + 2u_1\partial_{13}u_3 + 2(\partial_{23}u_2)u_3\\ &+ 2(\partial_3 u_2)(\partial_2 u_3) + 2(\partial_2 u_2)(\partial_3 u_3) + 2u_2\partial_{23}u_3. \end{aligned} \tag{148}$$

We may also write the quadratic source as the full 3×3 contraction

$$\begin{aligned}(\partial_i u_j)(\partial_j u_i) = {}& (\partial_1 u_1)^2 + (\partial_2 u_2)^2 + (\partial_3 u_3)^2\\ &+ (\partial_1 u_2)(\partial_2 u_1) + (\partial_2 u_1)(\partial_1 u_2)\\ &+ (\partial_1 u_3)(\partial_3 u_1) + (\partial_3 u_1)(\partial_1 u_3)\\ &+ (\partial_2 u_3)(\partial_3 u_2) + (\partial_3 u_2)(\partial_2 u_3). \end{aligned} \tag{149}$$

We now reconstruct the pressure through the inverse Laplacian:

$$p = (-\Delta)^{-1}\partial_i\partial_j(u_iu_j), \tag{150}$$

and therefore

$$\nabla p = \nabla(-\Delta)^{-1}\partial_i\partial_j(u_iu_j). \tag{151}$$

The operator

$$\nabla(-\Delta)^{-1}\partial_i\partial_j \tag{152}$$

is a Calderón–Zygmund singular integral operator, so that

$$\|\nabla p\|_{L^q} \le C_q \|u_iu_j\|_{W^{1,q}}, \qquad 1 < q < \infty. \tag{153}$$

We now open the $W^{1,q}$ norm more explicitly:

$$\|u_iu_j\|_{W^{1,q}} = \|u_iu_j\|_{L^q} + \sum_{k=1}^{3}\|\partial_k(u_iu_j)\|_{L^q}. \tag{154}$$

Now

$$\partial_k(u_iu_j) = (\partial_k u_i)u_j + u_i(\partial_k u_j). \tag{155}$$

Hence

$$\begin{aligned}\|\partial_k(u_iu_j)\|_{L^q} &\le \|(\partial_k u_i)u_j\|_{L^q} + \|u_i(\partial_k u_j)\|_{L^q}\\ &\le \|\partial_k u_i\|_{L^q}\|u_j\|_{L^\infty} + \|u_i\|_{L^\infty}\|\partial_k u_j\|_{L^q}. \end{aligned} \tag{156}$$

Summing in k,

$$\begin{aligned}&\sum_{k=1}^{3}\|\partial_k(u_iu_j)\|_{L^q}\\ &\quad\le \sum_{k=1}^{3}\|\partial_k u_i\|_{L^q}\|u_j\|_{L^\infty} + \sum_{k=1}^{3}\|u_i\|_{L^\infty}\|\partial_k u_j\|_{L^q}\\ &\quad\le \|u_j\|_{L^\infty}\sum_{k=1}^{3}\|\partial_k u_i\|_{L^q} + \|u_i\|_{L^\infty}\sum_{k=1}^{3}\|\partial_k u_j\|_{L^q}. \end{aligned} \tag{157}$$

Also,

$$\|u_iu_j\|_{L^q} \le \|u_i\|_{L^\infty}\|u_j\|_{L^q}. \tag{158}$$

Combining these,

$$\|\nabla p\|_{L^q} \le C_q\Big(\|u\|_{L^\infty}\|u\|_{L^q} + \|u\|_{L^\infty}\|\nabla u\|_{L^q} \tag{159}$$

$$+ \|\nabla u\|_{L^\infty}\|u\|_{L^q}\Big). \tag{160}$$

Therefore, for $m > \frac{5}{2}$ and $q > 3$,

$$\|\nabla p\|_{L^q} \le C_m\|u\|_{H^m}\|\nabla u\|_{L^\infty} + C_m\|u\|_{H^m}^2. \tag{161}$$

10 Higher pressure derivatives: third, fourth, and fifth gradient blocks

Differentiate (137) once:

$$-\Delta(\partial_k p) = \partial_k\partial_i\partial_j(u_iu_j). \quad (162)$$

Differentiate again:

$$-\Delta(\partial_{k\ell} p) = \partial_{k\ell}\partial_i\partial_j(u_iu_j). \quad (163)$$

Thus, by Calderón–Zygmund,

$$\begin{aligned} \|\nabla^2 p\|_{L^q} &\le C_q \|\nabla(u\otimes u)\|_{L^q}, \\ \|\nabla^3 p\|_{L^q} &\le C_q \|\nabla^2(u\otimes u)\|_{L^q}. \end{aligned} \quad (164)$$

We now expand the second derivative of the product in full detail:

$$\begin{aligned} \partial_{k\ell}(u_iu_j) &= \partial_k\Big(\partial_\ell(u_iu_j)\Big) \\ &= \partial_k\Big((\partial_\ell u_i)u_j + u_i(\partial_\ell u_j)\Big) \\ &= (\partial_{k\ell}u_i)u_j + (\partial_\ell u_i)(\partial_k u_j) \\ &\quad + (\partial_k u_i)(\partial_\ell u_j) + u_i(\partial_{k\ell}u_j). \end{aligned} \quad (165)$$

Thus

$$\begin{aligned} \|\nabla^3 p\|_{L^q} &\le C_q\Big(\|(\nabla^2 u)\,u\|_{L^q} + \|(\nabla u)(\nabla u)\|_{L^q} \\ &\quad + \|u\,\nabla^2 u\|_{L^q}\Big) \\ &\le C_q\Big(\|u\|_{L^\infty}\|\nabla^2 u\|_{L^q} + \|\nabla u\|_{L^\infty}\|\nabla u\|_{L^q} \\ &\quad + \|\nabla^2 u\|_{L^q}\|u\|_{L^\infty}\Big). \end{aligned} \quad (166)$$

Now differentiate twice more:

$$-\Delta(\partial_{abk\ell} p) = \partial_{abk\ell}\partial_i\partial_j(u_iu_j). \quad (167)$$

We now expand $\partial_{abk\ell}(u_iu_j)$ more slowly:

$$\begin{aligned} \partial_{abk\ell}(u_iu_j) &= \partial_a\Big(\partial_{bk\ell}(u_iu_j)\Big) \\ &= \partial_a\Big((\partial_{bk\ell}u_i)u_j + (\partial_{bk}u_i)(\partial_\ell u_j) \\ &\quad + (\partial_{b\ell}u_i)(\partial_k u_j) + (\partial_b u_i)(\partial_{k\ell}u_j) \\ &\quad + (\partial_{k\ell}u_i)(\partial_b u_j) + (\partial_k u_i)(\partial_{b\ell}u_j) \\ &\quad + (\partial_\ell u_i)(\partial_{bk}u_j) + u_i(\partial_{bk\ell}u_j)\Big). \end{aligned} \quad (168)$$

Differentiating term by term:

$$\begin{aligned} \partial_{abk\ell}(u_iu_j) &= (\partial_{abk\ell}u_i)u_j + (\partial_{bk\ell}u_i)(\partial_a u_j) \\ &\quad + (\partial_{abk}u_i)(\partial_\ell u_j) + (\partial_{bk}u_i)(\partial_{a\ell}u_j) \\ &\quad + (\partial_{ab\ell}u_i)(\partial_k u_j) + (\partial_{b\ell}u_i)(\partial_{ak}u_j) \\ &\quad + (\partial_{ab}u_i)(\partial_{k\ell}u_j) + (\partial_b u_i)(\partial_{ak\ell}u_j) \\ &\quad + (\partial_{ak\ell}u_i)(\partial_b u_j) + (\partial_{k\ell}u_i)(\partial_{ab}u_j) \\ &\quad + (\partial_{ak}u_i)(\partial_{b\ell}u_j) + (\partial_k u_i)(\partial_{ab\ell}u_j) \\ &\quad + (\partial_{a\ell}u_i)(\partial_{bk}u_j) + (\partial_\ell u_i)(\partial_{abk}u_j) \\ &\quad + (\partial_a u_i)(\partial_{bk\ell}u_j) + u_i(\partial_{abk\ell}u_j). \end{aligned} \quad (169)$$

Therefore

$$\begin{aligned} \|\nabla^5 p\|_{L^q} &\le C_q\Big(\|u\|_{L^\infty}\|\nabla^4 u\|_{L^q} + \|\nabla u\|_{L^\infty}\|\nabla^3 u\|_{L^q} \\ &\quad + \|\nabla^2 u\|_{L^{2q}}\|\nabla^2 u\|_{L^{2q}} + \|\nabla^3 u\|_{L^q}\|\nabla u\|_{L^\infty} \\ &\quad + \|\nabla^4 u\|_{L^q}\|u\|_{L^\infty}\Big). \end{aligned} \quad (170)$$

For one more large formal block, write

$$\begin{aligned} &\partial_{abcdrs}(u_iu_j) \\ &= (\partial_{abcdrs}u_i)u_j + (\partial_{abcdr}u_i)(\partial_s u_j) + (\partial_{abcds}u_i)(\partial_r u_j) \\ &\quad + (\partial_{abcrs}u_i)(\partial_d u_j) + (\partial_{abdrs}u_i)(\partial_c u_j) + (\partial_{acdrs}u_i)(\partial_b u_j) \\ &\quad + (\partial_{bcdrs}u_i)(\partial_a u_j) + (\partial_{abcd}u_i)(\partial_{rs}u_j) + (\partial_{abcr}u_i)(\partial_{ds}u_j) \\ &\quad + (\partial_{abcs}u_i)(\partial_{dr}u_j) + (\partial_{abdr}u_i)(\partial_{cs}u_j) + (\partial_{abds}u_i)(\partial_{cr}u_j) \\ &\quad + (\partial_{acdr}u_i)(\partial_{bs}u_j) + (\partial_{acds}u_i)(\partial_{br}u_j) + (\partial_{bcdr}u_i)(\partial_{as}u_j) \\ &\quad + (\partial_{bcds}u_i)(\partial_{ar}u_j) + \cdots + u_i(\partial_{abcdrs}u_j). \end{aligned} \quad (171)$$

11 Anisotropic differentiated hierarchy

We now differentiate the Navier–Stokes system with respect to the vertical variable x_3:

$$\partial_t(\partial_3 u) + \partial_3\big((u\cdot\nabla)u\big) + \nabla(\partial_3 p) - \nu\Delta(\partial_3 u) = 0. \quad (172)$$

Expand the nonlinear derivative:

$$\begin{aligned} \partial_3\big((u\cdot\nabla)u_i\big) &= \partial_3\Big(u_1\partial_1 u_i + u_2\partial_2 u_i + u_3\partial_3 u_i\Big) \\ &= (\partial_3 u_1)\partial_1 u_i + u_1\partial_3\partial_1 u_i \\ &\quad + (\partial_3 u_2)\partial_2 u_i + u_2\partial_3\partial_2 u_i \\ &\quad + (\partial_3 u_3)\partial_3 u_i + u_3\partial_{33}u_i. \end{aligned} \quad (173)$$

Grouping terms,

$$\partial_3\big((u\cdot\nabla)u_i\big) = (\partial_3 u_j)\partial_j u_i + u_j\partial_j(\partial_3 u_i). \quad (174)$$

Therefore,

$$\partial_3\big((u\cdot\nabla)u\big) = (\partial_3 u\cdot\nabla)u + (u\cdot\nabla)(\partial_3 u). \quad (175)$$

Now apply ∂^α, with $|\alpha| \le m-1$, to (172):

$$\partial_t \partial^\alpha \partial_3 u + \partial^\alpha \partial_3\big((u\cdot\nabla)u\big) + \nabla\partial^\alpha\partial_3 p - \nu\Delta\partial^\alpha\partial_3 u = 0. \tag{176}$$

Testing against $\partial^\alpha\partial_3 u$ and summing over $|\alpha|\le m-1$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|\partial_3 u\|^2_{H^{m-1}} + \nu\|\nabla\partial_3 u\|^2_{H^{m-1}}\\ &= -\sum_{|\alpha|\le m-1}\int_\Omega \partial^\alpha\Big((\partial_3 u\cdot\nabla)u\Big)\cdot\partial^\alpha\partial_3 u\,dx\\ &\quad -\sum_{|\alpha|\le m-1}\int_\Omega \partial^\alpha\Big((u\cdot\nabla)(\partial_3 u)\Big)\cdot\partial^\alpha\partial_3 u\,dx.\end{aligned} \tag{177}$$

The transport term is handled exactly as before and cancels after integration by parts, leaving

$$\tfrac{d}{dt}\|\partial_3 u\|^2_{H^{m-1}} + 2\nu\|\nabla\partial_3 u\|^2_{H^{m-1}} \le C_m\|\nabla u\|_{L^\infty}\|\partial_3 u\|^2_{H^{m-1}}. \tag{178}$$

Now consider repeated vertical differentiation. For $r\ge 1$,

$$\partial_t(\partial_3^r u_i) + \partial_3^r(u_j\partial_j u_i) + \partial_i\partial_3^r p - \nu\partial_{jj}\partial_3^r u_i = 0. \tag{179}$$

Using Leibniz,

$$\partial_3^r(u_j\partial_j u_i) = \sum_{\ell=0}^{r}\binom{r}{\ell}(\partial_3^\ell u_j)\,\partial_j\partial_3^{r-\ell}u_i. \tag{180}$$

Thus the full equation becomes

$$\begin{aligned}\partial_t(\partial_3^r u_i) &+ u_j\partial_j\partial_3^r u_i + \partial_i\partial_3^r p - \nu\partial_{jj}\partial_3^r u_i\\ &= -\sum_{\ell=1}^{r}\binom{r}{\ell}(\partial_3^\ell u_j)\,\partial_j\partial_3^{r-\ell}u_i.\end{aligned} \tag{181}$$

Testing against $\partial_3^r u_i$, integrating, and summing in i,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|\partial_3^r u\|^2_{L^2} + \nu\|\nabla\partial_3^r u\|^2_{L^2}\\ &= -\sum_{\ell=1}^{r}\binom{r}{\ell}\int_\Omega(\partial_3^\ell u_j)\,\partial_j\partial_3^{r-\ell}u_i\,\partial_3^r u_i\,dx.\end{aligned} \tag{182}$$

Summing for $r=1,\dots,m$,

$$\tfrac{d}{dt}\textstyle\sum_{r=1}^m\|\partial_3^r u\|^2_{L^2} + 2\nu\sum_{r=1}^m\|\nabla\partial_3^r u\|^2_{L^2} \le C_m\|\nabla u\|_{L^\infty}\|u\|^2_{H^m}. \tag{183}$$

Finally, for mixed anisotropic derivatives we write

$$\begin{aligned}\partial_1^a&\partial_2^b\partial_3^c(u_j\partial_j u_i)\\ &= \sum_{\beta_1=0}^{a}\sum_{\beta_2=0}^{b}\sum_{\beta_3=0}^{c}\binom{a}{\beta_1}\binom{b}{\beta_2}\binom{c}{\beta_3}\partial_1^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3}u_j\\ &\quad\cdot\partial_j\partial_1^{a-\beta_1}\partial_2^{b-\beta_2}\partial_3^{c-\beta_3}u_i.\end{aligned} \tag{184}$$

For $(a,b,c)=(5,1,3)$ we open it a bit more:

$$\begin{aligned}&\partial_1^5\partial_2\partial_3^3(u_j\partial_j u_i)\\ &= u_j\partial_j\partial_1^5\partial_2\partial_3^3 u_i + 5(\partial_1 u_j)\partial_j\partial_1^4\partial_2\partial_3^3 u_i + (\partial_2 u_j)\partial_j\partial_1^5\partial_3^3 u_i\\ &\quad + 3(\partial_3 u_j)\partial_j\partial_1^5\partial_2\partial_3^2 u_i + 10(\partial_{11}u_j)\partial_j\partial_1^3\partial_2\partial_3^3 u_i\\ &\quad + 5(\partial_{12}u_j)\partial_j\partial_1^4\partial_3^3 u_i + 15(\partial_{13}u_j)\partial_j\partial_1^4\partial_2\partial_3^2 u_i\\ &\quad + 3(\partial_{23}u_j)\partial_j\partial_1^5\partial_3^2 u_i + 3(\partial_{33}u_j)\partial_j\partial_1^5\partial_2\partial_3 u_i + \cdots.\end{aligned} \tag{185}$$

12 Kernel representation and differentiated Biot–Savart blocks

Starting from the Biot–Savart representation,

$$u_i(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\epsilon_{ijk}\frac{x_j-y_j}{|x-y|^3}\omega_k(y)\,dy. \tag{186}$$

Differentiating with respect to x_ℓ (derivative passes under the integral),

$$\begin{aligned}\partial_\ell u_i(x) &= -\frac{1}{4\pi}\partial_\ell\int_{\mathbb{R}^3}\epsilon_{ijk}\frac{x_j-y_j}{|x-y|^3}\omega_k(y)\,dy\\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\epsilon_{ijk}\partial_\ell\left(\frac{x_j-y_j}{|x-y|^3}\right)\omega_k(y)\,dy\\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\epsilon_{ijk}\partial_\ell\left((x_j-y_j)\,|x-y|^{-3}\right)\omega_k(y)\,dy.\end{aligned} \tag{187}$$

Applying the product rule,

$$\begin{aligned}\partial_\ell&\left((x_j-y_j)\,|x-y|^{-3}\right)\\ &= \partial_\ell(x_j-y_j)\,|x-y|^{-3} + (x_j-y_j)\,\partial_\ell(|x-y|^{-3})\\ &= \delta_{j\ell}|x-y|^{-3} + (x_j-y_j)\,\partial_\ell(|x-y|^{-3}).\end{aligned} \tag{188}$$

Differentiating the radial kernel,

$$\begin{aligned}\partial_\ell&(|x-y|^{-3})\\ &= \partial_\ell\left[\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\right)^{-3/2}\right]\\ &= -\frac{3}{2}|x-y|^{-5}\cdot 2(x_\ell-y_\ell) = -3(x_\ell-y_\ell)|x-y|^{-5}.\end{aligned} \tag{189}$$

Substituting back,

$$\partial_\ell\left(\frac{x_j-y_j}{|x-y|^3}\right) = \frac{\delta_{j\ell}}{|x-y|^3} - 3\frac{(x_j-y_j)(x_\ell-y_\ell)}{|x-y|^5}. \tag{190}$$

Therefore,

$$\begin{aligned}&\partial_\ell u_i(x)\\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\epsilon_{ijk}\left[\frac{\delta_{j\ell}}{|x-y|^3} - 3\frac{(x_j-y_j)(x_\ell-y_\ell)}{|x-y|^5}\right]\omega_k(y)\,dy.\end{aligned} \tag{191}$$

Now differentiating again (second derivative),

$$\partial_{m\ell} u_i(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \epsilon_{ijk} \partial_m \left[\frac{\delta_{j\ell}}{|x-y|^3} - 3\frac{(x_j-y_j)(x_\ell-y_\ell)}{|x-y|^5} \right] \omega_k(y)\, dy. \tag{192}$$

Applying product rule inside,

$$\begin{aligned}\partial_m \left((x_j-y_j)(x_\ell-y_\ell)|x-y|^{-5}\right) &= \delta_{jm}(x_\ell-y_\ell)|x-y|^{-5} + (x_j-y_j)\delta_{\ell m}|x-y|^{-5} \\ &\quad + (x_j-y_j)(x_\ell-y_\ell)\partial_m(|x-y|^{-5}).\end{aligned} \tag{193}$$

Differentiating again the kernel,

$$\partial_m(|x-y|^{-5}) = -5(x_m-y_m)|x-y|^{-7}. \tag{194}$$

Substituting,

$$\begin{aligned}\partial_m \left((x_j-y_j)(x_\ell-y_\ell)|x-y|^{-5}\right) &= \delta_{jm}(x_\ell-y_\ell)|x-y|^{-5} + \delta_{\ell m}(x_j-y_j)|x-y|^{-5} \\ &\quad - 5(x_j-y_j)(x_\ell-y_\ell)(x_m-y_m)|x-y|^{-7}.\end{aligned} \tag{195}$$

Substituting into second derivative,

$$\begin{aligned}\partial_m\partial_\ell \left(\frac{x_j-y_j}{|x-y|^3}\right) &= -3\delta_{j\ell}(x_m-y_m)|x-y|^{-5} \\ &\quad - 3\delta_{jm}(x_\ell-y_\ell)|x-y|^{-5} \\ &\quad - 3\delta_{\ell m}(x_j-y_j)|x-y|^{-5} \\ &\quad + 15(x_j-y_j)(x_\ell-y_\ell)(x_m-y_m)|x-y|^{-7}.\end{aligned} \tag{196}$$

Thus,

$$\begin{aligned}\partial_{m\ell} u_i(x) &= -\frac{1}{4\pi} \int_{\mathbb{R}^3} \epsilon_{ijk} \Big[-3\delta_{j\ell}(x_m-y_m)|x-y|^{-5} \\ &\quad - 3\delta_{jm}(x_\ell-y_\ell)|x-y|^{-5} - 3\delta_{\ell m}(x_j-y_j)|x-y|^{-5} \\ &\quad + 15(x_j-y_j)(x_\ell-y_\ell)(x_m-y_m)|x-y|^{-7}\Big] \omega_k(y)\, dy.\end{aligned} \tag{197}$$

Before applying the Bony decomposition, recall that the main obstruction in the energy estimate comes from the interaction between high and low frequencies. The decomposition separates these contributions and allows us to apply optimal estimates in Besov spaces.

Applying Bony decomposition,

$$u_j\partial_j u_i = T_{u_j}(\partial_j u_i) + T_{\partial_j u_i}(u_j) + R(u_j, \partial_j u_i), \tag{198}$$

where

$$T_f g = \sum_{k\ge -1} S_{k-1} f\, \Delta_k g, \qquad R(f,g) = \sum_{|k-\ell|\le 1} \Delta_k f\, \Delta_\ell g. \tag{199}$$

Expanding explicitly,

$$\begin{aligned}u_j\partial_j u_i = \sum_{k\ge -1} S_{k-1}u_j\, \Delta_k(\partial_j u_i) + \sum_{k\ge -1} S_{k-1}(\partial_j u_i)\, \Delta_k u_j \\ + \sum_{|k-\ell|\le 1} \Delta_k u_j\, \Delta_\ell(\partial_j u_i).\end{aligned} \tag{200}$$

Differentiating (Leibniz rule in frequency space),

$$\begin{aligned}\partial^\alpha(u_j\partial_j u_i) &= \partial^\alpha \left(\sum_{k\ge -1} S_{k-1}u_j\, \Delta_k(\partial_j u_i)\right) \\ &\quad + \partial^\alpha \left(\sum_{k\ge -1} S_{k-1}(\partial_j u_i)\, \Delta_k u_j\right) \\ &\quad + \partial^\alpha \left(\sum_{|k-\ell|\le 1} \Delta_k u_j\, \Delta_\ell(\partial_j u_i)\right).\end{aligned} \tag{201}$$

Testing against $\partial^\alpha u_i$ and integrating,

$$\begin{aligned}\int_\Omega \partial^\alpha(u_j\partial_j u_i)\, \partial^\alpha u_i\, dx &= (\text{low-high interaction}) \\ &\quad + (\text{high-low interaction}) \\ &\quad + (\text{high-high interaction}).\end{aligned} \tag{202}$$

Applying Hölder + Bernstein + Sobolev embedding,

$$\left| \int_\Omega \partial^\alpha(u_j\partial_j u_i)\, \partial^\alpha u_i\, dx \right| \le C_m \|\nabla u\|_{L^\infty} \|u\|_{H^m}^2 . \tag{203}$$

13 Bootstrap variable and conditional global scheme

Define the bootstrap functional

$$\mathcal{X}_m(t) := \|u(t)\|_{H^m}^2 + \mathcal{M}_m(t). \tag{204}$$

Differentiating in time,

$$\begin{aligned}\mathcal{X}_m'(t) &= \frac{d}{dt}\|u(t)\|_{H^m}^2 + \mathcal{M}_m'(t)\\ &\leq C_m\|\nabla u(t)\|_{L^\infty}\|u(t)\|_{H^m}^2\\ &\quad + C_m(1+\mathcal{M}_m(t))\mathcal{M}_m(t)\\ &\quad + C_m\|u(t)\|_{H^m}^2,\end{aligned} \tag{205}$$

where we used the high-order energy inequality together with the differential control on $\mathcal{M}_m(t)$.

Using the bound (84),

$$\|\nabla u(t)\|_{L^\infty} \leq C_m(1+\mathcal{M}_m(t)), \tag{206}$$

we substitute into (205).

Expanding term by term,

$$\begin{aligned}\mathcal{X}_m'(t) &\leq C_m(1+\mathcal{M}_m(t))\|u(t)\|_{H^m}^2\\ &\quad + C_m(1+\mathcal{M}_m(t))\mathcal{M}_m(t) + C_m\|u(t)\|_{H^m}^2\\ &= C_m\|u(t)\|_{H^m}^2 + C_m\mathcal{M}_m(t)\|u(t)\|_{H^m}^2\\ &\quad + C_m\mathcal{M}_m(t) + C_m\mathcal{M}_m(t)^2 + C_m\|u(t)\|_{H^m}^2\\ &= 2C_m\|u(t)\|_{H^m}^2 + C_m\mathcal{M}_m(t)\|u(t)\|_{H^m}^2\\ &\quad + C_m\mathcal{M}_m(t) + C_m\mathcal{M}_m(t)^2.\end{aligned} \tag{207}$$

Since

$$\mathcal{X}_m(t) = \|u(t)\|_{H^m}^2 + \mathcal{M}_m(t), \tag{208}$$

we have the simple bounds

$$\|u(t)\|_{H^m}^2 \leq \mathcal{X}_m(t), \qquad \mathcal{M}_m(t) \leq \mathcal{X}_m(t). \tag{209}$$

Substituting these into (207),

$$\begin{aligned}\mathcal{X}_m'(t) &\leq 2C_m\mathcal{X}_m(t) + C_m\mathcal{X}_m(t)^2\\ &\quad + C_m\mathcal{X}_m(t) + C_m\mathcal{X}_m(t)^2\\ &= 3C_m\mathcal{X}_m(t) + 2C_m\mathcal{X}_m(t)^2.\end{aligned} \tag{210}$$

Absorbing constants into C_m,

$$\mathcal{X}_m'(t) \leq C_m\mathcal{X}_m(t) + C_m\mathcal{X}_m(t)^2. \tag{211}$$

If we retain the dissipation from the high-order estimate,

$$\frac{d}{dt}\mathcal{X}_m(t) + 2\nu\|\nabla u(t)\|_{H^m}^2 \leq C_m(1+\mathcal{X}_m(t))\mathcal{X}_m(t), \tag{212}$$

and expanding the right-hand side,

$$C_m(1+\mathcal{X}_m(t))\mathcal{X}_m(t) = C_m\mathcal{X}_m(t) + C_m\mathcal{X}_m(t)^2. \tag{213}$$

Thus,

$$\frac{d}{dt}\mathcal{X}_m(t) + 2\nu\|\nabla u(t)\|_{H^m}^2 \leq C_m\mathcal{X}_m(t) + C_m\mathcal{X}_m(t)^2. \tag{214}$$

Dropping the nonnegative dissipation term,

$$\frac{d}{dt}\mathcal{X}_m(t) \leq C_m\mathcal{X}_m(t) + C_m\mathcal{X}_m(t)^2. \tag{215}$$

Now divide by $\mathcal{X}_m(t)^2$,

$$\frac{\mathcal{X}_m'(t)}{\mathcal{X}_m(t)^2} \leq C_m\frac{1}{\mathcal{X}_m(t)} + C_m. \tag{216}$$

Multiplying by -1 and using the derivative of the inverse,

$$\frac{d}{dt}\left(\frac{1}{\mathcal{X}_m(t)}\right) = -\frac{\mathcal{X}_m'(t)}{\mathcal{X}_m(t)^2}, \tag{217}$$

we obtain

$$\frac{d}{dt}\left(\frac{1}{\mathcal{X}_m(t)}\right) \geq -C_m\frac{1}{\mathcal{X}_m(t)} - C_m. \tag{218}$$

Define

$$Y(t) := \frac{1}{\mathcal{X}_m(t)}. \tag{219}$$

Then

$$Y'(t) + C_mY(t) \geq -C_m. \tag{220}$$

Multiplying by the integrating factor e^{C_mt},

$$\frac{d}{dt}\left(e^{C_mt}Y(t)\right) \geq -C_me^{C_mt}. \tag{221}$$

Integrating from 0 to t,

$$e^{C_mt}Y(t) - Y(0) \geq -\int_0^t C_me^{C_ms}\,ds. \tag{222}$$

Solving explicitly,

$$\int_0^t C_me^{C_ms}\,ds = e^{C_mt} - 1, \tag{223}$$

we obtain

$$\mathcal{X}_m(t) \leq \frac{\mathcal{X}_m(0)e^{C_mt}}{1-\mathcal{X}_m(0)\left(e^{C_mt}-1\right)}. \tag{224}$$

Hence the solution remains controlled as long as

$$1-\mathcal{X}_m(0)\left(e^{C_mt}-1\right) > 0. \tag{225}$$

14 Conditional continuation theorem

Theorem 14.1 (Conditional continuation principle). *Let*

$$u_0 \in H^m(\Omega), \qquad m > \frac{5}{2}, \qquad \operatorname{div} u_0 = 0, \tag{226}$$

and let

$$u \in C([0,T^*); H^m(\Omega)) \cap L^2_{\mathrm{loc}}([0,T^*); H^{m+1}(\Omega)) \tag{227}$$

be the maximal strong solution. Assume

$$\|\nabla u(t)\|_{L^\infty} \le C_m\big(1 + \mathcal{M}_m(t)\big), \tag{228}$$

and

$$\int_0^{T^*} \mathcal{M}_m(t)\, dt < \infty. \tag{229}$$

Then

$$\sup_{0\le t<T^*} \|u(t)\|_{H^m} < \infty, \qquad T^* = +\infty. \tag{230}$$

Proof. Using (95) and then (228),

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_{H^m}^2 &+ 2\nu\|\nabla u(t)\|_{H^m}^2 \\ &\le C_m \|\nabla u(t)\|_{L^\infty}\|u(t)\|_{H^m}^2 \\ &\le C_m\big(1+\mathcal{M}_m(t)\big)\|u(t)\|_{H^m}^2 .\end{aligned} \tag{231}$$

Dropping the nonnegative dissipation term,

$$\frac{d}{dt}\|u(t)\|_{H^m}^2 \le C_m\big(1+\mathcal{M}_m(t)\big)\|u(t)\|_{H^m}^2 . \tag{232}$$

Dividing by $\|u(t)\|_{H^m}^2$,

$$\frac{\frac{d}{dt}\|u(t)\|_{H^m}^2}{\|u(t)\|_{H^m}^2} \le C_m + C_m\mathcal{M}_m(t). \tag{233}$$

Applying the logarithmic derivative,

$$\frac{d}{dt}\log\big(\|u(t)\|_{H^m}^2\big) \le C_m + C_m\mathcal{M}_m(t). \tag{234}$$

Integrating from 0 to t,

$$\begin{aligned}\log\big(\|u(t)\|_{H^m}^2\big) &- \log\big(\|u_0\|_{H^m}^2\big) \\ &\le \int_0^t C_m\, ds + \int_0^t C_m\mathcal{M}_m(s)\, ds \\ &= C_m t + C_m\int_0^t \mathcal{M}_m(s)\, ds.\end{aligned} \tag{235}$$

Exponentiating,

$$\|u(t)\|_{H^m}^2 \le \|u_0\|_{H^m}^2 \exp\left(C_m t + C_m\int_0^t \mathcal{M}_m(s)\, ds\right). \tag{236}$$

Using (229),

$$\sup_{0\le t<T^*}\left(C_m t + C_m\int_0^t \mathcal{M}_m(s)\, ds\right) < \infty, \tag{237}$$

hence

$$\sup_{0\le t<T^*}\|u(t)\|_{H^m}^2 < \infty, \qquad \sup_{0\le t<T^*}\|u(t)\|_{H^m} < \infty. \tag{238}$$

This contradicts the blow-up alternative

$$T^* < \infty \quad \Longrightarrow \quad \limsup_{t\uparrow T^*}\|u(t)\|_{H^m} = +\infty. \tag{239}$$

Therefore

$$T^* = +\infty. \tag{240}$$

□

Recent developments on logarithmically improved regularity criteria, such as those in [14], provide further evidence supporting the refined continuation framework outlined in this section.

15 Existence, uniqueness, and singularity exclusion blocks

Let u and v be strong solutions with the same initial data, and define

$$w = u - v, \qquad \pi = p_u - p_v. \tag{241}$$

Subtracting

$$\partial_t u_i + u_j\partial_j u_i + \partial_i p_u - \nu\partial_{jj}u_i = 0, \tag{242}$$

$$\partial_t v_i + v_j\partial_j v_i + \partial_i p_v - \nu\partial_{jj}v_i = 0, \tag{243}$$

gives

$$\partial_t(u_i - v_i) + u_j\partial_j u_i - v_j\partial_j v_i + \partial_i(p_u - p_v) - \nu\partial_{jj}(u_i - v_i) = 0. \tag{244}$$

Thus

$$\partial_t w_i + u_j\partial_j u_i - v_j\partial_j v_i + \partial_i\pi - \nu\partial_{jj}w_i = 0. \tag{245}$$

Expanding the nonlinear difference,

$$\begin{aligned}u_j\partial_j u_i - v_j\partial_j v_i &= u_j\partial_j u_i - u_j\partial_j v_i + u_j\partial_j v_i - v_j\partial_j v_i \\ &= u_j\partial_j(u_i - v_i) + (u_j - v_j)\partial_j v_i \\ &= u_j\partial_j w_i + w_j\partial_j v_i.\end{aligned} \tag{246}$$

Hence

$$\partial_t w_i + u_j\partial_j w_i + w_j\partial_j v_i + \partial_i\pi - \nu\partial_{jj}w_i = 0. \tag{247}$$

Also,

$$\partial_i w_i = \partial_i u_i - \partial_i v_i = 0. \tag{248}$$

Testing against w_i and integrating,

$$\int_\Omega \partial_t w_i\, w_i\, dx + \int_\Omega u_j \partial_j w_i\, w_i\, dx + \int_\Omega w_j \partial_j v_i\, w_i\, dx + \int_\Omega \partial_i \pi\, w_i\, dx - \nu \int_\Omega \partial_{jj} w_i\, w_i\, dx = 0. \tag{249}$$

Summing in i,

$$\sum_{i=1}^{3} \int_\Omega \partial_t w_i\, w_i\, dx + \sum_{i=1}^{3} \int_\Omega u_j \partial_j w_i\, w_i\, dx + \sum_{i=1}^{3} \int_\Omega w_j \partial_j v_i\, w_i\, dx + \sum_{i=1}^{3} \int_\Omega \partial_i \pi\, w_i\, dx - \nu \sum_{i=1}^{3} \int_\Omega \partial_{jj} w_i\, w_i\, dx = 0. \tag{250}$$

For the time term,

$$\sum_{i=1}^{3} \int_\Omega \partial_t w_i\, w_i\, dx = \frac{1}{2}\frac{d}{dt} \sum_{i=1}^{3} \int_\Omega |w_i|^2\, dx = \frac{1}{2}\frac{d}{dt} \|w\|_{L^2}^2 . \tag{251}$$

For the transport term,

$$\begin{aligned}\sum_{i=1}^{3} \int_\Omega u_j \partial_j w_i\, w_i\, dx &= \frac{1}{2} \sum_{i=1}^{3} \int_\Omega u_j \partial_j (|w_i|^2)\, dx \\ &= \frac{1}{2} \int_\Omega u_j \partial_j \Big(\sum_{i=1}^{3} |w_i|^2 \Big)\, dx \\ &= \frac{1}{2} \int_\Omega u \cdot \nabla |w|^2\, dx \\ &= -\frac{1}{2} \int_\Omega (\operatorname{div} u) |w|^2\, dx \\ &= 0. \end{aligned}\tag{252}$$

For the pressure term,

$$\sum_{i=1}^{3} \int_\Omega \partial_i \pi\, w_i\, dx = -\int_\Omega \pi\, \partial_i w_i\, dx = 0. \tag{253}$$

For the viscous term,

$$-\nu \sum_{i=1}^{3} \int_\Omega \partial_{jj} w_i\, w_i\, dx = \nu \sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Omega |\partial_j w_i|^2\, dx = \nu \|\nabla w\|_{L^2}^2 . \tag{254}$$

Therefore

$$\frac{1}{2}\frac{d}{dt} \|w\|_{L^2}^2 + \nu \|\nabla w\|_{L^2}^2 = -\int_\Omega w_j \partial_j v_i\, w_i\, dx. \tag{255}$$

Estimating,

$$\begin{aligned}\left| \int_\Omega w_j \partial_j v_i\, w_i\, dx \right| &\le \int_\Omega |w_j|\, |\partial_j v_i|\, |w_i|\, dx \\ &\le \|\nabla v\|_{L^\infty} \int_\Omega |w|^2\, dx \\ &= \|\nabla v\|_{L^\infty} \|w\|_{L^2}^2 . \end{aligned}\tag{256}$$

Hence

$$\frac{1}{2}\frac{d}{dt} \|w\|_{L^2}^2 + \nu \|\nabla w\|_{L^2}^2 \le \|\nabla v\|_{L^\infty} \|w\|_{L^2}^2 . \tag{257}$$

Multiplying by 2,

$$\frac{d}{dt} \|w\|_{L^2}^2 + 2\nu \|\nabla w\|_{L^2}^2 \le 2 \|\nabla v\|_{L^\infty} \|w\|_{L^2}^2 . \tag{258}$$

Dropping the dissipation,

$$\frac{d}{dt} \|w\|_{L^2}^2 \le 2 \|\nabla v\|_{L^\infty} \|w\|_{L^2}^2 . \tag{259}$$

Applying the logarithmic derivative,

$$\frac{d}{dt} \log\big(\|w\|_{L^2}^2\big) \le 2 \|\nabla v\|_{L^\infty} . \tag{260}$$

Integrating,

$$\log\big(\|w(t)\|_{L^2}^2\big) - \log\big(\|w(0)\|_{L^2}^2\big) \le 2 \int_0^t \|\nabla v(s)\|_{L^\infty}\, ds. \tag{261}$$

Exponentiating,

$$\|w(t)\|_{L^2}^2 \le \|w(0)\|_{L^2}^2 \exp\left(2 \int_0^t \|\nabla v(s)\|_{L^\infty}\, ds \right). \tag{262}$$

If $w(0) = 0$, then

$$\begin{aligned} \|w(t)\|_{L^2}^2 = 0, &\qquad \|w(t)\|_{L^2} = 0, \\ w(t) = 0, &\qquad u(t) = v(t). \end{aligned}\tag{263}$$

This energy-based uniqueness argument, originally developed in the classical works [10, 15, 16], provides a standard route to prove uniqueness in the strong solution class. Subsequent refined uniqueness and regularity criteria, including results involving one slow variable and vortex-stretching estimates, have been established in [7, 11, 13, 17].

Applying ∂^α, $|\alpha| \le m-1$, to (247),

$$\partial_t \partial^\alpha w_i + \partial^\alpha (u_j \partial_j w_i) + \partial^\alpha (w_j \partial_j v_i) + \partial_i \partial^\alpha \pi - \nu \partial_{jj} \partial^\alpha w_i = 0. \tag{264}$$

Using Leibniz,

$$\partial^\alpha(u_j\partial_j w_i) = u_j\partial_j\partial^\alpha w_i + \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}(\partial^\beta u_j)\partial_j\partial^{\alpha-\beta}w_i, \tag{265}$$

$$\partial^\alpha(w_j\partial_j v_i) = w_j\partial_j\partial^\alpha v_i + \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}(\partial^\beta w_j)\partial_j\partial^{\alpha-\beta}v_i. \tag{266}$$

Testing against $\partial^\alpha w_i$, integrating, and summing,

$$\begin{aligned}\frac{d}{dt}\|w\|^2_{H^{m-1}} + 2\nu\|\nabla w\|^2_{H^{m-1}}\\ \le C_m\Big(\|\nabla u\|_{L^\infty} + \|\nabla v\|_{L^\infty}\Big)\|w\|^2_{H^{m-1}}.\end{aligned} \tag{267}$$

16 Cylindrical-coordinate expansion and axisymmetric structural block

$$\begin{aligned}x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \qquad x_3 = z,\\ r = \sqrt{x_1^2+x_2^2}, \qquad z = x_3.\end{aligned} \tag{268}$$

$$\begin{aligned}e_r = (\cos\theta, \sin\theta, 0), \qquad e_\theta = (-\sin\theta, \cos\theta, 0),\\ e_z = (0,0,1).\end{aligned} \tag{269}$$

$$u = u_r e_r + u_\theta e_\theta + u_z e_z. \tag{270}$$

$$\nabla = e_r\partial_r + e_\theta\frac{1}{r}\partial_\theta + e_z\partial_z. \tag{271}$$

$$\operatorname{div} u = \frac{1}{r}\partial_r(ru_r) + \frac{1}{r}\partial_\theta u_\theta + \partial_z u_z. \tag{272}$$

Opening the first term,

$$\frac{1}{r}\partial_r(ru_r) = \frac{1}{r}(u_r + r\partial_r u_r) = \frac{u_r}{r} + \partial_r u_r. \tag{273}$$

Hence

$$\operatorname{div} u = \partial_r u_r + \frac{u_r}{r} + \frac{1}{r}\partial_\theta u_\theta + \partial_z u_z. \tag{274}$$

$$(u\cdot\nabla) = u_r\partial_r + \frac{u_\theta}{r}\partial_\theta + u_z\partial_z. \tag{275}$$

$$(u\cdot\nabla)u_r = u_r\partial_r u_r + \frac{u_\theta}{r}\partial_\theta u_r + u_z\partial_z u_r, \tag{276}$$

$$(u\cdot\nabla)u_\theta = u_r\partial_r u_\theta + \frac{u_\theta}{r}\partial_\theta u_\theta + u_z\partial_z u_\theta, \tag{277}$$

$$(u\cdot\nabla)u_z = u_r\partial_r u_z + \frac{u_\theta}{r}\partial_\theta u_z + u_z\partial_z u_z. \tag{278}$$

$$\Delta u_r = \partial_r^2 u_r + \frac{1}{r}\partial_r u_r + \frac{1}{r^2}\partial_\theta^2 u_r + \partial_z^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\partial_\theta u_\theta, \tag{279}$$

$$\Delta u_\theta = \partial_r^2 u_\theta + \frac{1}{r}\partial_r u_\theta + \frac{1}{r^2}\partial_\theta^2 u_\theta + \partial_z^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\partial_\theta u_r, \tag{280}$$

$$\Delta u_z = \partial_r^2 u_z + \frac{1}{r}\partial_r u_z + \frac{1}{r^2}\partial_\theta^2 u_z + \partial_z^2 u_z. \tag{281}$$

Therefore

$$\begin{cases}\partial_t u_r + u_r\partial_r u_r + \frac{u_\theta}{r}\partial_\theta u_r + u_z\partial_z u_r - \frac{u_\theta^2}{r}\\ \quad = -\partial_r p + \nu\left(\partial_r^2 u_r + \frac{1}{r}\partial_r u_r + \frac{1}{r^2}\partial_\theta^2 u_r + \partial_z^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\partial_\theta u_\theta\right),\\ \partial_t u_\theta + u_r\partial_r u_\theta + \frac{u_\theta}{r}\partial_\theta u_\theta + u_z\partial_z u_\theta + \frac{u_r u_\theta}{r}\\ \quad = -\frac{1}{r}\partial_\theta p + \nu\left(\partial_r^2 u_\theta + \frac{1}{r}\partial_r u_\theta + \frac{1}{r^2}\partial_\theta^2 u_\theta + \partial_z^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\partial_\theta u_r\right),\\ \partial_t u_z + u_r\partial_r u_z + \frac{u_\theta}{r}\partial_\theta u_z + u_z\partial_z u_z\\ \quad = -\partial_z p + \nu\left(\partial_r^2 u_z + \frac{1}{r}\partial_r u_z + \frac{1}{r^2}\partial_\theta^2 u_z + \partial_z^2 u_z\right),\\ \partial_r u_r + \frac{u_r}{r} + \frac{1}{r}\partial_\theta u_\theta + \partial_z u_z = 0.\end{cases} \tag{282}$$

In the axisymmetric reduction,

$$\partial_\theta u_r = \partial_\theta u_\theta = \partial_\theta u_z = 0. \tag{283}$$

Thus

$$\begin{cases}\partial_t u_r + u_r\partial_r u_r + u_z\partial_z u_r - \frac{u_\theta^2}{r}\\ \quad = -\partial_r p + \nu\left(\partial_r^2 u_r + \frac{1}{r}\partial_r u_r + \partial_z^2 u_r - \frac{u_r}{r^2}\right),\\ \partial_t u_\theta + u_r\partial_r u_\theta + u_z\partial_z u_\theta + \frac{u_r u_\theta}{r}\\ \quad = \nu\left(\partial_r^2 u_\theta + \frac{1}{r}\partial_r u_\theta + \partial_z^2 u_\theta - \frac{u_\theta}{r^2}\right),\\ \partial_t u_z + u_r\partial_r u_z + u_z\partial_z u_z\\ \quad = -\partial_z p + \nu\left(\partial_r^2 u_z + \frac{1}{r}\partial_r u_z + \partial_z^2 u_z\right),\\ \partial_r u_r + \frac{u_r}{r} + \partial_z u_z = 0.\end{cases} \tag{284}$$

$$\Phi = \frac{\omega_r}{r}, \qquad \Gamma = \frac{\omega_\theta}{r}, \qquad \Psi = \frac{u_\theta}{r}. \tag{285}$$

$$\begin{aligned}\omega_r = \frac{1}{r}\partial_\theta u_z - \partial_z u_\theta, \qquad \omega_\theta = \partial_z u_r - \partial_r u_z,\\ \omega_z = \frac{1}{r}\partial_r(ru_\theta) - \frac{1}{r}\partial_\theta u_r.\end{aligned} \tag{286}$$

Under axisymmetry,

$$\omega_r = -\partial_z u_\theta, \quad \omega_\theta = \partial_z u_r - \partial_r u_z, \quad \omega_z = \partial_r u_\theta + \frac{u_\theta}{r}. \tag{287}$$

Global regularity results for large solutions with special geometric or anisotropic structures, including axisymmetric and one-slow-variable settings, have been established in [7, 17].

17 Small-data perturbative compatibility

$$\mathcal{Z}_m(0) \le \varepsilon_m. \tag{288}$$

$$\frac{d}{dt}\mathcal{Z}_m(t) \le C_m\mathcal{Z}_m(t) + C_m\mathcal{Z}_m(t)^2. \tag{289}$$

Assume the bootstrap bound

$$\mathcal{Z}_m(t) \le 2\varepsilon_m, \qquad 0 \le t \le T. \tag{290}$$

Then

$$\mathcal{Z}_m(t)^2 \le 2\varepsilon_m\,\mathcal{Z}_m(t). \tag{291}$$

Substituting into (289),

$$\begin{aligned}\frac{d}{dt}\mathcal{Z}_m(t) &\le C_m\mathcal{Z}_m(t) + C_m\mathcal{Z}_m(t)^2\\ &\le C_m\mathcal{Z}_m(t) + 2C_m\varepsilon_m\mathcal{Z}_m(t)\\ &= C_m(1+2\varepsilon_m)\mathcal{Z}_m(t).\end{aligned} \tag{292}$$

Applying the logarithmic derivative,

$$\frac{d}{dt}\log\mathcal{Z}_m(t) \le C_m(1+2\varepsilon_m). \tag{293}$$

Integrating,

$$\log\mathcal{Z}_m(t) - \log\mathcal{Z}_m(0) \le C_m(1+2\varepsilon_m)t. \tag{294}$$

Exponentiating,

$$\mathcal{Z}_m(t) \le \mathcal{Z}_m(0)e^{C_m(1+2\varepsilon_m)t}. \tag{295}$$

Keeping dissipation,

$$\frac{d}{dt}\mathcal{Z}_m(t) + \nu\mathcal{E}_m(t) \le C_m\mathcal{Z}_m(t)^2. \tag{296}$$

Assume

$$C_m\mathcal{Z}_m(t)^2 \le \frac{\nu}{2}\mathcal{E}_m(t) + C_m\varepsilon_m\mathcal{Z}_m(t). \tag{297}$$

Then

$$\frac{d}{dt}\mathcal{Z}_m(t) + \nu\mathcal{E}_m(t) \le \frac{\nu}{2}\mathcal{E}_m(t) + C_m\varepsilon_m\mathcal{Z}_m(t), \tag{298}$$

hence

$$\frac{d}{dt}\mathcal{Z}_m(t) + \frac{\nu}{2}\mathcal{E}_m(t) \le C_m\varepsilon_m\mathcal{Z}_m(t). \tag{299}$$

18 Conclusion

In this work, we developed a formal analytic framework for the three-dimensional incompressible Navier–Stokes equations, with special emphasis on the explicit expansion of the nonlinear structure, the vorticity channel, the pressure reconstruction mechanism, the anisotropic differentiated hierarchy, and the high-order Sobolev continuation scheme. The main strategy was to keep the algebra visible, to avoid excessive compression of multilinear interactions, and to write the differentiated identities in a form where each nonlinear contribution could be tracked separately.

The starting point was the classical incompressible Navier–Stokes system

$$\partial_t u + (u\cdot\nabla)u + \nabla p - \nu\Delta u = 0, \quad \operatorname{div} u = 0, \tag{300}$$

together with the corresponding strong Sobolev framework

$$u \in C([0,T];H^m(\Omega)) \cap L^2_{\mathrm{loc}}([0,T];H^{m+1}(\Omega)),\ m > \frac{5}{2}. \tag{301}$$

From this point of view, the continuation problem is reduced to the control of the nonlinear growth of the higher-order Sobolev norm, namely to inequalities of the form

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2_{H^m} &+ 2\nu\|\nabla u(t)\|^2_{H^m}\\ &\le C_m\|\nabla u(t)\|_{L^\infty}\|u(t)\|^2_{H^m}.\end{aligned} \tag{302}$$

The analysis carried out throughout the paper isolated three principal channels:

$$\begin{aligned}&\text{vorticity},\\ &\text{pressure},\\ &\text{anisotropic vertical differentiation}.\end{aligned} \tag{303}$$

These were assembled into the mixed control quantity

$$\mathcal{M}_m(t) = \|\omega(t)\|_{L^\infty} + \|\nabla p(t)\|_{L^q} + \|\partial_3 u(t)\|_{H^{m-1}},\ q > 3, \tag{304}$$

and the whole continuation mechanism was formally organized around the estimate

$$\|\nabla u(t)\|_{L^\infty} \le C_m\big(1 + \mathcal{M}_m(t)\big). \tag{305}$$

Combined with the high-order differentiated energy inequality, this yields

$$\frac{d}{dt}\|u(t)\|^2_{H^m} \le C_m\big(1+\mathcal{M}_m(t)\big)\|u(t)\|^2_{H^m}, \tag{306}$$

and therefore, after integration,

$$\|u(t)\|^2_{H^m} \le \|u_0\|^2_{H^m}\exp\left(C_m t + C_m\int_0^t \mathcal{M}_m(s)\,ds\right). \tag{307}$$

Consequently, if

$$\int_0^{T^*}\mathcal{M}_m(t)\,dt < \infty, \tag{308}$$

then the strong norm cannot blow up in finite time.

On the vorticity side, the explicit form

$$\partial_t \omega + (u \cdot \nabla)\omega - \nu \Delta \omega = (\omega \cdot \nabla)u \tag{309}$$

shows that the principal obstruction comes from the stretching term

$$(\omega \cdot \nabla)u, \tag{310}$$

which couples the growth of the vorticity amplitude to the Lipschitz norm of the velocity. On the pressure side, the elliptic identity

$$-\Delta p = \partial_i \partial_j (u_i u_j) \tag{311}$$

makes visible the nonlocal quadratic feedback of the convective tensor. On the anisotropic side, the repeated differentiation with respect to the vertical variable produced a hierarchy of estimates of the form

$$\begin{aligned}\frac{d}{dt} \|\partial_3 u(t)\|^2_{H^{m-1}} + 2\nu \|\nabla \partial_3 u(t)\|^2_{H^{m-1}} \\ \leq C_m \|\nabla u(t)\|_{L^\infty} \|\partial_3 u(t)\|^2_{H^{m-1}},\end{aligned} \tag{312}$$

which suggests that anisotropic control may play a nontrivial role in the strong continuation problem.

The formal closure systems introduced in the later sections were designed to enlarge the analytic picture and to display, in explicit multilinear form, the interaction between the Sobolev norm of the velocity, the differentiated vorticity, the pressure derivatives, and the vertical derivative hierarchy. In that sense, the paper was not intended to compress the argument into a minimal proof sketch, but rather to expand the structure so that the nonlinear transport, stretching, pressure, and commutator mechanisms remain directly visible on the page.

The rigorous part of the argument establishes a conditional continuation mechanism: if the mixed Lipschitz control can be rigorously closed in terms of the quantity $\mathcal{M}_m(t)$, and if $\mathcal{M}_m$ remains integrable on finite time intervals, then the strong solution continues globally. In this form, the strategy leads to the conditional statement

$$\begin{aligned}&\Big[\|\nabla u(t)\|_{L^\infty} \leq C_m(1 + \mathcal{M}_m(t)) \Big] \quad \text{and} \\ &\Big[\int_0^{T^*} \mathcal{M}_m(t)\, dt < \infty \Big] \Longrightarrow T^* = +\infty.\end{aligned} \tag{313}$$

At the same time, mathematical honesty requires stating clearly that the decisive step is still open. The unresolved point is not the formal propagation of the Sobolev norm once a suitable mixed estimate is granted; rather, it is the rigorous closure of the estimate itself, together with a genuinely closed evolution inequality for the mixed continuation functional. In other words, the program identifies a route, organizes the major nonlinear contributions, and makes the continuation mechanism explicit, but it does not yet constitute an unconditional proof of global regularity for the three-dimensional Navier–Stokes equations.

Therefore, the final outcome of the present manuscript is best understood as follows: it provides an expanded analytic program, a large explicit system of differentiated identities, a conditional continuation framework, and a structured decomposition of the main nonlinear obstructions; however, the decisive global closure step remains to be fully justified. The problem is not solved here in the strict sense of a completed millennium-proof, but the architecture of one possible route is laid out in substantial algebraic and differential detail.

This is precisely where the paper stands: not as a definitive resolution, but as a dense formal continuation program in which the pressure channel, the vorticity channel, and the anisotropic channel are pushed into a common analytic framework. Any future successful closure of this framework would have direct consequences for existence, smoothness, uniqueness, and singularity exclusion in the three-dimensional incompressible Navier–Stokes problem.

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