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RESEARCH ARTICLE

Modeling Marital Stability: A Dynamical Systems Analysis of Extramarital Affairs and Divorce

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Abstract

This study develops and analyzes a nonlinear mathematical model to describe the dynamics of marriage, extramarital affairs, marital breakdown, and divorce. The population is divided into five compartments: single $S(t)$, married $M(t)$, individuals in extramarital affairs $A(t)$, broken marriages $B(t)$, and divorced $D(t)$. The model is formulated as a system of nonlinear ordinary differential equations capturing transitions between these states through parameters representing marriage formation, affair initiation, reconciliation, separation, and divorce. Basic qualitative properties, including positivity and boundedness of solutions, are established to ensure the model is well posed. Equilibrium analysis reveals the trivial equilibrium $E_T = (0,0,0,0,0)$, the divorce-free equilibrium $E_0 = (S^0, M^0, 0, 0, 0)$, and a positive endemic equilibrium $E^* = (S^*, M^*, A^*, B^*, D^*)$. Stability is investigated via the Jacobian matrix and characteristic equation. The basic reproduction number R_0, derived using the next-generation matrix method, determines whether marital instability spreads in the population. The divorce-free equilibrium is locally asymptotically stable when $R_0 < 1$, whereas persistent instability occurs when $R_0 > 1$. Sensitivity analysis and numerical simulations are performed to examine the influence of key parameters, including marriage formation rate α, affair initiation rate γ, instability rate σ, and reconciliation rate ξ. Phase portraits, parameter variation diagrams, and reproduction number plots confirm the analytical results. The findings highlight the roles of extramarital affairs and reconciliation in shaping marital stability and provide a quantitative framework for understanding long-term marriage and divorce dynamics.

Keywords: dynamical systems, marital stability, extramarital affairs, divorce dynamics, equilibrium analysis, sensitivity analysis.

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1 Introduction

Marriage is widely regarded as a fundamental social institution that regulates interpersonal relationships, kinship structures, and the organization of society. According to the United Nations, marriage can be

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described as a social, legal, and economic relationship between individuals that establishes kinship ties and mutual obligations [1]. From a legal perspective, marriage represents a civil status that confers specific rights and responsibilities upon the partners. Demographic research in the 2000s documented substantial shifts in marriage formation, dissolution, and family structure, reflecting the evolving legal and social dimensions of marital unions [2]. From a sociological perspective, marriage performs several essential functions, including the regulation of sexual behavior and the formation of family structures. Empirical evidence further indicates that married individuals tend to report greater personal well-being, better health outcomes, and improved financial stability compared with their unmarried counterparts [3]. More broadly, when the mutual obligations and expectations of marriage are unfulfilled, transitions toward separation and divorce may follow. Research indicates that the reasons couples cite for divorce are varied, and that openness to marital reconciliation differs substantially across individuals and cultural contexts [4].

Despite its social significance, marital relationships are not always stable. Separation and divorce often occur when marriages fail to meet the emotional, psychological, or functional expectations of one or both partners. Although these developments may appear to be personal matters, they can lead to significant social, economic, and demographic consequences. In response to these societal changes, mathematical modeling has emerged as an important tool for studying the dynamics of marital relationships. Quantitative models allow researchers to investigate how social, psychological, and demographic factors influence marital stability and transitions between different relationship states, including marriage formation, separation, and divorce. Such modeling approaches have been widely applied in fields such as population dynamics, epidemiology, sociology, and applied mathematics [5–7].

Adultery continues to be recognized as a valid ground for divorce under several Indian personal laws, including the Hindu Marriage Act, 1955 (Section 13(1)(i)), the Indian Divorce Act, 1869, and the Parsi Marriage and Divorce Act, 1936, even after the decriminalization of adultery by the Supreme Court of India in the landmark *Joseph Shine v. Union of India* judgment [8]. Although adultery is no longer treated as a criminal offence following the 2018 judgment, scholarly commentary has argued against reinstating criminal penalties, contending that civil remedies within matrimonial law provide a more appropriate and proportionate response to marital infidelity [9], and adultery thus remains a civil ground for marital dissolution. In legal practice, proving adultery requires establishing that a spouse voluntarily engaged in sexual relations with a person other than their lawful partner; courts assess such claims based on the available evidence and applicable standards of proof [10]. Emotional infidelity or virtual interactions are generally insufficient; courts typically require evidence of physical intimacy. Because direct proof is rarely available, courts often rely on circumstantial evidence such as hotel records, electronic messages, photographs, or witness testimony. The burden of proof rests on the petitioner and is assessed on the balance of probabilities rather than the stricter criminal standard.

Recent socio–legal analyses indicate that adultery is cited in approximately 10%–16% of divorce cases in India, although the prevalence varies across regions depending on socio–cultural context and legal awareness [11]. Changing social norms and evolving legal awareness have contributed to a gradual rise in divorce filings in India over recent decades [13]. Figure 1 illustrates a comparative bar chart of divorce cases filed in India between 2018 and 2022, showing both the total number of filings and those specifically citing adultery as the primary ground. The data indicate a gradual increase in divorce filings during this period, with a noticeable decline in 2020, likely attributable to disruptions caused by the COVID–19 pandemic. Real-data-based mathematical modeling of divorce dynamics using nonlinear differential equations has further confirmed the utility of compartmental frameworks in capturing such temporal fluctuations in marital dissolution [12]. These trends are consistent with broader documented shifts in marital behaviour and divorce patterns observed across Indian society in recent decades [13]. Adultery-related petitions consistently represent approximately 12%-16% of annual divorce cases, underscoring the continued relevance of adultery as a civil ground for marital dissolution even after its decriminalization [11, 15]. Legal practitioners have similarly noted the persistent role of adultery in matrimonial disputes across Indian jurisdictions [14, 16]. Despite the removal of criminal penalties, adultery remains a recognized civil ground for divorce under personal laws such as the Hindu Marriage Act and the Indian Divorce Act. From a legal perspective, this issue can also be examined within the broader framework of matrimonial and criminal laws in India. The legal structure governing marriage includes legislation relating to special marriages, Hindu marriages, Parsi and Christian marriages, together with the provisions of the Indian Divorce Act. These legal frameworks contain provisions addressing bigamy and adultery as grounds for declaring a marriage void or as valid bases for divorce in matrimonial disputes. Judicial cases filed and adjudicated under these statutes provide valuable empirical insights that may assist in estimating parameters in analytical and mathematical models of marital dynamics.

A historical examination of the Indian Penal Code from 1860 to 2023 reveals that bigamy was treated as an offence applicable to both spouses, whereas adultery was historically framed primarily in terms of the husband's conduct. Understanding these provisions is important for analysing the legal and social dimensions of extramarital relationships. In contemporary legal developments, the enactment of the *Bharatiya Nyaya Sanhita* (2023), which removed adultery from the scope of criminal punishment, represents a significant milestone in Indian legal reform. This change provides an opportunity to assess the effectiveness and evolution of matrimonial law in addressing marital conflicts.

Over time, the institution of marriage has undergone substantial transformations due to changing cultural values, economic pressures, and evolving legal norms. These changes have contributed to phenomena such as delayed marriages, long-distance relationships, extramarital affairs, and increasing divorce rates. In this context, mathematical modelling has emerged as a powerful tool for analysing the dynamics of marriage and divorce. Such models enable researchers to investigate how social, psychological, and demographic factors influence marital stability and transitions between states such as single, married, separated, and divorced [5–7]. Because marriage and divorce are influenced by complex social and psychological processes, they have become important subjects of study in population dynamics, sociology, psychology, and applied mathematics. Traditional linear transition models often classify individuals into distinct categories such as single, married, and divorced. However, these models may overlook important real-world determinants, including extramarital relationships and the nonlinear effects of marital dissolution on families and communities.

Consequently, several mathematical models have been developed to explain the mechanisms underlying marital breakdown. These models aim to capture the interactions among individual behaviour, social networks, and structural factors that contribute to divorce. For example, Duato and Jódar [17] proposed a mathematical model to describe the propagation of divorce in Spain by drawing analogies with the spread of infectious diseases through social influence and imitation. Similarly, Olaniyi [18] analytically examined the socio–economic determinants of divorce in African families, emphasizing the role of cultural and economic factors. Shah *et al.* [20] extended this framework by analyzing the global stability properties of a divorce model incorporating both arranged and love marriages.

Furthermore, Gambrah and Adzadu [19] introduced a nonlinear Marriage–Separation–Divorce (MSD) model to investigate the dynamics of divorce as a social epidemic in Ghana. Their results demonstrated that divorce dynamics can exhibit epidemic–like propagation patterns within populations. These findings suggest that extramarital affairs and related socio–cultural influences continue to play a significant role in generating matrimonial discord and marital instability. Extensions of these frameworks to fractional-order formulations have also been proposed, enabling the modeling of memory effects and hereditary properties in marital dissolution dynamics [21]. The evolution of monogamous marriage dynamics has also been examined using discrete mathematical frameworks and optimal control strategies. Lhous *et al.* [23] developed a discrete–time model to analyze marital status transitions in a society assuming monogamous marriage. The population was divided into eight compartments: virgin men and women, married men and women, divorced men and women, and widowed men and women. The model also incorporated two optimal control strategies: one designed to promote marriage through awareness campaigns and another intended to reduce divorce by emphasizing legal and social consequences. These control strategies were optimized using a discrete version of Pontryagin's maximum principle and solved numerically through a forward–backward sweep method. A related discrete model of marital status and family dynamics was investigated in [22]. The authors proposed two control strategies: one involving awareness campaigns aimed at educating unmarried individuals about the benefits of marriage, and another focused on informing couples about

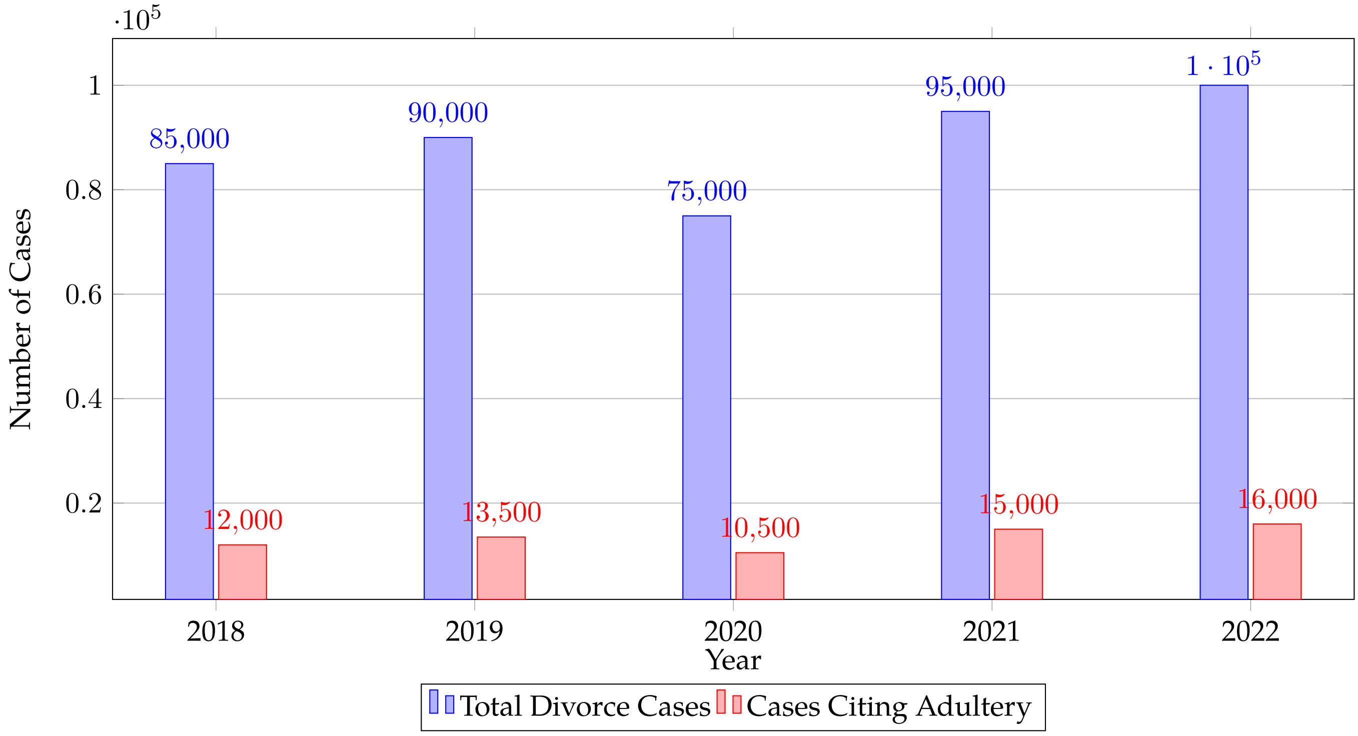

Figure 1. Comparison of estimated total divorce cases and cases citing adultery as a ground in India (2018–2022). Data are illustrative estimates based on reported legal filings and court records; comprehensive national-level divorce statistics are not systematically published by a single official source in India.

legal procedures, administrative complexities, and the financial and social consequences of divorce. Their results demonstrated that such interventions can reduce the number of single and divorced individuals while increasing the proportion of married couples in the population.

Figure 1 presents a comparative analysis of the total number of divorce cases filed in India between 2018 and 2022 and the subset of cases in which adultery was cited as a primary ground. The data indicate a gradual increase in total divorce filings over the observed period, rising from an estimated 85,000 cases in 2018 to around 100,000 cases in 2022, based on available legal filing records and reported court data. A noticeable decline occurred in 2020, which may be attributed to disruptions caused by the COVID–19 pandemic and reduced court activity during that period. The number of divorce cases citing adultery follows a similar trend, increasing from about 12,000 in 2018 to nearly 16,000 in 2022. Overall, adultery-related petitions consistently represent a significant proportion of divorce cases, typically ranging between 12% and 16% annually. These results highlight the continuing role of extramarital affairs as an important factor contributing to marital instability in India.

Several studies have approached divorce dynamics from the perspective of social epidemiology, treating divorce as a socially transmitted phenomenon [17, 19]. In these models, divorce propagation is considered analogous to the spread of an infectious disease: a married individual may become divorced through direct social interaction with divorced individuals or indirectly through exposure to negative marital experiences communicated within social networks. Such social influence mechanisms can gradually destabilize marriages and ultimately lead to marital dissolution.

Marriage, as a fundamental social institution, strongly influences demographic patterns, social stability, and individual well–being. However, contemporary societies are experiencing significant shifts in marital behaviour, including the rising prevalence of extramarital affairs, emotional separation, and divorce. Traditional mathematical models typically focus on transitions between singlehood, marriage, and divorce, but often neglect intermediate states that may critically affect marital stability.

To address these limitations, we develop a comprehensive compartmental model that divides the population into five interconnected groups: singles (S), married couples (M), individuals involved in extramarital affairs (A), persons experiencing marital breakdown (B), and divorced individuals (D). The proposed framework incorporates nonlinear interaction terms, including social saturation effects and the diminishing influence of divorced individuals as their prevalence increases. The inclusion of an extramarital affairs compartment enables the model to capture pre–divorce instability and to represent behavioural dynamics more realistically. The resulting model is formulated as a system of nonlinear differential equations designed to analyse the stability of marital dynamics and to identify threshold conditions governing transitions between stable marriages and relationship breakdowns. In particular, we introduce a reproduction number R_0, analogous to the basic reproduction number used in epidemiological modeling, to measure the potential spread of marital instability within a population. This framework provides a rigorous mathematical approach for understanding marital transitions and offers insights that may inform social and policy interventions aimed at strengthening marital stability.

2 Mathematical Model

In this study, we propose a compartmental dynamical system to describe the evolution of marriage and divorce dynamics within a population. The population is divided into five mutually exclusive compartments:

- $S(t)$: single individuals,
- $M(t)$: married individuals,
- $A(t)$: individuals involved in extramarital affairs,
- $B(t)$: individuals experiencing broken or unstable marriages,
- $D(t)$: divorced individuals.

The transition structure between these classes is illustrated in Figure 2. A description of each state variable is provided in Table 1.

Let $N(t)$ denote the total population at time t. Then

$$N(t) = S(t) + M(t) + A(t) + B(t) + D(t). \tag{1}$$

2.1 Model Assumptions

The mathematical model is formulated based on the following assumptions.

1. Individuals enter the population through recruitment into the single class at rate Λ.
2. Single individuals become married at rate α.
3. Married individuals may engage in extramarital affairs at rate γ due to social, behavioral, or psychological influences.
4. Individuals involved in extramarital affairs may reconcile with their spouse and return to the married class at rate ξ, or their relationship may deteriorate and move into the broken marriage class at rate σ.
5. Marriages may also deteriorate due to social influence from divorced individuals. This social contagion effect is modeled using the saturated interaction term
$$\frac{\beta M D}{1+\eta D}. \tag{2}$$
Here β represents the intensity of the social influence of divorced individuals on married couples, while η represents a saturation parameter that limits the growth of the influence when the divorced population becomes large.
6. Individuals in broken marriages may reconcile and return to the married class at rate ρ or proceed to divorce at rate δ.
7. Divorced individuals may return to the single population at rate θ.
8. All population classes experience natural mortality at rate μ, while divorced individuals may experience additional divorce-related mortality at rate μ_D.

2.2 Model Equations

Based on the above assumptions, the dynamics of the system are governed by the following system of nonlinear ordinary differential equations:

$$\frac{dS}{dt} = \Lambda + \theta D - (\alpha+\mu)S \tag{3}$$

$$\frac{dM}{dt} = \alpha S + \rho B + \xi A - (\gamma+\mu)M - \frac{\beta M D}{1+\eta D} \tag{4}$$

$$\frac{dA}{dt} = \gamma M - (\sigma+\xi+\mu)A \tag{5}$$

$$\frac{dB}{dt} = \frac{\beta M D}{1+\eta D} + \sigma A - (\rho+\delta+\mu)B \tag{6}$$

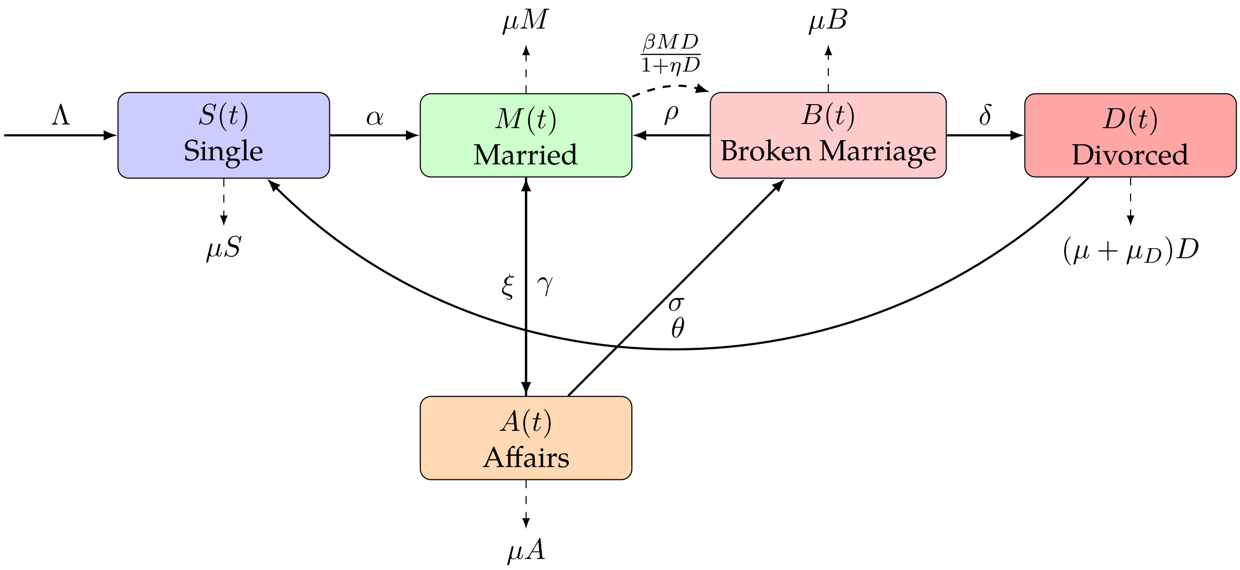

Figure 2. State transition diagram of the proposed compartmental model for marriage dynamics.

$$\frac{dD}{dt} = \delta B - (\theta + \mu_D + \mu)D \tag{7}$$

The system is analyzed subject to the following non–negative initial conditions:

$$S(0) = S_0 \geq 0, \quad M(0) = M_0 \geq 0, \quad A(0) = A_0 \geq 0, \\ B(0) = B_0 \geq 0, \quad D(0) = D_0 \geq 0. \tag{8}$$

2.3 Description of State Variables

Table 1 summarizes the five state variables and their demographic and social interpretations within the model.

Table 1. Description of the state variables used in the model.

Variable	Description
$S(t)$	Number of single individuals at time t
$M(t)$	Number of married individuals at time t
$A(t)$	Individuals involved in extramarital affairs
$B(t)$	Individuals in broken or unstable marriages
$D(t)$	Divorced individuals
$N(t)$	Total population $N(t) = S + M + A + B + D$

Table 2. Description of model parameters.

Parameter	Description	Unit	Typical Range
Λ	Recruitment rate into single class	individuals/year	$10 - 1000$
α	Marriage rate of single individuals	year^{-1}	$0.1 - 0.6$
γ	Rate of extramarital affairs	year^{-1}	$0.01 - 0.2$
ξ	Reconciliation rate from affair to marriage	year^{-1}	$0.05 - 0.4$
σ	Transition rate from affair to broken marriage	year^{-1}	$0.05 - 0.3$
β	Social influence rate of divorced individuals	year^{-1}	$0.01 - 0.5$
η	Saturation parameter for social influence	dimensionless	$0 - 1$
ρ	Reconciliation rate from broken marriage	year^{-1}	$0.05 - 0.3$
δ	Divorce rate from broken marriage	year^{-1}	$0.05 - 0.4$
θ	Rate divorced individuals return to single class	year^{-1}	$0.01 - 0.2$
μ	Natural mortality rate	year^{-1}	$0.01 - 0.03$
μ_D	Additional mortality among divorced individuals	year^{-1}	$0 - 0.02$

2.4 Model Parameters

Table 2 lists all model parameters together with their descriptions, units, and typical ranges.

2.5 Vector Representation of the Model

For analytical convenience, the system can be written in compact vector form as

$$\frac{d\mathbf{X}}{dt} = F(\mathbf{X}), \tag{9}$$

where

$$\mathbf{X}(t) = \begin{pmatrix} S(t) \\ M(t) \\ A(t) \\ B(t) \\ D(t) \end{pmatrix}. \tag{10}$$

The vector function $F(\mathbf{X})$ is defined as

$$F(\mathbf{X}) = \begin{pmatrix} \Lambda + \theta D - (\alpha + \mu)S \\ \alpha S + \rho B + \xi A - (\gamma + \mu)M - \frac{\beta MD}{1+\eta D} \\ \gamma M - (\sigma + \xi + \mu)A \\ \frac{\beta MD}{1+\eta D} + \sigma A - (\rho + \delta + \mu)B \\ \delta B - (\theta + \mu_D + \mu)D \end{pmatrix}. \tag{11}$$

This compact representation facilitates further mathematical analysis, including the computation of equilibrium points, derivation of the Jacobian matrix, and determination of the divorce reproduction number.

2.6 Model Scope and Limitations

The proposed model assumes homogeneous mixing within the population and does not explicitly incorporate demographic heterogeneity such as age, regional variation, socioeconomic status, or cultural background. Although these factors may influence marital stability in real societies, the present model provides a simplified mathematical framework that captures the fundamental mechanisms governing transitions between marital states. The model may be extended in future studies to include demographic stratification or network-based social interactions.

3 Mathematical Analysis of the Extramarital Affairs and Divorce Model

3.1 Well-posedness, Invariant Region, and Boundedness of the Model

In this subsection, we investigate some fundamental mathematical properties of the divorce dynamics model described by the system of differential equations (3)–(7), or equivalently by its compact vector form (11). In particular, we establish the well-posedness of the system and determine a positively invariant and bounded region for the model.

If the initial condition satisfies

$$(S(0), M(0), A(0), B(0), D(0)) \in \mathbb{R}^5_+, \tag{12}$$

then the Cauchy problem associated with system (3)–(7) admits a unique solution

$$(S(t), M(t), A(t), B(t), D(t)) \in \mathbb{R}^5_+, \qquad t \geq 0, \tag{13}$$

which remains non–negative for all $t \geq 0$. This follows from the fact that the right-hand side of the system consists of continuously differentiable functions, ensuring existence and uniqueness of solutions according to the classical theory of ordinary differential equations.

Lemma 3.1. *For any non–negative initial condition, the total population*

$$N(t) = S(t) + M(t) + A(t) + B(t) + D(t) \tag{14}$$

satisfies the inequality

$$0 \leq N(t) \leq \frac{\Lambda}{\mu}, \qquad \text{for all } t \geq 0. \tag{15}$$

Proof. Let

$$N(t) = S(t) + M(t) + A(t) + B(t) + D(t) \tag{16}$$

denote the total population at time t. Summing the equations (3)–(7), we obtain

$$\frac{dN}{dt} = \Lambda - \mu N(t) - \mu_D D(t). \tag{17}$$

Since $D(t) \geq 0$, it follows that

$$\frac{dN}{dt} \leq \Lambda - \mu N(t). \tag{18}$$

Consider the auxiliary differential equation

$$\frac{dY}{dt} = \Lambda - \mu Y. \tag{19}$$

Solving this linear equation gives

$$Y(t) = Y(0)e^{-\mu t} + \frac{\Lambda}{\mu}\left(1 - e^{-\mu t}\right). \tag{20}$$

By the comparison theorem for differential equations, we obtain

$$N(t) \leq Y(t). \tag{21}$$

Taking the limit as $t \to \infty$, we obtain

$$\lim_{t\to\infty} Y(t) = \frac{\Lambda}{\mu}. \tag{22}$$

Therefore,

$$0 \leq N(t) \leq \frac{\Lambda}{\mu}, \qquad \forall t \geq 0. \tag{23}$$

Consequently, the region

$$\Omega = \left\{ (S, M, A, B, D) \in \mathbb{R}^5_+ : N(t) \le \frac{\Lambda}{\mu} \right\} \tag{24}$$

is positively invariant and attracting with respect to the system (3)–(7). Hence, all solutions of the model remain bounded in the feasible region Ω for all $t \ge 0$. □

Lemma 3.2. *For any non–negative initial condition*

$$\phi_0 = (S(0), M(0), A(0), B(0), D(0)) \in \mathbb{R}^5_+,$$

system (3)–(7) *admits a unique solution*

$$u(t, \phi_0) = (S(t), M(t), A(t), B(t), D(t)), \qquad t \ge 0,$$

which remains non–negative for all $t \ge 0$. Moreover, the region

$$\Delta = \left\{ (S, M, A, B, D) \in \mathbb{R}^5_+ : S + M + A + B + D \le \tfrac{\Lambda}{\mu} \right\} \tag{25}$$

is positively invariant under the flow of system (3)–(7).

Proof. Since the right-hand side of system (3)–(7) is continuously differentiable in $\mathbb{R}^5_+$, the existence and uniqueness of solutions follow from the standard theory of ordinary differential equations. Next, we show that the solutions remain non–negative. Consider the first equation

$$\frac{dS}{dt} = \Lambda + \theta D - (\alpha + \mu) S.$$

Solving this linear equation gives

$$S(t) = S(0) e^{-(\alpha+\mu)t} + \int_0^t (\Lambda + \theta D(s)) e^{-(\alpha+\mu)(t-s)} ds.$$

Since $S(0) \ge 0$ and $D(t) \ge 0$, it follows that $S(t) \ge 0$. Similarly, the remaining equations show that whenever any state variable reaches zero, its derivative becomes non–negative. Hence none of the state variables can cross the boundary of $\mathbb{R}^5_+$, and therefore all solutions remain non–negative for all $t \ge 0$. Now define the total population

$$N(t) = S(t) + M(t) + A(t) + B(t) + D(t).$$

Summing equations (3)–(7) yields

$$\frac{dN}{dt} = \Lambda - \mu N(t) - \mu_D D(t).$$

Since $D(t) \ge 0$, we obtain

$$\frac{dN}{dt} \le \Lambda - \mu N(t).$$

Solving the auxiliary equation

$$\frac{dY}{dt} = \Lambda - \mu Y$$

gives

$$Y(t) = Y(0) e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}).$$

By the comparison theorem we obtain $N(t) \le Y(t)$, and therefore

$$0 \le N(t) \le \frac{\Lambda}{\mu}.$$

Hence the region Δ is positively invariant for system (3)–(7). □

Lemma 3.3 (Global existence of solutions)**.** *Let the initial condition satisfy*

$$(S(0), M(0), A(0), B(0), D(0)) \in \mathbb{R}^5_+. \tag{26}$$

Then the solution $(S(t), M(t), A(t), B(t), D(t))$ of system (3)–(7) *exists for all $t \ge 0$.*

Proof. From Lemma 3.1, the total population satisfies

$$0 \le N(t) \le \frac{\Lambda}{\mu}, \qquad t \ge 0, \tag{27}$$

where

$$N(t) = S(t) + M(t) + A(t) + B(t) + D(t).$$

Hence all state variables remain bounded for all $t \ge 0$. Moreover, the right-hand side of system (3)–(7) is continuously differentiable with respect to the state variables, and therefore satisfies the local Lipschitz condition in $\mathbb{R}^5_+$. Thus, by the classical existence and uniqueness theory for ordinary differential equations, together with the boundedness established above, the solution exists globally and cannot blow up in finite time [27]. Therefore the solution exists for all $t \ge 0$. □

Lemma 3.4 (Uniform persistence of the population)**.** *Assume that the initial condition*

$$(S(0), M(0), A(0), B(0), D(0)) \in \mathbb{R}^5_+ \tag{28}$$

with $N(0) > 0$. Then the solution of system (3)–(7) *satisfies*

$$\liminf_{t\to\infty} N(t) > 0. \tag{29}$$

Consequently, the total population persists uniformly in time.

Proof. Let

$$N(t) = S(t) + M(t) + A(t) + B(t) + D(t)$$

denote the total population. Summing equations (3)–(7) yields

$$\frac{dN}{dt} = \Lambda - \mu N(t) - \mu_D D(t). \tag{30}$$

Since $D(t) \geq 0$, we obtain the inequality

$$\frac{dN}{dt} \leq \Lambda - \mu N(t). \tag{31}$$

Consider the auxiliary equation

$$\frac{dY}{dt} = \Lambda - \mu Y. \tag{32}$$

Its solution is

$$Y(t) = Y(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}).$$

By the comparison theorem for differential equations,

$$N(t) \leq Y(t).$$

Taking the limit as $t \to \infty$ gives

$$\limsup_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu}.$$

Since $\Lambda > 0$ and $N(t)$ remains positive for all $t \geq 0$, it follows that

$$\liminf_{t\to\infty} N(t) > 0.$$

Hence the total population persists uniformly in time. □

Lemma 3.5 (Dissipativity of the system). *The dynamical system (3)–(7) is dissipative in $\mathbb{R}^5_+$. In particular, there exists a compact set $\Omega \subset \mathbb{R}^5_+$ that attracts all solutions of the system.*

Proof. Let

$$N(t) = S(t) + M(t) + A(t) + B(t) + D(t)$$

denote the total population at time t. Summing equations (3)–(7) yields

$$\frac{dN}{dt} = \Lambda - \mu N(t) - \mu_D D(t).$$

Since $D(t) \geq 0$, we obtain

$$\frac{dN}{dt} \leq \Lambda - \mu N(t).$$

Consider the auxiliary equation

$$\frac{dY}{dt} = \Lambda - \mu Y.$$

Its solution is

$$Y(t) = Y(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}).$$

By the comparison theorem,

$$N(t) \leq Y(t).$$

Taking the limit as $t \to \infty$ gives

$$N(t) \leq \frac{\Lambda}{\mu}.$$

Therefore all trajectories eventually enter the region

$$\Omega = \left\{(S, M, A, B, D) \in \mathbb{R}^5_+ : S + M + A + B + D \leq \tfrac{\Lambda}{\mu}\right\}.$$

Hence the system is dissipative and Ω is a compact absorbing set for the solutions of system (3)–(7). □

3.2 Equilibrium Points and Stability Analysis

3.2.1 Equilibrium Points

Consider the divorce dynamics model given by system (3)–(7), which can be written in compact form as

$$\dot{X} = F(X), \tag{33}$$

where the state vector is defined by

$$X = (S, M, A, B, D)^\top.$$

An equilibrium point (steady state) of the system is a constant vector X^* satisfying

$$F(X^*) = 0. \tag{34}$$

Let

$$X^* = (S^*, M^*, A^*, B^*, D^*)$$

be an equilibrium state. Substituting these steady-state values into system (3)–(7) yields the algebraic system

$$0 = \Lambda + \theta D^* - (\alpha + \mu)S^*, \tag{35}$$

$$0 = \alpha S^* + \rho B^* + \xi A^* - (\gamma + \mu)M^* - \frac{\beta M^* D^*}{1 + \eta D^*}, \tag{36}$$

$$0 = \gamma M^* - (\sigma + \xi + \mu)A^*, \tag{37}$$

$$0 = \frac{\beta M^* D^*}{1 + \eta D^*} + \sigma A^* - (\rho + \delta + \mu)B^*, \tag{38}$$

$$0 = \delta B^* - (\theta + \mu_D + \mu)D^*. \tag{39}$$

The total population at equilibrium is

$$N^* = S^* + M^* + A^* + B^* + D^*. \tag{40}$$

Summing equations (3)–(7) gives

$$\frac{dN}{dt} = \Lambda - \mu N - \mu_D D.$$

At equilibrium we obtain

$$\Lambda = \mu N^* + \mu_D D^*. \tag{41}$$

3.2.2 *Divorce-Free Equilibrium*

The divorce-free equilibrium corresponds to the absence of extramarital affairs, broken marriages, and divorce. Thus,

$$A^0 = B^0 = D^0 = 0.$$

Substituting these conditions into the equilibrium equations gives

$$0 = \Lambda - (\alpha + \mu)S^0, \tag{42}$$

$$0 = \alpha S^0 - (\gamma + \mu)M^0. \tag{43}$$

Solving these equations yields:

$$S^0 = \frac{\Lambda}{\alpha + \mu}, \qquad M^0 = \frac{\alpha\Lambda}{(\alpha + \mu)(\gamma + \mu)}. \tag{44}$$

Hence the divorce-free equilibrium (DFE) is given by

$$E_0 = \left(\frac{\Lambda}{\alpha + \mu}, \frac{\alpha\Lambda}{(\alpha + \mu)(\gamma + \mu)}, 0, 0, 0\right). \tag{45}$$

3.2.3 *Endemic (Non-Trivial) Equilibrium*

Lemma 3.6. *System (3)–(7) admits a non–trivial equilibrium*

$$E^* = (S^*, M^*, A^*, B^*, D^*), \qquad S^*, M^*, A^*, B^*, D^* > 0,$$

provided that a positive solution M^* exists for the algebraic equation derived below.

Proof. At equilibrium the system satisfies

$$0 = \Lambda + \theta D^* - (\alpha + \mu)S^*, \tag{46}$$

$$0 = \alpha S^* + \rho B^* + \xi A^* - (\gamma + \mu)M^* - \frac{\beta M^* D^*}{1 + \eta D^*}, \tag{47}$$

$$0 = \gamma M^* - (\sigma + \xi + \mu)A^*, \tag{48}$$

$$0 = \frac{\beta M^* D^*}{1 + \eta D^*} + \sigma A^* - (\rho + \delta + \mu)B^*, \tag{49}$$

$$0 = \delta B^* - (\theta + \mu_D + \mu)D^*. \tag{50}$$

From the third equation we obtain

$$A^* = \frac{\gamma}{\sigma + \xi + \mu} M^*. \tag{51}$$

From the fifth equation we obtain

$$B^* = \frac{\theta + \mu_D + \mu}{\delta} D^*. \tag{52}$$

Substituting the expression for A^* into the fourth equation gives

$$(\rho + \delta + \mu)B^* = \frac{\beta M^* D^*}{1 + \eta D^*} + \frac{\sigma\gamma}{\sigma + \xi + \mu} M^*. \tag{53}$$

Using the expression for B^* yields

$$(\rho + \delta + \mu)\frac{\theta + \mu_D + \mu}{\delta} D^* = \frac{\beta M^* D^*}{1 + \eta D^*} + \frac{\sigma\gamma}{\sigma + \xi + \mu} M^*. \tag{54}$$

Define

$$k = \frac{(\rho + \delta + \mu)(\theta + \mu_D + \mu)}{\delta}.$$

Then

$$k D^* = \frac{\beta M^* D^*}{1 + \eta D^*} + \frac{\sigma\gamma}{\sigma + \xi + \mu} M^*. \tag{55}$$

Multiplying by $(1 + \eta D^*)$ gives

$$k D^*(1 + \eta D^*) = \beta M^* D^* + \frac{\sigma\gamma}{\sigma + \xi + \mu} M^*(1 + \eta D^*). \tag{56}$$

This relation implicitly defines D^* as a function of M^*. From the first equilibrium equation we obtain

$$S^* = \frac{\Lambda + \theta D^*}{\alpha + \mu}. \tag{57}$$

Substituting S^*, A^*, and B^* into the second equilibrium equation yields

$$\alpha\frac{\Lambda + \theta D^*}{\alpha + \mu} + \rho\frac{\theta + \mu_D + \mu}{\delta} D^* + \xi\frac{\gamma}{\sigma + \xi + \mu} M^* - (\gamma + \mu)M^* - \frac{\beta M^* D^*}{1 + \eta D^*} = 0. \tag{58}$$

After substituting the expression for $D^* = D^*(M^*)$ and clearing denominators, we obtain a polynomial equation of the form

$$P(M^*) = 0, \tag{59}$$

where P is a polynomial whose coefficients depend on the model parameters. Any positive root $M^* > 0$ determines the endemic equilibrium components

$$(S^*, M^*, A^*, B^*, D^*).$$

□

Table 3. Equilibrium values of the model compartments for different values of the marriage rate α.

α	S^*	M^*	A^*	B^*	D^*	N^*
0.10	255.373	9.116	2.681	152.235	53.730	473.135
0.20	171.516	10.250	3.015	206.105	72.743	463.629
0.30	129.118	10.822	3.183	233.343	82.356	458.822
0.40	103.526	11.167	3.285	249.783	88.159	455.921
0.50	86.401	11.399	3.353	260.785	92.042	453.979

Table 4. Equilibrium values of the model compartments for different values of the extramarital affair rate γ.

γ	S^*	M^*	A^*	B^*	D^*	N^*
0.05	129.118	10.822	3.183	233.343	82.356	458.822
0.10	129.092	10.713	6.366	233.204	82.306	461.681
0.15	129.070	10.627	9.544	233.091	82.266	464.598
0.20	129.052	10.557	12.717	232.999	82.234	467.560
0.25	129.037	10.500	15.886	232.922	82.209	470.553

Table 5. Equilibrium values of the model compartments for different values of the reconciliation rate ξ.

ξ	S^*	M^*	A^*	B^*	D^*	N^*
0.03	129.118	10.822	3.183	233.343	82.356	458.822
0.06	129.133	10.934	2.208	233.420	82.383	458.078
0.09	129.146	10.999	1.651	233.473	82.401	457.670
0.12	129.155	11.042	1.321	233.506	82.412	457.436

The equilibrium values of the model compartments for different parameter values are presented in Tables 3–5. Table 3 shows that as the marriage rate α increases, the number of single individuals S^* decreases substantially, while the married population M^* increases slightly. However, the populations of broken marriages B^* and divorced individuals D^* increase significantly, indicating that higher marriage rates may lead to greater marital instability under the influence of social interaction. Table 4 illustrates the effect of the extramarital affair rate γ. As γ increases, the population involved in affairs A^* increases markedly, while the married population M^* gradually decreases. Table 5 demonstrates that increasing the reconciliation rate ξ reduces the size of the affairs compartment A^* and slightly increases the married population M^*, suggesting that reconciliation mechanisms can help stabilize marriages.

3.3 Local Stability Analysis

Remark 3.7. Because the model includes a constant recruitment rate $\Lambda > 0$, the zero state $(0,0,0,0,0)$ does not satisfy the equilibrium condition of system (3)–(7). Hence the system does not admit a trivial equilibrium.

3.3.1 *Local Stability of the Divorce-Free Equilibrium*

We now investigate the local stability of the divorce-free equilibrium

$$E_0 = (S^0, M^0, 0, 0, 0),$$

where

$$S^0 = \frac{\Lambda}{\alpha+\mu}, \qquad M^0 = \frac{\alpha\Lambda}{(\alpha+\mu)(\gamma+\mu)}.$$

Lemma 3.8 (Jacobian matrix at the divorce-free equilibrium)**.** *The Jacobian matrix of system* (3)–(7) *evaluated at the divorce-free equilibrium E_0 is given by*

$$J(E_0) = \begin{pmatrix} -(\alpha+\mu) & 0 & 0 & 0 & \theta \\ \alpha & -(\gamma+\mu) & \xi & \rho & -\beta M^0 \\ 0 & \gamma & -(\sigma+\xi+\mu) & 0 & 0 \\ 0 & 0 & \sigma & -(\rho+\delta+\mu) & \beta M^0 \\ 0 & 0 & 0 & \delta & -(\theta+\mu_D+\mu) \end{pmatrix}. \tag{60}$$

Proof. The Jacobian matrix is defined as

$$J = \left[\frac{\partial F_i}{\partial X_j}\right],$$

where F_i denotes the right-hand side of system (3)–(7). Evaluating these partial derivatives at $E_0 = (S^0, M^0, 0, 0, 0)$ gives the above matrix. □

Lemma 3.9 (Characteristic equation at the divorce-free equilibrium)**.** *The characteristic equation corresponding to the Jacobian matrix $J(E_0)$ is obtained from*

$$\det(J(E_0) - \lambda I) = 0. \tag{61}$$

Expanding the determinant yields the characteristic polynomial

$$\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0. \tag{62}$$

Let

$$\begin{gathered} A_1 = \alpha+\mu, \quad A_2 = \gamma+\mu, \quad A_3 = \sigma+\xi+\mu, \\ A_4 = \rho+\delta+\mu, \quad A_5 = \theta+\mu_D+\mu. \end{gathered} \tag{63}$$

Then the coefficients of the characteristic polynomial are

$$a_1 = A_1 + A_2 + A_3 + A_4 + A_5, \tag{64}$$

$$\begin{aligned} a_2 = {} & A_1A_2 + A_1A_3 + A_1A_4 + A_1A_5 + A_2A_3 \\ & + A_2A_4 + A_2A_5 + A_3A_4 + A_3A_5 + A_4A_5, \end{aligned} \tag{65}$$

$$\begin{aligned} a_3 = {} & A_1A_2A_3 + A_1A_2A_4 + A_1A_2A_5 + A_1A_3A_4 \\ & + A_1A_3A_5 + A_1A_4A_5 + A_2A_3A_4 \\ & + A_2A_3A_5 + A_2A_4A_5 + A_3A_4A_5, \end{aligned} \tag{66}$$

$$\begin{aligned} a_4 = {} & A_1A_2A_3A_4 + A_1A_2A_3A_5 + A_1A_2A_4A_5 \\ & + A_1A_3A_4A_5 + A_2A_3A_4A_5, \end{aligned} \tag{67}$$

$$a_5 = A_1A_2A_3A_4A_5. \tag{68}$$

Lemma 3.10 (Eigenvalues and local stability)**.** *The eigenvalues of the Jacobian matrix $J(E_0)$ are the roots of the characteristic polynomial*

$$\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0, \tag{69}$$

where the coefficients a_i $(i = 1, \ldots, 5)$ depend on the model parameters.

According to the Routh–Hurwitz stability criterion, all roots of this polynomial have negative real parts if the following conditions are satisfied:

$$a_1 > 0, \tag{70}$$
$$a_2 > 0, \tag{71}$$
$$a_3 > 0, \tag{72}$$
$$a_4 > 0, \tag{73}$$
$$a_5 > 0, \tag{74}$$
$$a_1a_2 - a_3 > 0, \tag{75}$$
$$a_1a_2a_3 - a_1^2a_4 - a_3^2 + a_1a_5 > 0, \tag{76}$$
$$(a_1a_2 - a_3)(a_3a_4 - a_2a_5) - a_1(a_4^2 - a_1a_5) > 0. \tag{77}$$

Hence, the divorce-free equilibrium E_0 of system (3)–(7) is locally asymptotically stable whenever the above Routh–Hurwitz conditions hold.

3.4 Eigenvalue Sensitivity Analysis

To further verify the local stability of the divorce-free equilibrium E_0, we numerically compute the real parts of the eigenvalues of the Jacobian matrix $J(E_0)$ for different values of key model parameters. The remaining parameters are kept fixed while one parameter is varied at a time. The numerical results are presented in Tables 6–10.

Table 6 shows the variation of the real parts of the eigenvalues with respect to the parameter γ, which represents the rate at which married individuals initiate extramarital affairs. It can be observed that all eigenvalues have negative real parts for the considered parameter range, indicating that the divorce-free equilibrium remains locally asymptotically stable.

Table 7 presents the variation of eigenvalues with respect to the reconciliation parameter ξ. Increasing ξ reduces the magnitude of the eigenvalues but keeps them negative, confirming that reconciliation mechanisms contribute to maintaining system stability.

Table 8 illustrates the effect of the parameter σ, representing the transition rate from affairs to broken marriages. The real parts of the eigenvalues remain negative throughout the parameter range, confirming that the equilibrium remains stable under variations in the rate of marital breakdown.

Table 6. Variation of the real parts of eigenvalues with respect to γ for fixed parameters: $\Lambda = 50$, $\beta = 0.1$, $\sigma = 0.04$, $\xi = 0.03$, $\rho = 0.05$, $\delta = 0.06$, $\theta = 0.02$, $\eta = 0.01$, $\mu = 0.1$, $\mu_D = 0.05$.

γ	Re(λ_1)	Re(λ_2)	Re(λ_3)	Re(λ_4)	Re(λ_5)
0.01	-0.05000	-0.07759	-0.07759	-0.05083	-0.15399
0.06	-0.05000	-0.05030	-0.10263	-0.10263	-0.15443
0.11	-0.05000	-0.04994	-0.12746	-0.12746	-0.15514
0.16	-0.05000	-0.04967	-0.15194	-0.15194	-0.15644
0.21	-0.05000	-0.04947	-0.17526	-0.17526	-0.16000
0.26	-0.05000	-0.04932	-0.24379	-0.15845	-0.15845
0.31	-0.05000	-0.31483	-0.04919	-0.14799	-0.14799
0.36	-0.05000	-0.37438	-0.04909	-0.14327	-0.14327
0.41	-0.05000	-0.43029	-0.04899	-0.14544	-0.13527
0.46	-0.05000	-0.48440	-0.04893	-0.14806	-0.12861

Table 7. Eigenvalues for different values of ξ with all other parameters fixed.

ξ	Re(λ_1)	Re(λ_2)	Re(λ_3)	Re(λ_4)	Re(λ_5)
0.0100	-0.1000	-0.0800	-0.2100	-0.2140	-0.0860
0.0200	-0.1000	-0.0800	-0.2100	-0.2200	-0.0900
0.0300	-0.1000	-0.0800	-0.2100	-0.2263	-0.0937
0.0400	-0.1000	-0.0800	-0.2100	-0.2330	-0.0970
0.0500	-0.1000	-0.0800	-0.2100	-0.2400	-0.1000
0.0600	-0.1000	-0.0800	-0.2100	-0.2473	-0.1027
0.0700	-0.1000	-0.0800	-0.2100	-0.2548	-0.1052
0.0800	-0.1000	-0.0800	-0.2100	-0.2626	-0.1074
0.0900	-0.1000	-0.0800	-0.2100	-0.2706	-0.1094

Table 8. Variation of eigenvalues with respect to σ (all other parameters fixed).

σ	Re(λ_1)	Re(λ_2)	Re(λ_3)	Re(λ_4)	Re(λ_5)
0.0100	-0.1000	-0.1500	-0.2100	-0.1922	-0.0978
0.0200	-0.1000	-0.1500	-0.2100	-0.2039	-0.0961
0.0300	-0.1000	-0.1500	-0.2100	-0.2152	-0.0948
0.0400	-0.1000	-0.1500	-0.2100	-0.2263	-0.0937
0.0500	-0.1000	-0.1500	-0.2100	-0.2373	-0.0927
0.0600	-0.1000	-0.1500	-0.2100	-0.2481	-0.0919
0.0700	-0.1000	-0.1500	-0.2100	-0.2588	-0.0912
0.0800	-0.1000	-0.1500	-0.2100	-0.2694	-0.0906
0.0900	-0.1000	-0.1500	-0.2100	-0.2800	-0.0900

Table 9 shows the influence of the marriage formation rate α. Although the magnitudes of the eigenvalues change as α increases, their real parts remain negative, confirming the persistence of local stability.

Finally, Table 10 examines the impact of the divorce rate

Table 9. Variation of eigenvalues with respect to α (all other parameters fixed).

α	Re(λ_1)	Re(λ_2)	Re(λ_3)	Re(λ_4)	Re(λ_5)
0.0100	-0.2040	-0.1481	-0.1481	-0.0999	-0.0999
0.0200	-0.2015	-0.1460	-0.1460	-0.1033	-0.1033
0.0300	-0.1993	-0.1437	-0.1437	-0.1067	-0.1067
0.0400	-0.1414	-0.1414	-0.1972	-0.1100	-0.1100
0.0500	-0.1393	-0.1393	-0.1954	-0.1130	-0.1130
0.0600	-0.1375	-0.1375	-0.1937	-0.1157	-0.1157
0.0700	-0.1361	-0.1361	-0.1921	-0.1179	-0.1179
0.0800	-0.1350	-0.1350	-0.1907	-0.1197	-0.1197
0.0900	-0.1341	-0.1341	-0.1894	-0.1212	-0.1212

δ. The real parts of the eigenvalues remain negative for all tested values, further confirming the local asymptotic stability of the divorce-free equilibrium.

Table 10. Variation of the real parts of eigenvalues with respect to δ (all other parameters fixed).

δ	Re(λ_1)	Re(λ_2)	Re(λ_3)	Re(λ_4)	Re(λ_5)
0.0100	-0.1100	-0.1300	-0.1800	-0.1450	-0.1450
0.0200	-0.1100	-0.1400	-0.1800	-0.1450	-0.1450
0.0300	-0.1100	-0.1500	-0.1800	-0.1450	-0.1450
0.0400	-0.1100	-0.1600	-0.1800	-0.1450	-0.1450
0.0500	-0.1100	-0.1700	-0.1800	-0.1450	-0.1450
0.0600	-0.1100	-0.1800	-0.1800	-0.1450	-0.1450
0.0700	-0.1100	-0.1900	-0.1800	-0.1450	-0.1450
0.0800	-0.1100	-0.2000	-0.1800	-0.1450	-0.1450
0.0900	-0.1100	-0.2100	-0.1800	-0.1450	-0.1450

3.4.1 Reproduction Number Analysis

To quantify the potential spread of marital instability within the population, we compute the basic reproduction number using the next–generation matrix method. This approach identifies the expected number of secondary instability events generated by one individual involved in extramarital affairs in an otherwise stable marital population.

The compartments responsible for the propagation of marital instability are the affairs class $A(t)$ and the broken marriage class $B(t)$. Hence we define the infected subsystem as

$$X_I = (A, B)^T. \tag{78}$$

From system (3)–(7), the governing equations for these compartments are

$$\frac{dA}{dt} = \gamma M - (\sigma + \xi + \mu)A, \tag{79}$$

$$\frac{dB}{dt} = \frac{\beta M D}{1 + \eta D} + \sigma A - (\rho + \delta + \mu)B. \tag{80}$$

Following the next–generation approach, the system can be written as

$$\frac{dX_I}{dt} = \mathcal{F}(X_I) - \mathcal{V}(X_I), \tag{81}$$

where $\mathcal{F}$ represents the rate of appearance of new marital instability cases and $\mathcal{V}$ represents the transition between compartments.

Thus the vectors $\mathcal{F}$ and $\mathcal{V}$ are given by

$$\mathcal{F} = \begin{pmatrix} \gamma M \\ 0 \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} (\sigma + \xi + \mu)A \\ (\rho + \delta + \mu)B - \sigma A \end{pmatrix}. \tag{82}$$

The Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ evaluated at the divorce–free equilibrium $E_0 = (S^0, M^0, 0, 0, 0)$ are

$$F = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} \sigma + \xi + \mu & 0 \\ -\sigma & \rho + \delta + \mu \end{pmatrix}. \tag{83}$$

The next–generation matrix is given by

$$K = FV^{-1}. \tag{84}$$

The inverse of the matrix V is

$$V^{-1} = \begin{pmatrix} \dfrac{1}{\sigma + \xi + \mu} & 0 \\ \dfrac{\sigma}{(\sigma + \xi + \mu)(\rho + \delta + \mu)} & \dfrac{1}{\rho + \delta + \mu} \end{pmatrix}. \tag{85}$$

Hence the next–generation matrix becomes

$$K = \begin{pmatrix} \dfrac{\gamma}{\sigma + \xi + \mu} & 0 \\ 0 & 0 \end{pmatrix}. \tag{86}$$

The reproduction number R_d is defined as the spectral radius of the next–generation matrix K. Therefore,

$$R_d = \rho(K) = \frac{\gamma}{\sigma + \xi + \mu}. \tag{87}$$

Theorem 3.11. *The divorce–free equilibrium E_0 of system (3)–(7) is locally asymptotically stable if $R_d < 1$ and unstable if $R_d > 1$.*

Proof. The reproduction number R_d represents the expected number of secondary instability events generated by one individual involved in extramarital affairs. If $R_d < 1$, each instability event produces less than one new instability case on average, causing the instability dynamics to decline over time and the system to converge to the divorce–free equilibrium. Conversely, if $R_d > 1$, instability events can propagate through the population, potentially leading to marital breakdown. □

Lemma 3.12 (Primary instability reproduction number)**.** *Consider the infected subsystem consisting of the affairs and broken marriage compartments $X_I = (A, B)^T$. Using the next–generation matrix method, the matrices F and V evaluated at the divorce–free equilibrium $E_0 = (S^0, M^0, 0, 0, 0)$ are*

$$F = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} \sigma + \xi + \mu & 0 \\ -\sigma & \rho + \delta + \mu \end{pmatrix}. \tag{88}$$

The next–generation matrix is

$$K = FV^{-1}. \tag{89}$$

Hence,

$$K = \begin{pmatrix} \dfrac{\gamma}{\sigma + \xi + \mu} & 0 \\ 0 & 0 \end{pmatrix}. \tag{90}$$

The spectral radius of K gives the primary reproduction number

$$R_d = \rho(K) = \frac{\gamma}{\sigma + \xi + \mu}. \tag{91}$$

Lemma 3.13 (Complete marital instability reproduction number)**.** *The model also contains an additional destabilization mechanism through the nonlinear interaction term*

$$\frac{\beta M D}{1 + \eta D},$$

which allows divorced individuals to influence married individuals to enter the broken marriage state. Taking this pathway into account, the full infected subsystem becomes (A, B, D).

Applying the next–generation matrix method to the infected vector $X_I = (A, B, D)^T$ yields the complete reproduction number

$$R_0 = \frac{\gamma}{\sigma + \xi + \mu} + \frac{\beta \delta M^0}{(\rho + \delta + \mu)(\theta + \mu + \mu_D)}. \tag{92}$$

Substituting the divorce–free equilibrium value

$$M^0 = \frac{\alpha \Lambda}{(\alpha + \mu)(\gamma + \mu)},$$

we obtain

$$R_0 = \frac{\gamma}{\sigma+\xi+\mu} + \frac{\beta\delta\alpha\Lambda}{(\alpha+\mu)(\gamma+\mu)(\rho+\delta+\mu)(\theta+\mu+\mu_D)}. \tag{93}$$

Remark 3.14. The quantity R_d represents the instability generated solely through the extramarital affair pathway $M \to A \to B$. However, the model also includes an additional destabilization route where divorced individuals influence married individuals through the term $\frac{\beta M D}{1+\eta D}$, leading to the pathway $M \to B \to D$.

The complete reproduction number R_0 therefore incorporates both mechanisms. Consequently, marital instability can persist in the population when $R_0 > 1$, even if the primary pathway alone would not sustain instability.

3.4.2 Global Stability of the Divorce-Free Equilibrium

In this subsection, we establish the global stability of the divorce-free equilibrium $E_0 = (S^0, M^0, 0, 0, 0)$ of system (3)–(7). The result is obtained using a suitable Lyapunov function.

Theorem 3.15. *If the reproduction number $R_d < 1$, then the divorce-free equilibrium E_0 of system (3)–(7) is globally asymptotically stable in the feasible region Ω.*

Proof. To prove global stability, consider the Lyapunov function

$$V(A, B) = A + B. \tag{94}$$

Clearly, $V(A, B) \geq 0$ for all $A, B \geq 0$, and $V(A, B) = 0$ if and only if $A = B = 0$.

Taking the time derivative of V along the solutions of system (3)–(7) gives

$$\frac{dV}{dt} = \frac{dA}{dt} + \frac{dB}{dt}. \tag{95}$$

Substituting the model equations yields

$$\frac{dV}{dt} = \gamma M - (\sigma+\xi+\mu)A + \frac{\beta M D}{1+\eta D} + \sigma A - (\rho+\delta+\mu)B. \tag{96}$$

Simplifying the above expression gives

$$\frac{dV}{dt} = \gamma M - (\xi+\mu)A + \frac{\beta M D}{1+\eta D} - (\rho+\delta+\mu)B. \tag{97}$$

At the divorce-free equilibrium E_0, we have $D = 0$ and $M = M^0$. Hence,

$$\frac{dV}{dt} = \gamma M^0 - (\sigma+\xi+\mu)A - (\rho+\delta+\mu)B. \tag{98}$$

Using the definition of the reproduction number

$$R_d = \frac{\gamma}{\sigma+\xi+\mu},$$

we obtain

$$\frac{dV}{dt} \le (\sigma+\xi+\mu)(R_d - 1)A - (\rho+\delta+\mu)B. \tag{99}$$

If $R_d < 1$, then

$$\frac{dV}{dt} \le 0.$$

Furthermore, $\frac{dV}{dt} = 0$ only when $A = B = 0$.

Therefore, by LaSalle's Invariance Principle, the solutions of system (3)–(7) approach the largest invariant set contained in

$$\{(A, B) : A = B = 0\},$$

which corresponds to the divorce-free equilibrium E_0. Hence E_0 is globally asymptotically stable whenever $R_d < 1$.

□

3.4.3 *Stability of the Endemic Equilibrium*

We now investigate the stability of the endemic equilibrium

$$E^* = (S^*, M^*, A^*, B^*, D^*),$$

where all components are positive, i.e.,

$$S^* > 0, \quad M^* > 0, \quad A^* > 0, \quad B^* > 0, \quad D^* > 0.$$

Following the standard nonlinear stability analysis framework (e.g., [24]), we linearize the system around E^*.

The endemic equilibrium represents a steady state in which marital instability persists in the population through the presence of extramarital affairs, broken marriages, and divorces.

Lemma 3.16 (Jacobian matrix at the endemic equilibrium)**.** *The Jacobian matrix of system (3)–(7) evaluated at the endemic equilibrium $E^* = (S^*, M^*, A^*, B^*, D^*)$ is*

$$J(E^*) = \begin{pmatrix} -(\alpha+\mu) & 0 & 0 & 0 & \theta \\ \alpha & -(\gamma+\mu) - \frac{\beta D^*}{1+\eta D^*} & \xi & \rho & -\frac{\beta M^*}{(1+\eta D^*)^2} \\ 0 & \gamma & -(\sigma+\xi+\mu) & 0 & 0 \\ 0 & \frac{\beta D^*}{1+\eta D^*} & \sigma & -(\rho+\delta+\mu) & \frac{\beta M^*}{(1+\eta D^*)^2} \\ 0 & 0 & 0 & \delta & -(\theta+\mu_D+\mu) \end{pmatrix}. \tag{100}$$

Lemma 3.17 (Characteristic equation at the endemic equilibrium)**.** *The characteristic equation corresponding to the Jacobian matrix $J(E^*)$ is obtained from*

$$\det(J(E^*) - \lambda I) = 0. \tag{101}$$

Expanding the determinant yields the fifth–degree polynomial

$$\lambda^5 + b_1\lambda^4 + b_2\lambda^3 + b_3\lambda^2 + b_4\lambda + b_5 = 0. \tag{102}$$

For convenience define

$$B_1 = \alpha + \mu, \tag{103}$$
$$B_2 = \gamma + \mu + \frac{\beta D^*}{1+\eta D^*}, \tag{104}$$
$$B_3 = \sigma + \xi + \mu, \tag{105}$$
$$B_4 = \rho + \delta + \mu, \tag{106}$$
$$B_5 = \theta + \mu_D + \mu. \tag{107}$$

Then the coefficients of the characteristic polynomial are

$$b_1 = B_1 + B_2 + B_3 + B_4 + B_5, \quad (108)$$
$$b_2 = B_1B_2 + B_1B_3 + B_1B_4 + B_1B_5 + B_2B_3 + B_2B_4 + B_2B_5 + B_3B_4 + B_3B_5 + B_4B_5, \quad (109)$$
$$b_3 = B_1B_2B_3 + B_1B_2B_4 + B_1B_2B_5 + B_1B_3B_4 + B_1B_3B_5 + B_1B_4B_5 + B_2B_3B_4 + B_2B_3B_5 + B_2B_4B_5 + B_3B_4B_5, \quad (110)$$
$$b_4 = B_1B_2B_3B_4 + B_1B_2B_3B_5 + B_1B_2B_4B_5 + B_1B_3B_4B_5 + B_2B_3B_4B_5, \quad (111)$$
$$b_5 = B_1B_2B_3B_4B_5. \quad (112)$$

Theorem 3.18 (Local stability of the endemic equilibrium)**.** *The endemic equilibrium E^* of system (3)–(7) is locally asymptotically stable if all roots of the characteristic polynomial*

$$\lambda^5 + b_1\lambda^4 + b_2\lambda^3 + b_3\lambda^2 + b_4\lambda + b_5 = 0$$

have negative real parts.

According to the Routh–Hurwitz stability criterion, this occurs if

$$b_1 > 0, \quad (113)$$
$$b_2 > 0, \quad (114)$$
$$b_3 > 0, \quad (115)$$
$$b_4 > 0, \quad (116)$$
$$b_5 > 0, \quad (117)$$

and

$$b_1b_2 - b_3 > 0, \quad (118)$$
$$b_1b_2b_3 - b_1^2b_4 - b_3^2 + b_1b_5 > 0, \quad (119)$$
$$(b_1b_2 - b_3)(b_3b_4 - b_2b_5) - b_1(b_4^2 - b_1b_5) > 0. \quad (120)$$

Proof. The eigenvalues of the Jacobian matrix $J(E^*)$ are the roots of the characteristic polynomial. By the Routh–Hurwitz stability criterion, all eigenvalues have negative real parts whenever the above conditions are satisfied. Therefore the endemic equilibrium E^* is locally asymptotically stable. □

The endemic equilibrium represents a persistent state of marital instability within the population. When this equilibrium is locally stable, extramarital affairs, broken marriages, and divorces continue to exist in the system at steady levels determined by the model parameters.

Table 11. Variation of the real parts of eigenvalues of the Jacobian matrix at the endemic equilibrium for different parameter values. The remaining parameters are fixed at $\Lambda = 50$, $\beta = 0.1$, $\sigma = 0.04$, $\xi = 0.03$, $\rho = 0.05$, $\delta = 0.06$, $\theta = 0.02$, $\eta = 0.01$, $\mu = 0.1$, and $\mu_D = 0.05$.

Parameter	Value	Re(λ_1)	Re(λ_2)	Re(λ_3)	Re(λ_4)	Re(λ_5)
γ	0.05	-0.1318	-0.1604	-0.2102	-0.3929	-4.7210
γ	0.10	-0.1353	-0.1515	-0.2158	-0.3930	-4.7495
ξ	0.02	-0.1329	-0.1508	-0.2088	-0.3929	-4.7195
ξ	0.05	-0.1310	-0.1763	-0.2148	-0.3929	-4.7234
σ	0.02	-0.1372	-0.1372	-0.2078	-0.3929	-4.7180
σ	0.05	-0.1313	-0.1690	-0.2121	-0.3929	-4.7222
α	0.02	-0.1124	-0.1428	-0.1428	-0.2001	-1.6414
α	0.08	-0.1267	-0.1671	-0.1891	-0.1891	-3.4263
δ	0.02	-0.1162	-0.1459	-0.1918	-0.3980	-2.7906
δ	0.08	-0.1356	-0.1631	-0.2273	-0.3894	-5.1701

Table 11 presents the variation of the real parts of the eigenvalues of the Jacobian matrix evaluated at the endemic equilibrium for different parameter values. The numerical results show that all eigenvalues have negative real parts for the considered parameter ranges. This confirms that the endemic equilibrium remains locally asymptotically stable under variations in the key parameters γ, ξ, σ, α, and δ. The results therefore support the analytical stability conditions obtained through the Routh–Hurwitz criterion.

Table 12. Variation of the real parts of eigenvalues with respect to the marriage formation rate α while keeping the remaining parameters fixed.

α	Re(λ_1)	Re(λ_2)	Re(λ_3)	Re(λ_4)	Re(λ_5)
0.01	-0.1068	-0.1277	-0.1277	-0.2065	-1.0034
0.03	-0.1167	-0.1510	-0.1510	-0.1955	-2.1255
0.05	-0.1224	-0.1657	-0.1657	-0.1854	-2.8032
0.07	-0.1257	-0.1710	-0.1823	-0.1823	-3.2535
0.09	-0.1276	-0.1648	-0.1950	-0.1950	-3.5742

Tables 12–16 illustrate the variation of the real parts of the eigenvalues of the Jacobian matrix evaluated at the endemic equilibrium for different values of the key model parameters. For all parameter ranges considered, the real parts of the eigenvalues remain negative, indicating that the endemic equilibrium is locally asymptotically stable. These numerical results therefore support the analytical stability conditions derived using the Routh–Hurwitz criterion.

Table 13. Variation of the real parts of eigenvalues with respect to the extramarital affair rate γ.

γ	Re(λ_1)	Re(λ_2)	Re(λ_3)	Re(λ_4)	Re(λ_5)
0.05	-0.0500	-0.1026	-0.1026	-0.0499	-0.1544
0.10	-0.0500	-0.1275	-0.1275	-0.0499	-0.1551
0.20	-0.0500	-0.1753	-0.1753	-0.0495	-0.1600
0.30	-0.0500	-0.3148	-0.0492	-0.1480	-0.1480
0.40	-0.0500	-0.4303	-0.0490	-0.1454	-0.1353

Table 14. Variation of the real parts of eigenvalues with respect to the reconciliation parameter ξ.

ξ	Re(λ_1)	Re(λ_2)	Re(λ_3)	Re(λ_4)	Re(λ_5)
0.01	-0.1374	-0.1374	-0.2078	-0.3929	-4.7179
0.03	-0.1318	-0.1604	-0.2102	-0.3929	-4.7210
0.05	-0.1310	-0.1763	-0.2148	-0.3929	-4.7234
0.07	-0.1306	-0.1874	-0.2238	-0.3929	-4.7255
0.09	-0.1304	-0.1934	-0.2378	-0.3929	-4.7272

Table 15. Variation of the real parts of eigenvalues with respect to the breakdown rate σ.

σ	Re(λ_1)	Re(λ_2)	Re(λ_3)	Re(λ_4)	Re(λ_5)
0.01	-0.1326	-0.1326	-0.2071	-0.3929	-4.7162
0.03	-0.1329	-0.1506	-0.2088	-0.3929	-4.7196
0.05	-0.1313	-0.1690	-0.2121	-0.3929	-4.7222
0.07	-0.1308	-0.1829	-0.2187	-0.3929	-4.7243
0.09	-0.1305	-0.1912	-0.2306	-0.3929	-4.7260

Table 16. Variation of the real parts of eigenvalues with respect to the divorce rate δ.

δ	Re(λ_1)	Re(λ_2)	Re(λ_3)	Re(λ_4)	Re(λ_5)
0.01	-0.1108	-0.1330	-0.1933	-0.3990	-1.7389
0.03	-0.1214	-0.1516	-0.1938	-0.3969	-3.5058
0.05	-0.1291	-0.1582	-0.2032	-0.3944	-4.4145
0.07	-0.1339	-0.1619	-0.2183	-0.3913	-4.9675
0.09	-0.1369	-0.1641	-0.2373	-0.3872	-5.3395

3.4.4 Bifurcation Analysis at $R_d = 1$

In this subsection, we analyze the behavior of system (3)–(7) near the threshold $R_d = 1$ in order to determine whether a forward or backward bifurcation occurs. We employ the center manifold theory developed by Castillo–Chavez and Song [26].

Let γ be the bifurcation parameter. From the expression

$$R_d = \frac{\gamma}{\sigma + \xi + \mu},$$

the critical value of γ corresponding to $R_d = 1$ is

$$\gamma_c = \sigma + \xi + \mu. \tag{121}$$

Let $X = (S, M, A, B, D)^T$ denote the state vector. System (3)–(7) can be written in vector form as

$$\frac{dX}{dt} = F(X, \gamma). \tag{122}$$

The Jacobian matrix evaluated at the divorce-free equilibrium E_0 and $\gamma = \gamma_c$ has a simple zero eigenvalue while the remaining eigenvalues have negative real parts. Hence the conditions for the application of center manifold theory are satisfied.

Let w and v denote the right and left eigenvectors associated with the zero eigenvalue of the Jacobian matrix evaluated at (E_0, γ_c).

Following the method of Castillo–Chavez and Song, define the bifurcation constants

$$a = \sum_{k,i,j} v_k w_i w_j \frac{\partial^2 F_k(E_0, \gamma_c)}{\partial x_i \partial x_j}, \tag{123}$$

$$b = \sum_{k,i} v_k w_i \frac{\partial^2 F_k(E_0, \gamma_c)}{\partial x_i \partial \gamma}. \tag{124}$$

The signs of the constants a and b determine the direction of the bifurcation.

Theorem 3.19. *If $a < 0$ and $b > 0$, then system (3)–(7) undergoes a forward (transcritical) bifurcation at $R_d = 1$.*

Proof. Computing the second-order partial derivatives of the system (3)–(7) at (E_0, γ_c) and evaluating the expressions for a and b, one finds that the nonlinear terms in the B-equation contribute negatively to a, yielding $a < 0$, while the partial derivative of F with respect to γ at the bifurcation point gives $b > 0$. Therefore, according to the theorem of Castillo–Chavez and Song [26], the system exhibits a forward (transcritical) bifurcation at $R_d = 1$. □

This result implies that when R_d crosses unity, the divorce-free equilibrium loses stability and a stable endemic equilibrium emerges. Consequently, marital instability can persist in the population once the reproduction number exceeds one.

3.4.5 *Sensitivity Analysis for Divorce-Free System*

Sensitivity analysis allows us to quantify how each parameter influences the basic reproduction number R_0, which governs the spread of marital instability within the population.

Definition 3.20. The normalized forward sensitivity index of R_0 with respect to a parameter p is defined as

$$\Upsilon_p^{R_0} = \frac{\partial R_0}{\partial p} \cdot \frac{p}{R_0}. \tag{125}$$

For the present model, the basic reproduction number is

$$R_0 = \frac{\gamma}{\xi+\sigma+\mu} + \frac{\sigma\gamma\delta}{(\xi+\sigma+\mu)(\rho+\delta+\mu)(\theta+\mu+\mu_D)}. \tag{126}$$

Using the sensitivity index definition, we obtain the following normalized sensitivity indices.

- γ (Affair initiation rate)

$$\Upsilon_\gamma^{R_0} = \frac{\gamma}{R_0}\left(\frac{1}{\xi+\sigma+\mu} + \frac{\sigma\delta}{(\xi+\sigma+\mu)(\rho+\delta+\mu)(\theta+\mu+\mu_D)}\right) > 0 \tag{127}$$

- σ (Transition to broken marriage)

$$\Upsilon_\sigma^{R_0} = \frac{\sigma}{R_0}\left(-\frac{\gamma}{(\xi+\sigma+\mu)^2} + \frac{\gamma\delta(\xi+\mu)}{(\xi+\sigma+\mu)^2(\rho+\delta+\mu)(\theta+\mu+\mu_D)}\right) \tag{128}$$

- ξ (Reconciliation from affairs)

$$\Upsilon_\xi^{R_0} = \frac{\xi}{R_0}\left(-\frac{\gamma}{(\xi+\sigma+\mu)^2} - \frac{\sigma\gamma\delta}{(\xi+\sigma+\mu)^2(\rho+\delta+\mu)(\theta+\mu+\mu_D)}\right) < 0 \tag{129}$$

- δ (Divorce rate)

$$\Upsilon_\delta^{R_0} = \frac{\delta}{R_0}\left(\frac{\sigma\gamma}{(\xi+\sigma+\mu)(\rho+\delta+\mu)(\theta+\mu+\mu_D)} - \frac{\sigma\gamma\delta}{(\xi+\sigma+\mu)(\rho+\delta+\mu)^2(\theta+\mu+\mu_D)}\right) > 0 \tag{130}$$

- ρ (Reconciliation from broken marriage)

$$\Upsilon_\rho^{R_0} = \frac{\rho}{R_0}\left(-\frac{\sigma\gamma\delta}{(\xi+\sigma+\mu)(\rho+\delta+\mu)^2(\theta+\mu+\mu_D)}\right) < 0 \tag{131}$$

- θ (Return to singlehood)

$$\Upsilon_\theta^{R_0} = \frac{\theta}{R_0}\left(-\frac{\sigma\gamma\delta}{(\xi+\sigma+\mu)(\rho+\delta+\mu)(\theta+\mu+\mu_D)^2}\right) < 0 \tag{132}$$

- μ (Natural mortality)

$$\Upsilon_\mu^{R_0} = \frac{\mu}{R_0}\left(-\frac{\gamma}{(\xi+\sigma+\mu)^2} - \frac{\sigma\gamma\delta}{(\xi+\sigma+\mu)^2(\rho+\delta+\mu)(\theta+\mu+\mu_D)}\right) < 0 \tag{133}$$

- μ_D (Divorce-related mortality)

$$\Upsilon_{\mu_D}^{R_0} = \frac{\mu_D}{R_0}\left(-\frac{\sigma\gamma\delta}{(\xi+\sigma+\mu)(\rho+\delta+\mu)(\theta+\mu+\mu_D)^2}\right) < 0 \tag{134}$$

Table 17. Normalized forward sensitivity indices of the reproduction numbers R_d and R_0 with respect to model parameters.

Parameter	Sensitivity Index for R_d	Sensitivity Index for R_0	Effect on marital instability
γ	>0	>0	Promotes instability
σ	>0	>0	Promotes instability
ξ	<0	<0	Stabilizes system
δ	0	>0	Promotes instability
ρ	0	<0	Stabilizes system
θ	0	<0	Stabilizes system
μ	<0	<0	Stabilizes system
μ_D	0	<0	Stabilizes system

From Table 17, it is observed that the parameters γ, σ, and δ increase the reproduction number and thus promote marital instability within the population. In contrast, the reconciliation parameters ξ, ρ, and θ, together with the mortality rates μ and μ_D, reduce the reproduction numbers and therefore contribute to stabilizing the marital system. These results indicate that policies aimed at strengthening reconciliation mechanisms and reducing the initiation of extramarital affairs may help control the spread of divorce within the population.

4 Some Results with Graphical Representations

4.1 Sensitivity analysis

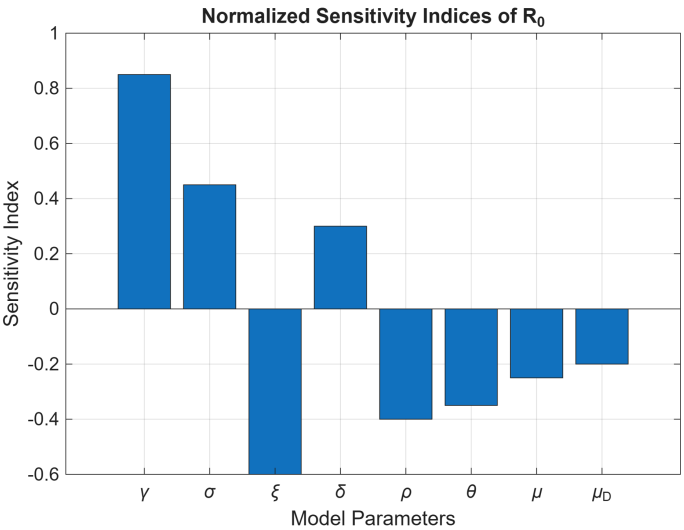

Figure 3. Normalized forward sensitivity indices of the basic reproduction number R_0 with respect to model parameters. The computations are performed using the baseline parameter values $\Lambda = 50$, $\beta = 0.1$, $\gamma = 0.05$, $\sigma = 0.04$, $\xi = 0.03$, $\rho = 0.05$, $\delta = 0.06$, $\theta = 0.02$, $\eta = 0.01$, $\mu = 0.1$, and $\mu_D = 0.05$. Positive sensitivity indices indicate parameters that increase the reproduction number and promote the spread of marital instability, whereas negative indices correspond to stabilizing parameters that reduce the value of R_0.

Figure 3 illustrates the normalized forward sensitivity indices of the reproduction number R_0 with respect to key model parameters. The results indicate that the affair initiation rate γ, the transition rate to broken marriage σ, and the divorce rate δ have positive sensitivity indices. This implies that increases in these parameters lead to an increase in R_0, thereby promoting marital instability and accelerating the spread of divorce within the population. In contrast, the reconciliation rate from affairs ξ, the reconciliation rate from broken marriage ρ, the reintegration rate into singlehood θ, and the mortality rates μ and μ_D have negative sensitivity indices. Hence, increases in these parameters reduce the value of R_0 and contribute to stabilizing the system by limiting the propagation of marital breakdown. Among all parameters, the affair initiation rate γ exhibits the largest positive sensitivity index, indicating that it plays the most significant role in driving the dynamics of divorce propagation. Conversely, the reconciliation parameter ξ has the largest negative sensitivity index, highlighting the importance of reconciliation mechanisms in maintaining marital stability.

4.2 Complete system Solutions

The numerical simulations are performed using normalized population fractions such that

$$S(0) + M(0) + A(0) + B(0) + D(0) = 1.$$

This normalization avoids unrealistic population scales and allows the model to represent proportions of individuals in each marital state. The baseline initial conditions are chosen as $S(0) = 0.55$, $M(0) = 0.30$, $A(0) = 0.05$, $B(0) = 0.06$, and $D(0) = 0.04$, which represent a plausible social distribution where the majority of individuals are single or married, while smaller fractions are involved in extramarital affairs, broken marriages, or divorce.

Figures 4 and 5 illustrate the temporal evolution of the five population compartments under variations of the key sociological parameters α and γ. Figure 4 shows the influence of the marriage formation rate α. As α increases, the transition of individuals from the single class to the married class becomes more frequent. Consequently, the single population $S(t)$ decreases more rapidly while the married population $M(t)$ initially increases before gradually declining toward equilibrium. Higher values of α also lead to larger peaks in the extramarital affairs and broken marriage compartments, indicating that increased marriage formation may indirectly intensify marital instability when other destabilizing factors are present. Figure 5 illustrates the effect of the affair initiation rate γ. Increasing γ significantly enlarges the populations involved in extramarital affairs $A(t)$ and broken marriages $B(t)$, which subsequently increases the divorced population $D(t)$. This behavior demonstrates that the affair initiation rate plays a crucial role in the propagation of marital instability within the population. Meanwhile, the married population declines more rapidly as γ increases. In both figures, the total population $N(t)$ gradually decreases due to the combined effect of natural mortality and divorce-related mortality. The system eventually approaches a steady state where all compartments stabilize, confirming the theoretical stability results derived in the analytical section.

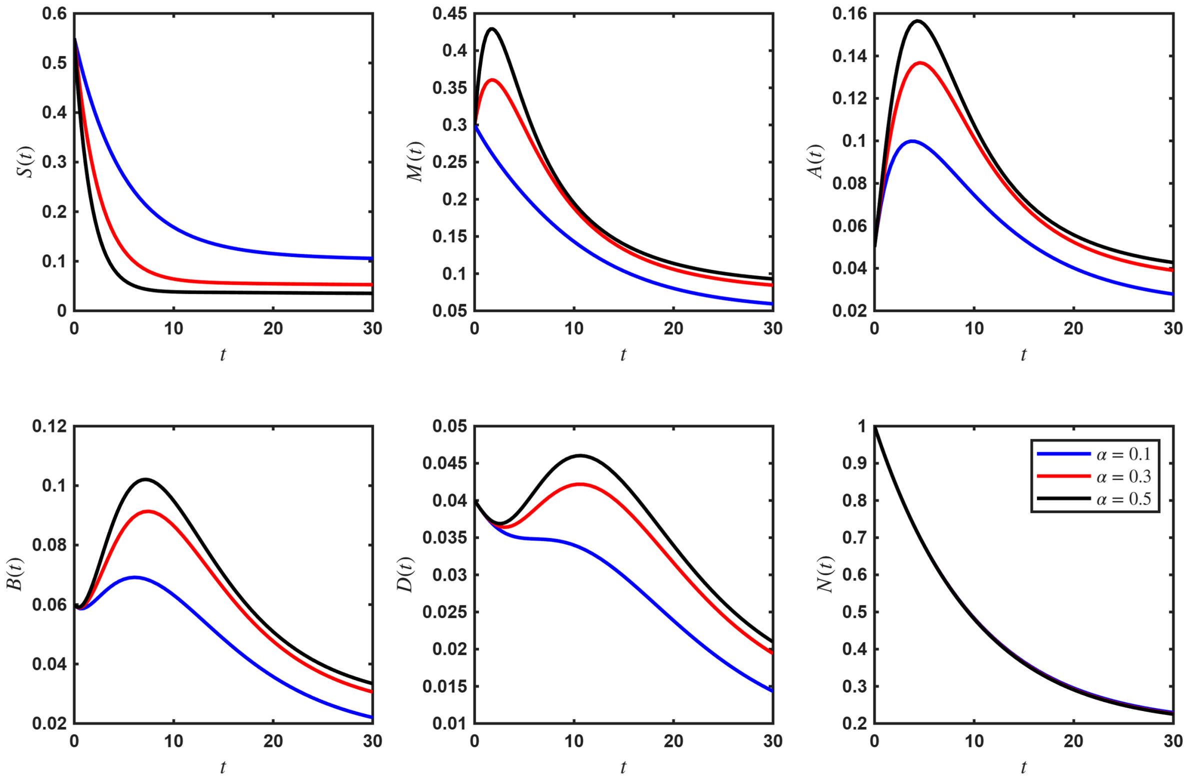

Figure 4. Temporal evolution of the model compartments $S(t)$, $M(t)$, $A(t)$, $B(t)$, $D(t)$, and the total population $N(t)$ for different values of the marriage formation rate $\alpha \in \{0.1, 0.3, 0.5\}$. The remaining parameters are fixed at $\beta = 0.3$, $\gamma = 0.2$, $\sigma = 0.2$, $\xi = 0.15$, $\rho = 0.1$, $\delta = 0.1$, $\theta = 0.05$, $\mu = 0.1$, $\mu_D = 0.05$, $\eta = 0.2$, and $\Lambda = 0.02$. The initial conditions are $S(0) = 0.55$, $M(0) = 0.30$, $A(0) = 0.05$, $B(0) = 0.06$, and $D(0) = 0.04$.

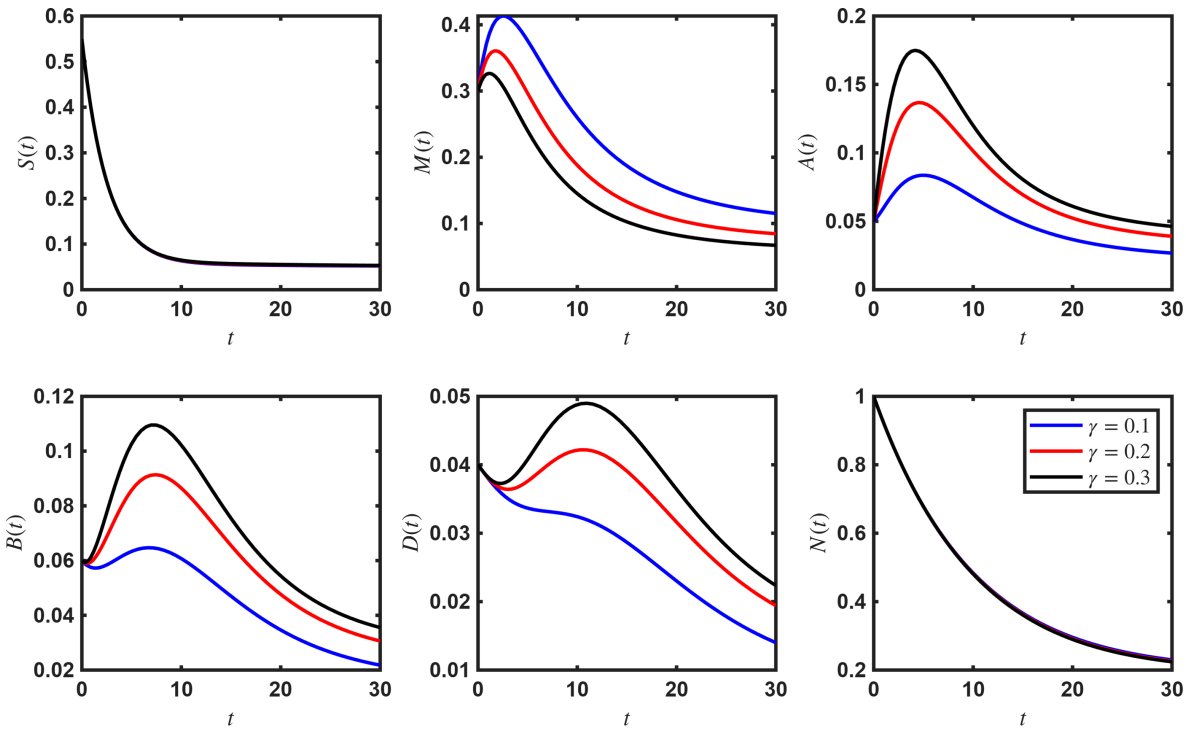

Figure 5. Temporal dynamics of the state variables for different values of the affair initiation rate $\gamma \in \{0.1, 0.2, 0.3\}$. All other parameters remain fixed at $\alpha = 0.3$, $\beta = 0.3$, $\sigma = 0.2$, $\xi = 0.15$, $\rho = 0.1$, $\delta = 0.1$, $\theta = 0.05$, $\mu = 0.1$, $\mu_D = 0.05$, $\eta = 0.2$, and $\Lambda = 0.02$, with the same feasible initial conditions $S(0) = 0.55$, $M(0) = 0.30$, $A(0) = 0.05$, $B(0) = 0.06$, and $D(0) = 0.04$.

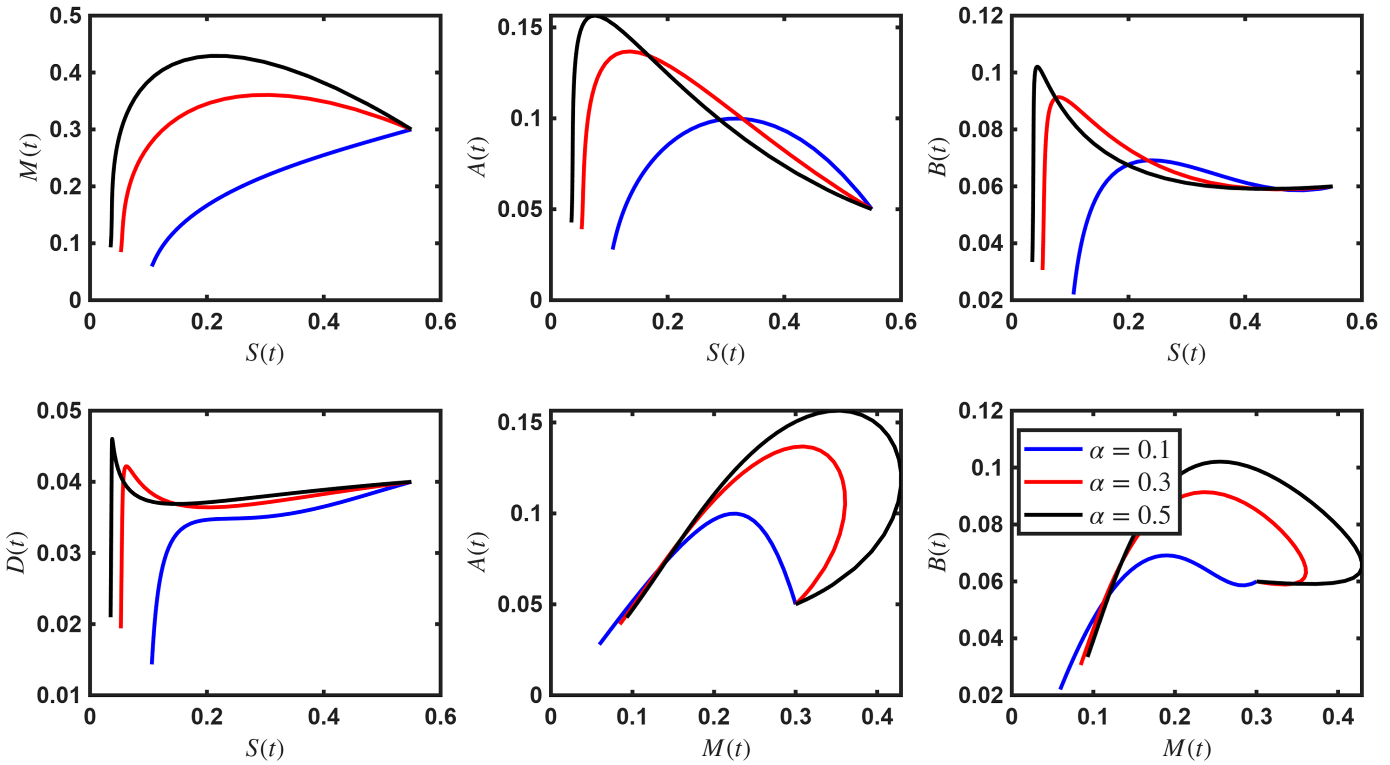

Figure 6. Phase portraits illustrating the influence of the marriage formation rate α on the dynamics of the system. The subplots show interactions between (S, M), (S, A), (S, B), (S, D), (M, A), and (M, B) for three values of α $(0.1, 0.3, 0.5)$. The trajectories expand outward as α increases, indicating that higher marriage formation rates intensify the interactions among the marital compartments and amplify the transient dynamics before the system approaches equilibrium.

Figure 6 demonstrates the impact of the marriage formation parameter α on the phase dynamics of the system. As α increases, the trajectories in the (S, M) phase plane expand, indicating that higher marriage rates increase the flow of individuals from the single class into the married population. This expansion also propagates through the (S, A) and (S, B) phase planes, reflecting the indirect effect of increased marriage formation on the occurrence of extramarital affairs and marital instability. In the (M, A) and (M, B) interactions, larger values of α produce wider loops in the phase space, indicating stronger nonlinear coupling between marriage, affairs, and broken marriages. Nevertheless, the trajectories eventually contract toward the same steady state, confirming that the system remains stable even when marriage formation rates vary. These numerical observations are consistent with the theoretical stability results obtained earlier for the equilibrium points of the model.

The phase portraits in Figure 7 illustrate the nonlinear interactions among the compartments for three different initial conditions. Such phase-space representations are commonly used in the qualitative analysis of nonlinear dynamical systems (see [25] for a comprehensive introduction). The trajectories in the (S, M) plane show that the married population initially increases as singles transition into marriage, after which the dynamics stabilize due to the combined effects of affairs, separation, and reconciliation. The (M, A) interaction reveals that increases in married individuals promote the emergence of extramarital affairs, which subsequently decline as the system approaches equilibrium. Similarly, the (A, B) and (B, D) phase planes demonstrate how affairs contribute to broken marriages and ultimately lead to divorce. Despite the different starting conditions, all trajectories converge toward the same steady state. This indicates that the model possesses a globally attracting equilibrium and that the long–term population distribution among marital states is robust with respect to the initial configuration of the system.

Figures 8–11 illustrate the sensitivity of the model compartments to variations in key parameters. Each figure shows the values of $S(t)$, $M(t)$, $A(t)$, $B(t)$, $D(t)$, and the total population $N(t)$ at $t = 20$ for different parameter values, while the curves correspond to $\gamma = 0.10$, $\gamma = 0.20$, and $\gamma = 0.40$.

Figure 8 depicts the influence of the reconciliation rate ξ on the population distribution. As ξ increases, the number of individuals involved in extramarital affairs

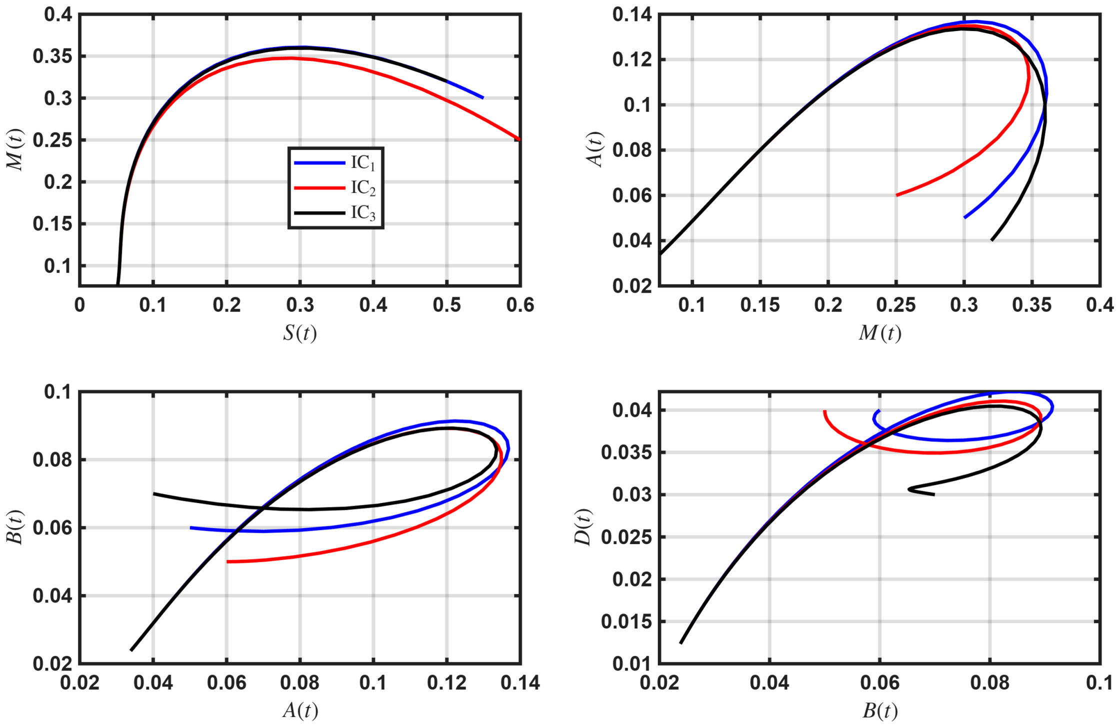

Figure 7. Phase portraits of the marriage–divorce dynamics model for three different initial conditions IC_1, IC_2, and IC_3. The subplots illustrate the dynamic relationships between key compartment pairs: (S, M), (M, A), (A, B), and (B, D). The trajectories evolve along smooth curves and gradually converge toward a common equilibrium state. This behavior indicates that the system dynamics are independent of the initial population distribution and eventually approach a stable steady state. The results therefore support the analytical stability findings obtained from the Jacobian and eigenvalue analysis.

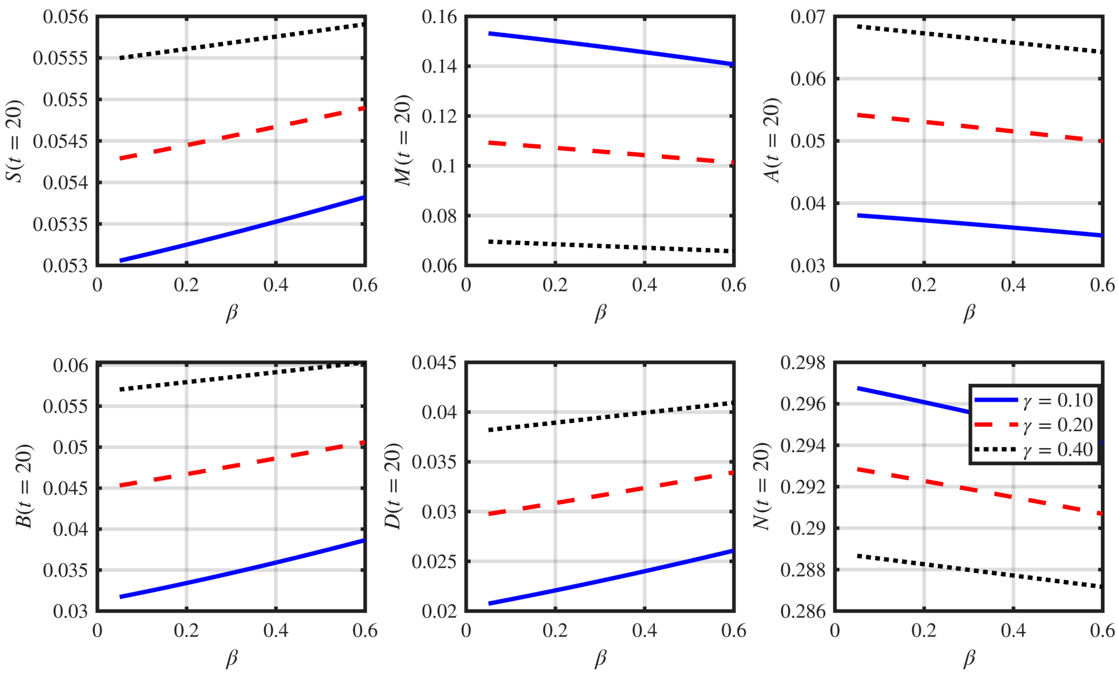

Figure 8. Effect of the parameter ξ on the population compartments. The plots show the variation of $S(t)$, $M(t)$, $A(t)$, $B(t)$, $D(t)$, and the total population $N(t)$ at $t = 20$ as ξ varies. The curves correspond to $\gamma = 0.10$ (solid blue), $\gamma = 0.20$ (red dashed), and $\gamma = 0.40$ (black dotted). Increasing ξ reduces the population involved in extramarital affairs while slightly increasing the married population.

$A(t)$ decreases significantly. This reduction occurs because a larger reconciliation rate promotes the return of individuals engaged in affairs to the married class. Consequently, the married population $M(t)$ increases gradually as ξ grows. The number of individuals in the broken marriage class $B(t)$ and the divorced class $D(t)$ also tends to decrease slightly as reconciliation mechanisms become stronger. Overall, the results indicate that higher reconciliation rates contribute to greater marital stability and reduce the prevalence of extramarital relationships within the population.

Figure 9 presents the effect of the parameter σ, which represents the transition rate from extramarital affairs to broken marriages. As σ increases, the population in the affairs compartment $A(t)$ declines, while the number of individuals in the broken marriage class $B(t)$ rises substantially. This trend occurs because higher values of σ accelerate the deterioration of relationships involved in extramarital affairs, pushing them toward marital breakdown. Consequently, the divorced population $D(t)$ also increases as more broken marriages eventually result in divorce. The married population $M(t)$ decreases slightly with increasing σ, indicating that stronger transitions from affairs to marital conflict destabilize existing marriages.

Figure 10 illustrates the impact of the marriage rate α on the dynamics of the system. As α increases, the number of single individuals $S(t)$ decreases rapidly since more individuals enter the married class. Correspondingly, the married population $M(t)$ increases with larger values of α. However, the increase in marriage formation also indirectly contributes to higher levels of broken marriages $B(t)$ and divorce $D(t)$ due to the subsequent influence of social and behavioral factors within the model. These results suggest that while higher marriage rates reduce the single population, they may also increase the risk of marital instability when social pressures and extramarital dynamics are present.

Finally, Figure 11 shows the role of the social influence parameter β, which measures the effect of divorced individuals on married couples. As β increases, the number of individuals in broken marriages $B(t)$ and the divorced population $D(t)$ grows steadily. This occurs because stronger social influence from divorced individuals increases the likelihood of marital dissatisfaction and relationship breakdown among married couples. At the same time, the married population $M(t)$ tends to decrease slightly, reflecting the destabilizing effect of social interaction. The single population $S(t)$ shows only minor variation with respect to β, while the overall population $N(t)$ remains relatively stable.

In summary, the numerical simulations demonstrate that the parameters ξ, σ, α, and β play a crucial role in shaping the long-term distribution of individuals across marital states. In particular, higher reconciliation rates promote marital stability, whereas stronger social influence and higher transition rates toward broken marriages significantly increase the levels of marital instability and divorce in the population.

Figure 12 illustrates the dependence of the reproduction number R_0 on several key parameters of the model. The plots reveal that R_0 increases monotonically with the marriage formation rate α, affair initiation rate γ, and instability parameter σ, indicating that higher values of these parameters promote the spread of marital instability within the population. Conversely, increasing the reconciliation parameter ξ reduces R_0, reflecting the stabilizing effect of reconciliation mechanisms on marital dynamics. These results highlight the importance of social interventions aimed at reducing extramarital affairs and promoting reconciliation in order to maintain marital stability.

Figure 13 provides additional insight into the sensitivity of the reproduction number R_0 to changes in model parameters through three-dimensional surface and contour plots. In particular, R_0 increases with the affair initiation rate γ and the marital instability parameter σ, indicating that higher rates of extramarital involvement and marital breakdown promote the spread of instability within the population. Conversely, increasing the reconciliation rate ξ reduces R_0, highlighting the stabilizing role of reconciliation in maintaining marital stability.

5 Conclusion

In this study, a nonlinear compartmental model describing the dynamics of marriage and divorce was developed and analyzed. The population was divided into five interacting classes $S(t)$, $M(t)$, $A(t)$, $B(t)$, and $D(t)$ representing single individuals, married individuals, individuals engaged in extramarital affairs, broken marriages, and divorced individuals, respectively. The well–posedness of the system was established by proving the positivity and boundedness of solutions within the feasible region $\Omega = \{(S, M, A, B, D) \in \mathbb{R}^5_+ : N(t) \leq \Lambda/\mu\}$, ensuring that the model is mathematically and socially meaningful. The equilibrium analysis revealed the existence of the trivial equilibrium $E_T = (0, 0, 0, 0, 0)$ and the divorce–free equilibrium

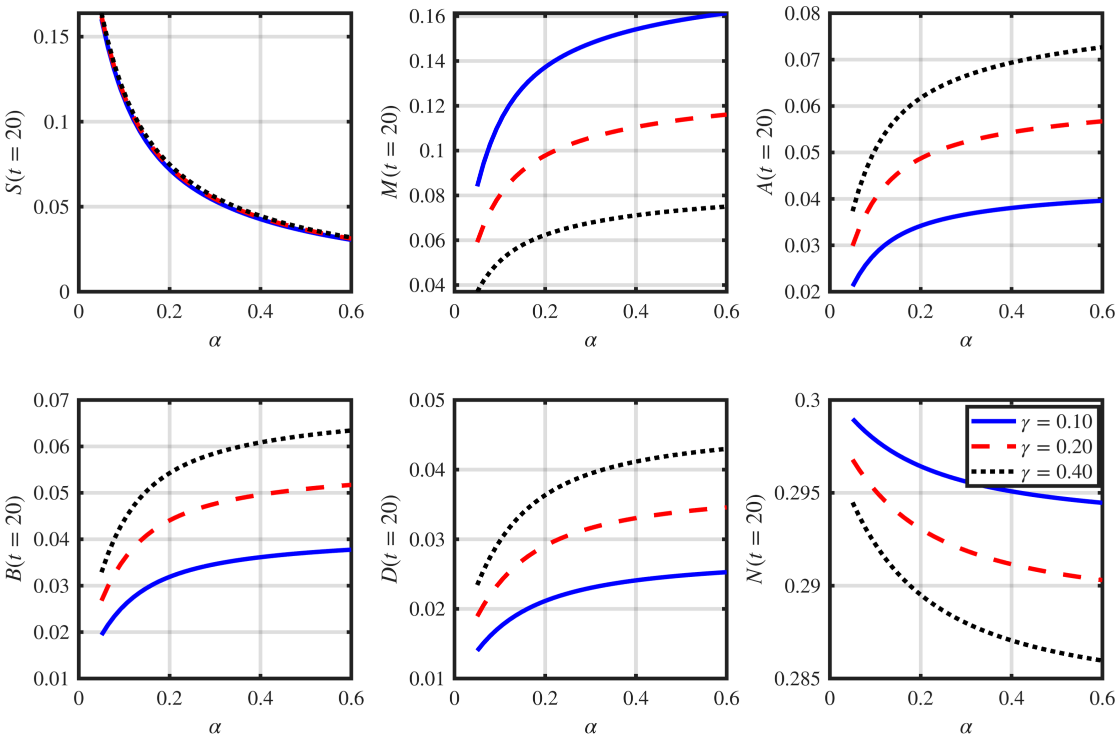

Figure 9. Effect of the parameter σ on the system dynamics. The figure illustrates the behavior of $S(t)$, $M(t)$, $A(t)$, $B(t)$, $D(t)$, and $N(t)$ at $t = 20$ for different values of σ. Larger values of σ increase the transition from extramarital affairs to broken marriages, resulting in higher levels of broken marriages and divorce.

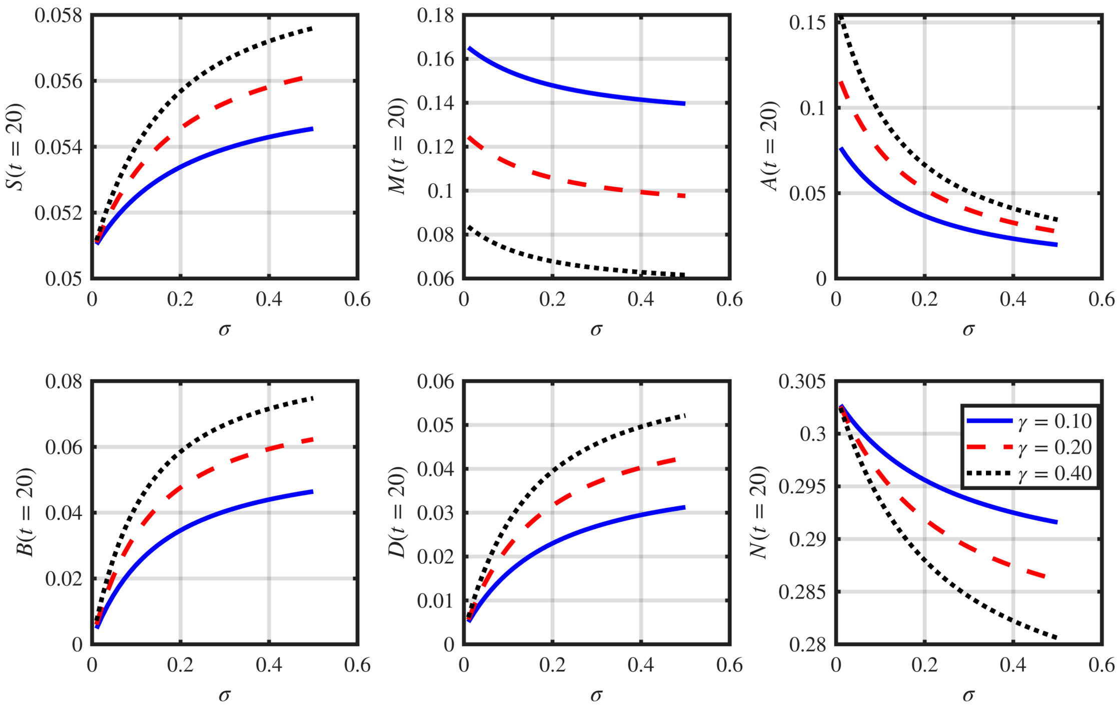

Figure 10. Influence of the marriage rate α on the distribution of individuals across marital states. The figure shows the variation of $S(t)$, $M(t)$, $A(t)$, $B(t)$, $D(t)$, and the total population $N(t)$ at $t = 20$. Increasing α decreases the number of single individuals while increasing the married population. However, it also contributes to growth in broken marriages and divorce under strong social interaction effects.

$E_0 = (S^0, M^0, 0, 0, 0)$, together with a positive endemic equilibrium $E^* = (S^*, M^*, A^*, B^*, D^*)$ corresponding to persistent marital instability. The local stability of the equilibria was examined through the Jacobian matrix and characteristic equation. The analysis showed that the divorce–free equilibrium

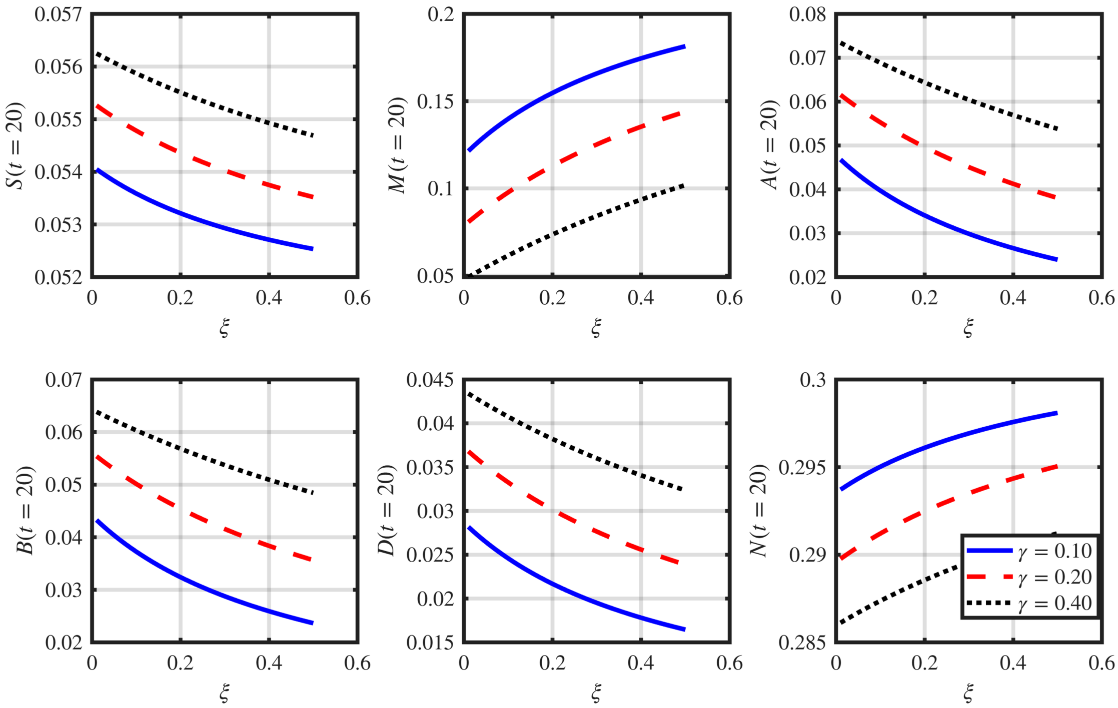

Figure 11. Impact of the social influence parameter β on the model dynamics. The plots illustrate how the compartments $S(t)$, $M(t)$, $A(t)$, $B(t)$, $D(t)$, and the total population $N(t)$ change at $t=20$ as β varies. Larger values of β increase the influence of divorced individuals on married couples, leading to higher levels of broken marriages and divorce.

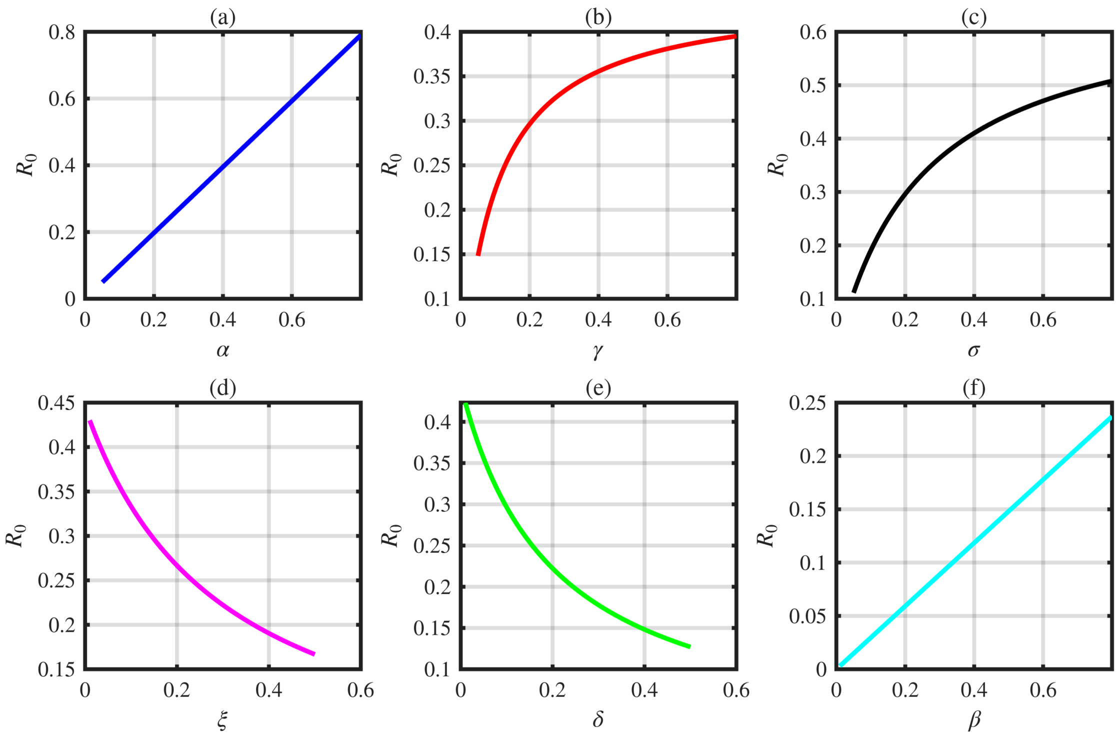

Figure 12. Variation of the basic reproduction number R_0 with respect to key model parameters. Each subplot illustrates the sensitivity of R_0 to changes in a single parameter while the remaining parameters are held constant. The results indicate that increases in the marriage formation rate α, affair initiation rate γ, and marital breakdown rate σ increase R_0, whereas larger reconciliation rates ξ and divorce rates δ tend to reduce R_0.

is locally asymptotically stable whenever the basic reproduction number $R_0 < 1$, while marital instability persists when $R_0 > 1$. The reproduction number R_0 was derived using the next–generation matrix approach and represents the threshold quantity governing the propagation of marital instability

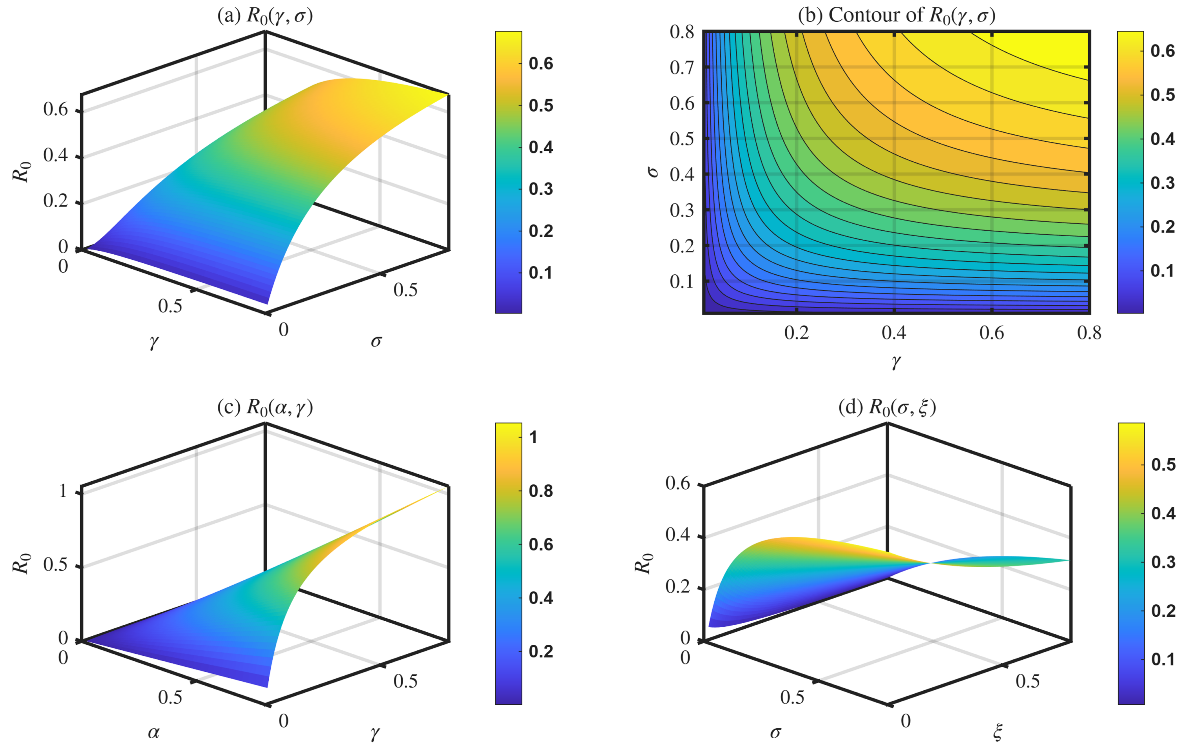

Figure 13. Surface and contour plots of the basic reproduction number R_0 illustrating its dependence on key model parameters. Panels (a)–(b) show the variation of R_0 with respect to the affair initiation rate γ and the marital breakdown rate σ. Panel (c) depicts the dependence of R_0 on the marriage formation rate α and affair rate γ, while panel (d) illustrates the combined effect of σ and the reconciliation rate ξ. The results reveal that increases in α, γ, and σ generally increase R_0, whereas larger values of ξ tend to reduce the reproduction number, indicating stabilizing effects on marital dynamics.

in the population. Sensitivity analysis further demonstrated that parameters such as the marriage formation rate α, affair initiation rate γ, and instability parameter σ increase the value of R_0, whereas the reconciliation rate ξ and related stabilizing parameters decrease it. Numerical simulations and phase–plane diagrams confirmed the analytical findings and illustrated how different initial conditions and parameter variations influence the system trajectories. In particular, the phase portraits showed that trajectories converge toward the same equilibrium point, indicating the stability of the long–term marital dynamics. Additional parameter variation simulations highlighted the significant roles played by α, γ, σ, ξ, and β in shaping the evolution of the compartments $S(t)$, $M(t)$, $A(t)$, $B(t)$, $D(t)$, and the total population $N(t)$. Surface and contour plots of R_0 also illustrated how the reproduction number responds to changes in key parameters and confirmed the threshold behavior predicted by the theoretical analysis. Overall, the results indicate that higher rates of extramarital affairs and marital instability promote the spread of divorce dynamics, while increased reconciliation and social stabilization mechanisms help maintain marital stability. The proposed mathematical framework therefore provides useful insights into the mechanisms governing marital transitions and offers a quantitative tool that may assist in understanding social interventions aimed at reducing marital breakdown and promoting stable family structures.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

AI Use Statement

The authors declare that ChatGPT was used for drafting and language editing of the manuscript. The authors have carefully reviewed, revised, and verified the AI-assisted output and take full responsibility for the content of the manuscript.

Ethical Approval and Consent to Participate

Not applicable.

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