RESEARCH ARTICLE

A Stiffly Stable Adaptive Fifth-Order Block Time-Stepping Method for Stiff Oscillatory Differential Equations

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Abstract

This paper presents an adaptive block time-stepping algorithm designed for the numerical integration of first-order stiff oscillatory problems, which typically involve multiple time scales and stringent stability conditions. The proposed method integrates a block formulation enabling the simultaneous evaluation of several solution points with an adaptive step size mechanism to effectively regulate local truncation errors. A predictor-corrector strategy is employed, where an explicit predictor generates initial estimates, and an implicit block corrector enhances stability for stiff systems. The resulting nonlinear equations are resolved using Newton's iterative method. The scheme is proven to be consistent, zero-stable, and achieves fifth-order accuracy. Its stability characteristics are further examined through A-stability analysis using locus boundary techniques, validating its applicability to stiff problems. Additionally, an error control procedure is incorporated to dynamically adjust the step size, ensuring an optimal balance between computational efficiency and solution accuracy. Numerical experiments on benchmark stiff oscillatory systems indicate that the proposed method outperforms the fourth-order variable step size block backward differentiation formula (VSBBDF4), as well as MATLAB solvers ODE15s and ODE23s, in terms of stability, accuracy, and computational efficiency.

Keywords: stiff ordinary differential equations, variable step size, block backward differentiation formula, A-stability, predictor-corrector method

1 Introduction

Stiff ordinary differential equations (ODEs) arise in a broad spectrum of scientific and engineering applications, including control systems, electrical circuit simulation, atmospheric and climate modeling, chemical kinetics, combustion processes, and

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biological systems. These problems often originate from mathematical models describing multiscale physical phenomena, where the governing dynamics involve both rapidly decaying transients and slowly varying solution components. In many cases, such systems also exhibit oscillatory behavior, leading to stiff oscillatory problems characterized by eigenvalues with large negative real parts coupled with non-negligible imaginary components. The coexistence of stiffness and oscillations presents significant analytical and computational challenges in the numerical integration of such systems [1].

A defining feature of stiff systems is the presence of widely separated time scales. This disparity imposes severe restrictions on the stability of numerical methods, particularly explicit schemes. Although explicit methods such as classical Runge-Kutta techniques are straightforward to implement and computationally inexpensive per step, they suffer from stringent step-size limitations when applied to stiff problems. Specifically, stability requirements force the use of extremely small step sizes, even when the solution exhibits smooth behavior over large intervals. As a result, explicit methods become highly inefficient and impractical for large-scale or long-time simulations involving stiff dynamics [2, 4–6].

To overcome these limitations, implicit numerical methods have been extensively developed and widely adopted due to their superior stability properties. Among the most prominent classes are backward differentiation formulas (BDFs), which were systematically introduced and analyzed by Gear [5]. BDF methods are linear multistep schemes that are particularly well-suited for stiff systems due to their favorable stability characteristics, including A-stability for low-order variants and strong stability behavior for higher-order implementations. Extensions of classical BDF methods, such as extended backward differentiation formulas (EBDFs) and their superclass variants (SEBDFs), have been proposed to improve accuracy and stability while maintaining computational efficiency [8, 9].

In addition to multistep approaches, other families of implicit methods have been developed for stiff ODEs. Rosenbrock-type methods, which are linearly implicit schemes derived from Runge-Kutta formulations, provide an attractive compromise between stability and computational cost by avoiding fully nonlinear system solves at each step [7, 10]. Similarly, implicit Runge-Kutta methods, particularly those based on Gauss–Legendre and Radau collocation formulas, offer high-order accuracy and excellent stability properties, including A-stability and L-stability. These methods have been extensively studied and are widely regarded as robust tools for stiff systems, although their practical implementation often involves solving large systems of nonlinear equations, which can be computationally demanding [4, 6, 11].

The theoretical foundation for the analysis of stiff ODE solvers has been comprehensively developed in the literature, most notably in the works of Hairer and Wanner [4, 11], which provide a rigorous treatment of stability theory, convergence analysis, and differential-algebraic systems. These contributions have played a central role in shaping modern numerical methods for stiff problems. In practical computational environments, these theoretical advances have been incorporated into widely used software packages. For instance, the MATLAB stiff solvers ODE15s and ODE23s implement variable-order, variable step-size algorithms based on numerical differentiation formulas (NDFs) and Rosenbrock-type techniques, respectively. These solvers dynamically adjust both the step size and method order to achieve an optimal balance between accuracy and efficiency, making them suitable for a wide range of stiff applications [12, 13].

Despite the success of these classical methods, several challenges remain. One major issue is the phenomenon of order reduction, which occurs when high-order methods fail to achieve their theoretical convergence rates in the presence of stiffness or oscillatory behavior. Additionally, many implicit methods require the repeated solution of nonlinear algebraic systems at each time step, leading to significant computational overhead, particularly for large-scale systems. These limitations have motivated ongoing research into the development of more efficient and robust numerical schemes that can maintain high-order accuracy while reducing computational cost [14].

A promising direction in this regard is the development of adaptive numerical methods, particularly those based on variable step-size strategies. Such methods dynamically adjust the step size in response to local error estimates and changes in solution behavior, thereby improving efficiency without sacrificing accuracy. Variable step-size techniques are especially valuable for stiff problems, where the degree of stiffness may vary significantly over the integration interval. By adapting the step size appropriately, these methods can effectively capture both fast and slow dynamics while minimizing unnecessary computations [3, 15–17].

In recent years, block methods have emerged as an effective framework for enhancing the efficiency of numerical integration. Unlike traditional one-step or multistep methods that compute a single solution point per step, block methods compute multiple solution points simultaneously within a single computational cycle. This approach offers several advantages, including reduced error propagation, improved parallelization potential, and enhanced computational efficiency [18–20].

Several notable contributions have advanced the development of variable step-size block methods. Ibrahim et al. [21] introduced a fourth-order variable step-size block backward differentiation formula (VBBDF), establishing a foundation for adaptive block multistep schemes. Suleiman et al. [22] extended this work by proposing a superclass formulation that allows for systematic construction of higher-order block methods. Ijam et al. [23] further developed this framework by introducing a fourth-order variable step-size superclass diagonally implicit method, demonstrating improved stability and efficiency for stiff systems. More recently, Bala et al. [24] proposed a third-order variable step-size superclass BBDF method, highlighting the growing interest in combining block structures with adaptive strategies.

Notwithstanding these advancements, there remains a notable gap in the literature regarding the development of higher-order block methods with adaptive step-size capability, particularly at the fifth-order level. Existing methods often involve trade-offs between accuracy, stability, and computational cost, and there is a clear need for schemes that integrate these desirable properties within a unified framework. In particular, the incorporation of additional flexibility in the method formulation, such as tunable parameters for stability control, has not been fully explored in the context of high-order block methods.

Motivated by these challenges, the present study proposes a fully implicit fifth-order variable step-size superclass block backward differentiation formula (VSBBDF5) for the numerical solution of stiff ODEs. The proposed method is constructed within a hybrid predictor-corrector framework, in which an explicit variable step-size predictor is derived using Taylor series expansion within a general linear multistep context. The predicted values are subsequently employed in the implicit corrector phase, leading to improved accuracy and convergence properties.

A key feature of the proposed method is the introduction of a stability control parameter, denoted by $\rho \in (-1, 1)$, which is embedded within the method coefficients. This parameter provides an additional degree of freedom that can be used to adjust the region of absolute stability, thereby enhancing the method's adaptability to different classes of stiff problems. By appropriately selecting the value of ρ, the proposed scheme can achieve improved stability performance compared to conventional fixed-coefficient methods.

Overall, the proposed VSBBDF5 method aims to bridge the gap between high-order accuracy, adaptive step-size capability, and enhanced stability control. It is expected to provide an efficient and reliable computational tool for solving a wide range of stiff ODEs encountered in scientific and engineering applications.

2 Derivation of the Method

To construct the fifth-order fully implicit variable step-size superclass of the block backward differentiation formula (VSBBDF5), the scheme is formulated as follows:

$$\sum_{j=0}^{5} \delta_{j,i,r} v_{n+j-3} = h\phi_{k,i,r}\left(f_{n+k} - \rho f_{n+k-1}\right), \; k = i = 1, 2. \tag{1}$$

The linear operator π_i corresponding to the VSBBDF5 scheme is introduced and defined as follows:

$$\begin{aligned}\pi_i\left[v\left(u_n\right), h\right]: \; & \delta_{0,i} v_{n-3} + \delta_{1,i} v_{n-2} + \delta_{2,i} v_{n-1} + \delta_{3,i} v_n \\ & + \delta_{4,i} v_{n+1} + \delta_{5,i} v_{n+2} - h\phi_{k,i,r}\left(f_{n+k} - \rho f_{n+k-1}\right) = 0, \\ & k = i = 1, 2.\end{aligned} \tag{2}$$

In order to obtain the first approximation v_{n+1}, we introduce the corresponding linear operator π_i, defined as follows:

$$\begin{aligned}&\delta_{0,1} v_{n-3} + \delta_{1,1} v_{n-2} + \delta_{2,1} v_{n-1} + \delta_{3,1} v_n \\ &+ \delta_{4,1} v_{n+1} + \delta_{5,1} v_{n+2} - h\phi_{1,1}\left(v'_{n+1} - \rho v'_n\right) = 0.\end{aligned} \tag{3}$$

By expanding equation (3) as a Taylor series about u_n and collecting terms according to their respective orders, we obtain the following expression:

$$\begin{aligned}&\Psi_{0,1}v(u_n)+\Psi_{1,1}hv'(u_n)+\Psi_{2,1}h^2v''(u_n)\\&\quad+\Psi_{3,1}h^3v'''(u_n)+\Psi_{4,1}h^4v^{(4)}(u_n)+\cdots=0.\end{aligned}\tag{4}$$

where

$$\left.\begin{aligned}\Psi_{0,1}&=\delta_{0,1}+\delta_{1,1}+\delta_{2,1}+\delta_{3,1}+\delta_{4,1}+\delta_{5,1}=0,\\ \Psi_{1,1}&=-3r\delta_{0,1}-2r\delta_{1,1}-r\delta_{2,1}+\delta_{4,1}+2\delta_{5,1}\\&\quad-\phi_{1,1}(1-\rho)=0,\\ \Psi_{2,1}&=\frac{9}{2}r^2\delta_{0,1}+2r^2\delta_{1,1}+\frac{1}{2}r^2\delta_{2,1}+\frac{1}{2}\delta_{4,1}\\&\quad+2\delta_{5,1}-\phi_{1,1}=0,\\ \Psi_{3,1}&=-\frac{27}{6}r^3\delta_{0,1}-\frac{4}{3}r^3\delta_{1,1}-\frac{1}{6}r^3\delta_{2,1}+\frac{1}{6}\delta_{4,1}\\&\quad+\frac{4}{3}\delta_{5,1}-\frac{1}{2}\phi_{1,1}=0,\\ \Psi_{4,1}&=\frac{81}{24}r^4\delta_{0,1}+\frac{2}{3}r^4\delta_{1,1}+\frac{1}{24}r^4\delta_{2,1}+\frac{1}{24}\delta_{4,1}\\&\quad+\frac{2}{3}\delta_{5,1}-\frac{1}{6}\phi_{1,1}=0,\\ \Psi_{5,1}&=-\frac{243}{120}r^5\delta_{0,1}-\frac{32}{120}r^5\delta_{1,1}-\frac{1}{120}r^5\delta_{2,1}+\frac{1}{120}\delta_{4,1}\\&\quad+\frac{32}{120}\delta_{5,1}-\frac{1}{24}\phi_{1,1}=0.\end{aligned}\right\}\tag{5}$$

In the derivation of the first point v_{n+1}, the coefficient $\delta_{4,1}$ is set to unity for normalization. The resulting system of simultaneous equations in (5) is then solved to determine the coefficients $\delta_{j,i}$ and $\phi_{j,i}$. Substituting these computed values into equation (3) leads to the following expression:

$$\begin{aligned}&v_{n+1}=\\&\frac{8r^4\rho+4r^4+12r^3\rho+12r^3+4r^2\rho+13r^2+6r+1}{6\left(36r^4\rho+24r^3\rho-33r^3-58r^2-33r-6\right)r^3}v_{n-3}\\&-\frac{18r^3\rho+9r^3+6r^2\rho+15r^2+7r+1}{4r^3\left(12r^3\rho-11r^2-12r-3\right)}v_{n-2}\\&+\frac{72r^4\rho+36r^4+60r^3\rho+60r^3+12r^2\rho+37r^2+10r+1}{2\left(12r^4\rho+24r^3\rho-11r^3-34r^2-27r-6\right)r^3}v_{n-1}\\&+\frac{108r^6\rho-36r^6+66r^5\rho-132r^5-134r^4\rho-193r^4}{12r^3\left(12r^3\rho-11r^2-12r-3\right)}v_n\\&+\frac{-114r^3\rho-144r^3-22r^2\rho-58r^2-12r-1}{12r^3\left(12r^3\rho-11r^2-12r-3\right)}v_n\\&+\frac{36r^5\rho+36r^5+30r^4\rho+96r^4+6r^3\rho+97r^3+97r^3+47r^2+11r+1}{4\left(36r^5\rho+96r^4\rho-33r^4+48r^3\rho-124r^3-149r^2-72r-12\right)}v_{n+2}\\&-\frac{6r^3+11r^2+6r+1}{12r^3\rho-11r^2-12r-3}hf_{n+1}\\&+\frac{6r^3+11r^2+6r+1}{2\left(12r^3\rho-11r^2-12r-3\right)}\rho hf_n.\end{aligned}\tag{6}$$

By following the same procedure employed in deriving the first point, the second point is subsequently obtained as follows:

$$\begin{aligned}&v_{n+2}=\\&-\frac{2}{3}\frac{2r^4\rho+4r^4+9r^3\rho+24r^3+14r^2\rho+52r^2+9r\rho+48r+2\rho+16}{\left(3r+1\right)\left(6r^3\rho-18r^3+11r^2\rho-88r^2+6r\rho-120r+\rho-48\right)r^3}v_{n-3}\\&+\frac{9r^4\rho+18r^4+36r^3\rho+96r^3+47r^2\rho+176r^2+24r\rho+128r+4\rho+32}{\left(3r+1\right)\left(6r^3\rho-18r^3+11r^2\rho-88r^2+6r\rho-120r+\rho-48\right)r^3}v_{n-2}\\&-\frac{2\left(18r^3\rho+36r^3+27r^2\rho+84r^2+84r^2+13r\rho+64r+2\rho+16\right)}{\left(6r^3\rho-18r^3+11r^2\rho-88r^2+6r\rho-120r+\rho-48\right)r^3}v_{n-1}\\&+\frac{1}{3}\frac{18r^6\rho+18r^6+99r^5\rho+132r^5+211r^4\rho+386r^4+225r^3\rho+576r^3}{\left(6r^3\rho-18r^3+11r^2\rho-88r^2+6r\rho-120r+\rho-48\right)r^3}v_n\\&+\frac{1}{3}\frac{127r^2\rho+464r^2+36r\rho+192r+4\rho+32}{\left(6r^3\rho-18r^3+11r^2\rho-88r^2+6r\rho-120r+\rho-48\right)r^3}v_n\\&-\frac{4\left(36r^5+33r^4\rho+228r^4+124r^3\rho+544r^3+149r^2\rho+608r^2+72r\rho+320r+12\rho+64\right)}{\left(6r^2+5r+1\right)\left(6r^3\rho-18r^3+11r^2\rho-88r^2+6r\rho-120r+\rho-48\right)}v_{n+1}\\&-\frac{4\left(3r^3+11r^2+12r+4\right)}{6r^3\rho-18r^3+11r^2\rho-88r^2+6r\rho-120r+\rho-48}hf_{n+2}\\&+\frac{4\left(3r^3+11r^2+12r+4\right)}{6r^3\rho-18r^3+11r^2\rho-88r^2+6r\rho-120r+\rho-48}\rho hf_{n+1}.\end{aligned}\tag{7}$$

An important consideration in the development of the proposed method is the careful choice of the superclass parameter ρ, since it has a direct impact on both the stability properties and the overall computational efficiency of the scheme. In this study, the value $\rho=-\frac{3}{4}$ is adopted, motivated by earlier investigations that have demonstrated its capability to produce stable and reliable numerical solutions for stiff differential equations, both from theoretical stability analyses [25, 26] and from practical applications to stiff dynamical systems [27].

The selection $\rho=-\frac{3}{4}$, which lies within the admissible interval $(-1,1)$, ensures that the method satisfies key theoretical conditions such as zero-stability and A-stability. In addition to these essential properties, treating ρ as a free parameter introduces a degree of flexibility into the formulation, enabling the method to be adjusted for improved stability regions or higher accuracy depending on the nature of the stiff problems under consideration.

With ρ fixed at $-\frac{3}{4}$, different choices of the step-size ratio r in equations (6) and (7) give rise to various members of VSBBDF5. Specifically, the cases $r=1$, $r=2$, and $r=\frac{5}{6}$ lead to distinct schemes, which are explicitly presented in equations (8)–(10), respectively.

For $r=1$

$$\left.\begin{aligned} v_{n+1} =& -\frac{3}{175}v_{n-3} + \frac{1}{10}v_{n-2} - \frac{6}{35}v_{n-1} + \frac{6}{5}v_n \\ &- \frac{39}{350}v_{n+2} + \frac{24}{35}hf_{n+1} + \frac{18}{35}hf_n, \\ v_{n+2} =& \frac{39}{584}v_{n-3} - \frac{30}{73}v_{n-2} + \frac{155}{146}v_{n-1} - \frac{105}{73}v_n \\ &+ \frac{1005}{584}v_{n+1} + \frac{30}{73}hf_{n+2} + \frac{45}{146}hf_{n+1}. \end{aligned}\right\} \tag{8}$$

For $r = 2$

$$\left.\begin{aligned} v_{n+1} =& -\frac{15}{18304}v_{n-3} + \frac{21}{4576}v_{n-2} + \frac{35}{9152}v_{n-1} \\ &+ \frac{5145}{4576}v_n - \frac{2415}{18304}v_{n+2} + \frac{105}{143}hf_{n+1} + \frac{315}{572}hf_n, \\ v_{n+2} =& \frac{3}{493}v_{n-3} - \frac{772}{17255}v_{n-2} + \frac{558}{3451}v_{n-1} \\ &- \frac{1812}{3451}v_n + \frac{3456}{2465}v_{n+1} + \frac{1536}{3451}hf_{n+2} + \frac{1152}{3451}hf_{n+1}. \end{aligned}\right\} \tag{9}$$

For $r = \frac{5}{6}$

$$\left.\begin{aligned} v_{n+1} =& -\frac{71104}{2093625}v_{n-3} + \frac{44793}{232625}v_{n-2} - \frac{1342656}{3954625}v_{n-1} \\ &+ \frac{298606}{232625}v_n - \frac{29246}{284733}v_{n+2} + \frac{1232}{1861}hf_{n+1} + \frac{924}{1861}hf_n, \\ v_{n+2} =& \frac{131648}{1045625}v_{n-3} - \frac{12306249}{16730000}v_{n-2} + \frac{1819584}{1045625}v_{n-1} \\ &- \frac{2054943}{1045625}v_n + \frac{49113}{26768}v_{n+1} + \frac{3366}{8365}hf_{n+2} + \frac{5049}{16730}hf_{n+1}. \end{aligned}\right\} \tag{10}$$

The schemes presented in equations (8)-(10), which constitute the VSBBDF5, achieve fifth-order accuracy, with their corresponding error constants expressed as follows:

$$\gamma_{6,1} = \begin{pmatrix} \frac{1}{70} \\ -\frac{17}{292} \end{pmatrix}, \gamma_{6,2} = \begin{pmatrix} -\frac{11}{292} \\ -\frac{3}{481} \end{pmatrix}, \gamma_{6,\frac{5}{6}} = \begin{pmatrix} -\frac{6}{387} \\ -\frac{1}{238} \end{pmatrix}.$$

The proposed method is formulated within a predictor-corrector framework to enhance both numerical stability and computational accuracy. In particular, the corrector component is implemented in the PECE (Predict-Evaluate-Correct-Evaluate) mode, wherein an initial approximation to the solution at the subsequent step is first predicted, followed by its evaluation, correction, and a final re-evaluation to achieve improved precision.

Given that the scheme is inherently of corrector type, the derivation of a compatible predictor is indispensable. Consequently, an explicit predictor formula is constructed in a manner consistent with the block structure and the order of the VSBBDF5 method. This predictor furnishes suitable initial approximations for the corrector stage, thereby facilitating convergence and enhancing the overall efficiency of the computational procedure.

3 Derivation of the Predictor Formulae

The initial approximation is obtained using an explicit predictor method, which is derived through Taylor series expansion applied to the relation:

$$\sum_{j=0}^{5} \alpha_{j,i,r} v_{n+j-3} = 0, \qquad i = 1, 2. \tag{11}$$

The formulation of equation (11) follows from the general k-step linear multistep method (LMM) by setting the coefficients $\phi_{k,i,r} = 0$ on the right-hand side, thereby yielding an explicit predictor scheme. To compute the coefficients corresponding to the first and second points of the predictor method in (11), we define the associated linear operator of equation (11) as follows:

$$\begin{aligned} \pi_i\left[v\left(u_n\right), h\right] =& \alpha_{0,i}v_{n-3} + \alpha_{1,i}v_{n-2} + \alpha_{2,i}v_{n-1} + \alpha_{3,i}v_n \\ &+ \alpha_{4,i}v_{n+1} + \alpha_{5,i}v_{n+2} = 0. \end{aligned} \tag{12}$$

First Point: $i = 1$

By setting the coefficient $\alpha_{5,1} = 0$ in (11), the linear operator (11) becomes:

$$\begin{aligned} &\alpha_{0,1}v\left(u_n - 3rh\right) + \alpha_{1,1}v\left(u_n - 2rh\right) \\ &+ \alpha_{2,1}v\left(u_n - rh\right) + \alpha_{3,1}v\left(u_n\right) + \alpha_{4,1}v\left(u_n + h\right) = 0. \end{aligned} \tag{13}$$

By expanding equation (13) in a Taylor series about the point u_n, the following system of simultaneous linear equations is obtained:

$$\left.\begin{aligned} \sigma_{0,1} &= \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} + \alpha_{4,1} = 0, \\ \sigma_{1,1} &= -3r\alpha_{0,1} - 2r\alpha_{1,1} - r\alpha_{2,1} + \alpha_{4,1} = 0, \\ \sigma_{2,1} &= \frac{9}{2}r^2\alpha_{0,1} + 2r^2\alpha_{1,1} + \frac{1}{2}r^2\alpha_{2,1} + \frac{1}{2}\alpha_{4,1} = 0, \\ \sigma_{3,1} &= -\frac{27}{6}r^3\alpha_{0,1} - \frac{4}{3}r^3\alpha_{1,1} - \frac{1}{6}r^3\alpha_{2,1} + \frac{1}{6}\alpha_{4,1} = 0. \end{aligned}\right\} \tag{14}$$

By normalizing $\alpha_{4,1} = 1$ and solving the resulting system of equations with the aid of the Maple 18 computational environment, the coefficients associated with the first point are determined, as summarized in Table 1.

Table 1. Coefficients of the predictor method for the first point.

$\alpha_{0,1}$	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{4,1}$
$\frac{1}{6}\frac{2r^2+3r+1}{r^3}$	$-\frac{1}{2}\frac{3r^2+4r+1}{r^3}$	$\frac{1}{2}\frac{6r^2+5r+1}{r^3}$	$-\frac{1}{6}\frac{6r^3+11r^2+6r+1}{r^3}$	1

Substituting the computed coefficients into equation (13) yields the expression corresponding to the first point, which is given as follows:

$$\begin{aligned} v_{n+1} = & -\frac{1}{6}\frac{2r^2+3r+1}{r^3}v_{n-3} + \frac{1}{2}\frac{3r^2+4r+1}{r^3}v_{n-2} \\ & -\frac{1}{2}\frac{6r^2+5r+1}{r^3}v_{n-1} + \frac{1}{6}\frac{6r^3+11r^2+6r+1}{r^3}v_n. \end{aligned} \tag{15}$$

Second Point: $i = 2$

Similarly, by assigning the coefficient $\alpha_{4,2} = 0$ in equation (12), the corresponding linear operator is reduced to the following form:

$$\begin{aligned} & \alpha_{0,2}v\left(u_n - 3rh\right) + \alpha_{1,2}v\left(u_n - 2rh\right) + \alpha_{2,2}v\left(u_n - rh\right) \\ & + \alpha_{3,2}v\left(u_n\right) + \alpha_{5,2}v\left(u_n + 2h\right) = 0. \end{aligned} \tag{16}$$

By expanding equation (16) in a Taylor series about the point u_n, the following system of simultaneous linear equations is obtained:

$$\left.\begin{aligned} \sigma_{0,2} &= \alpha_{0,2} + \alpha_{1,2} + \alpha_{2,2} + \alpha_{3,2} + \alpha_{5,2} = 0, \\ \sigma_{1,2} &= -3r\alpha_{0,2} - 2r\alpha_{1,2} - r\alpha_{2,2} + 2\alpha_{5,2} = 0, \\ \sigma_{2,2} &= \frac{9}{2}r^2\alpha_{0,2} + 2r^2\alpha_{1,2} + \frac{1}{2}r^2\alpha_{2,2} + 2\alpha_{5,2} = 0, \\ \sigma_{3,2} &= -\frac{9}{2}r^3\alpha_{0,2} - \frac{4}{3}r^3\alpha_{1,2} - \frac{1}{6}r^3\alpha_{2,2} + \frac{4}{3}\alpha_{5,2} = 0. \end{aligned}\right\} \tag{17}$$

By assigning $\alpha_{5,2} = 1$ and solving the resulting system of equations using the Maple 18 computational environment, the coefficients corresponding to the second point are obtained, as summarized in Table 2.

Table 2. Coefficients of the predictor method for the second point.

$\alpha_{0,2}$	$\alpha_{1,2}$	$\alpha_{2,2}$	$\alpha_{3,2}$	$\alpha_{5,2}$
$\frac{2}{3}\frac{r^2+3r+3}{r^3}$	$-\frac{3r^2+8r+4}{r^3}$	$\frac{2(3r^2+5r+2)}{r^3}$	$-\frac{1}{3}\frac{3r^3+11r^2+12r+4}{r^3}$	1

Substituting the computed coefficients into equation (16) yields the expression corresponding to the second point, which can be written as follows:

$$\begin{aligned} v_{n+2} = & -\frac{2}{3}\frac{r^2+3r+3}{r^3}v_{n-3} + \frac{3r^2+8r+4}{r^3}v_{n-2} \\ & -\frac{2\left(3r^2+5r+2\right)}{r^3}v_{n-1} + \frac{1}{3}\frac{3r^3+11r^2+12r+4}{r^3}v_n. \end{aligned} \tag{18}$$

Hence, by combining equations (15) and (18), the two-point Explicit Variable Step-Size Predictor (EVSP) method is obtained and can be expressed as follows:

$$\left.\begin{aligned} v_{n+1} = & -\frac{1}{6}\frac{2r^2+3r+1}{r^3}v_{n-3} + \frac{1}{2}\frac{3r^2+4r+1}{r^3}v_{n-2} \\ & -\frac{1}{2}\frac{6r^2+5r+1}{r^3}v_{n-1} + \frac{1}{6}\frac{6r^3+11r^2+6r+1}{r^3}v_n, \\ v_{n+2} = & -\frac{2}{3}\frac{r^2+3r+3}{r^3}v_{n-3} + \frac{3r^2+8r+4}{r^3}v_{n-2} \\ & -\frac{2\left(3r^2+5r+2\right)}{r^3}v_{n-1} + \frac{1}{3}\frac{3r^3+11r^2+12r+4}{r^3}v_n. \end{aligned}\right\} \tag{19}$$

By considering the step-size ratios $r = 1$, $r = 2$, and $r = \frac{5}{6}$, which correspond respectively to constant step size, step-size halving, and an increase in step size by a factor of 1.2 in equation (19), the following schemes are obtained:

For $r = 1$

$$\left.\begin{aligned} v_{n+1} &= -v_{n-3} + 4v_{n-2} - 6v_{n-1} + 4v_n, \\ v_{n+2} &= -4v_{n-3} + 15v_{n-2} - 20v_{n-1} + 10v_n. \end{aligned}\right\} \tag{20}$$

For $r = 2$

$$\left.\begin{aligned} v_{n+1} &= -\frac{5}{16}v_{n-3} + \frac{21}{16}v_{n-2} - \frac{35}{16}v_{n-1} + \frac{35}{16}v_n, \\ v_{n+2} &= -v_{n-3} + 4v_{n-2} - 6v_{n-1} + 4v_n. \end{aligned}\right\} \tag{21}$$

For $r = \frac{5}{6}$

$$\left.\begin{aligned} v_{n+1} &= -\frac{176}{125}v_{n-3} + \frac{693}{125}v_{n-2} - \frac{1008}{1125}v_{n-1} + \frac{616}{125}v_n, \\ v_{n+2} &= -\frac{748}{125}v_{n-3} + \frac{2754}{125}v_{n-2} - \frac{3564}{125}v_{n-1} + \frac{1683}{125}v_n. \end{aligned}\right\} \tag{22}$$

The derived two-point explicit variable step-size predictor (EVSP) method serves as an efficient mechanism for generating initial approximations required by the fully implicit VSBBDF5 corrector scheme. Owing to its explicit formulation, the predictor avoids the need for iterative solution procedures, thereby reducing computational overhead. The method incorporates variable step-size ratios, which enhances its adaptability to varying solution dynamics and improves its effectiveness when applied to problems with rapidly changing behavior.

Furthermore, the use of four previous solution points ensures that the predictor maintains a reasonable level of accuracy while remaining computationally inexpensive. In the overall predictor–corrector framework, the EVSP method provides reliable starting values that accelerate convergence of the implicit corrector and contribute to the stability and efficiency of the numerical integration process, particularly in the solution of stiff differential systems.

4 Stability Analysis of the Method

In this section, the stability behavior of the schemes given in (8)-(10) is analyzed, with particular attention devoted to zero-stability and A-stability. This investigation is based on the evaluation of the roots of the corresponding characteristic polynomials and how these roots are distributed relative to the unit circle in the complex plane.

To determine the stability regions, the classical linear test equation $v' = \lambda v$ is employed, where λ satisfies $\Re e(\lambda) < 0$. The analysis begins with method (8), for which the procedure is explicitly demonstrated. The same analytical framework is then applied in a similar manner to methods (9) and (10).

Definition 1 (Zero-Stability). A LMM is said to be zero-stable if its first characteristic polynomial satisfies the root condition. That is, the numerical solution remains bounded as the step size $h \to 0$, ensuring that small perturbations in the initial data do not lead to unbounded growth in the solution. In other words, a LMM is zero-stable if no root of the first characteristic polynomial lies outside the unit circle, and any root on the unit circle is simple (i.e., has multiplicity one) [28, 29].

Definition 2 (A-stability). A LMM is said to be A-stable if the stability region contains the entire left half-plane of the complex plane [30].

i. By introducing the test equation $v' = \lambda v$, subject to $\Re e(\lambda) < 0$, into scheme (8), the resulting expression is obtained as follows:

$$\left.\begin{aligned} v_{n+1} = &-\frac{3}{175}v_{n-3} + \frac{1}{10}v_{n-2} - \frac{6}{35}v_{n-1} + \frac{6}{5}v_n \\ &-\frac{39}{350}v_{n+2} + \frac{24}{35}h\lambda v_{n+1} + \frac{18}{35}h\lambda v_n, \\ v_{n+2} = &\frac{39}{584}v_{n-3} - \frac{30}{73}v_{n-2} + \frac{155}{146}v_{n-1} - \frac{105}{73}v_n \\ &+\frac{1005}{584}v_{n+1} + \frac{30}{73}h\lambda v_{n+2} + \frac{45}{146}h\lambda v_{n+1}. \end{aligned}\right\} \tag{23}$$

By rearranging the expression and grouping like terms in (23), we obtain:

$$\left.\begin{aligned} &\left(1 - \frac{24}{35}h\lambda\right)v_{n+1} + \frac{39}{350}v_{n+2} \\ &= -\frac{3}{175}v_{n-3} + \frac{1}{10}v_{n-2} - \frac{6}{35}v_{n-1} + \left(\frac{6}{5} + \frac{18}{35}h\lambda\right)v_n, \\ &\left(-\frac{1005}{584} - \frac{45}{146}h\lambda\right)v_{n+1} + \left(1 - \frac{30}{73}h\lambda\right)v_{n+2} \\ &= \frac{39}{584}v_{n-3} - \frac{30}{73}v_{n-2} + \frac{155}{146}v_{n-1} - \frac{105}{73}v_n. \end{aligned}\right\} \tag{24}$$

These equations can be expressed in matrix form as follows:

$$\begin{bmatrix} 1 - \frac{24}{35}\lambda h & \frac{39}{350} \\ -\frac{1005}{584} - \frac{45}{146}h\lambda & 1 - \frac{30}{73}h\lambda \end{bmatrix} \begin{bmatrix} v_{n+1} \\ v_{n+2} \end{bmatrix} = \begin{bmatrix} -\frac{3}{175} & \frac{1}{10} \\ \frac{39}{584} & -\frac{30}{73} \end{bmatrix} \begin{bmatrix} v_{n-3} \\ v_{n-2} \end{bmatrix} + \begin{bmatrix} -\frac{6}{35} & \frac{6}{5} + \frac{18}{35}h\lambda \\ \frac{155}{146} & -\frac{105}{73} \end{bmatrix} \begin{bmatrix} v_{n-1} \\ v_n \end{bmatrix}. \tag{25}$$

Let $H = \lambda h$ in equation (25). This leads to

$$\begin{bmatrix} 1 - \frac{24}{35}H & \frac{39}{350} \\ -\frac{1005}{584} - \frac{45}{146}H & 1 - \frac{30}{73}H \end{bmatrix} \begin{bmatrix} v_{n+1} \\ v_{n+2} \end{bmatrix} = \begin{bmatrix} -\frac{3}{175} & \frac{1}{10} \\ \frac{39}{584} & -\frac{30}{73} \end{bmatrix} \begin{bmatrix} v_{n-3} \\ v_{n-2} \end{bmatrix} + \begin{bmatrix} -\frac{6}{35} & \frac{6}{5} + \frac{18}{35}H \\ \frac{155}{146} & -\frac{105}{73} \end{bmatrix} \begin{bmatrix} v_{n-1} \\ v_n \end{bmatrix}. \tag{26}$$

Definition 3. If m denotes the number of blocks and s denotes the number of points in each block, then $n = ms$. By [31], we let

$$V_m = \begin{bmatrix} v_{n+1} \\ v_{n+2} \end{bmatrix} = \begin{bmatrix} v_{2m+1} \\ v_{2m+2} \end{bmatrix},$$

$$V_{m-1} = \begin{bmatrix} v_{n-1} \\ v_n \end{bmatrix} = \begin{bmatrix} v_{2m-1} \\ v_{2m} \end{bmatrix} = \begin{bmatrix} v_{2(m-1)-1} \\ v_{2(m-1)+2} \end{bmatrix},$$

$$V_{m-2} = \begin{bmatrix} v_{n-3} \\ v_{n-2} \end{bmatrix} = \begin{bmatrix} v_{2(m-2)+1} \\ v_{2(m-2)+2} \end{bmatrix}.$$

An equivalent representation of equation (26) is given by:

$$AV_m = BV_{m-1} + CV_{m-2}. \tag{27}$$

where

$$A = \begin{bmatrix} 1 - \frac{24}{35}H & \frac{39}{350} \\ -\frac{1005}{584} - \frac{45}{146}H & 1 - \frac{30}{73}H \end{bmatrix},$$

$$B = \begin{bmatrix} -\frac{6}{35} & \frac{6}{5} + \frac{18}{35}H \\ \frac{155}{146} & -\frac{105}{73} \end{bmatrix},$$

$$C = \begin{bmatrix} -\frac{3}{175} & \frac{1}{10} \\ \frac{39}{584} & -\frac{30}{73} \end{bmatrix}.$$

The stability polynomial $\chi(\epsilon, H)$, obtained by combining the first and second characteristic polynomials of the method, is derived by evaluating the determinant of the matrix expression $(A\epsilon^2 - B\epsilon - C)$, yielding:

$$\begin{aligned} \chi_1(\epsilon, H) &= \det\left(A\epsilon^2 - B\epsilon - C\right) \\ &= \left| \begin{bmatrix} 1 - \frac{24}{35}H & \frac{39}{350} \\ -\frac{1005}{584} - \frac{45}{146}H & 1 - \frac{30}{73}H \end{bmatrix} \epsilon^2 \right. \\ &\quad \left. - \begin{bmatrix} -\frac{6}{35} & \frac{6}{5} + \frac{18}{35}H \\ \frac{155}{146} & -\frac{105}{73} \end{bmatrix} \epsilon - \begin{bmatrix} -\frac{3}{175} & \frac{1}{10} \\ \frac{39}{584} & -\frac{30}{73} \end{bmatrix} \right| \\ &= \frac{48719}{40880}\epsilon^4 - \frac{11559}{10220}\epsilon^4 H - \frac{123}{365}\epsilon^3 - \frac{3123}{4088}\epsilon^2 \\ &\quad - \frac{176}{125}\epsilon^4 H^2 - \frac{459}{292}\epsilon^3 H - \frac{8217}{10220}\epsilon^2 H - \frac{233}{2555}\epsilon \\ &\quad + \frac{3}{8176} + \frac{81}{511}\epsilon^3 H^2 - \frac{351}{10220}\epsilon H. \end{aligned} \tag{28}$$

The absolute stability region of method (8) is obtained by plotting the locus of points defined by the corresponding stability function in the complex plane.

$$\chi_1(\epsilon, H) = 0. \tag{29}$$

$\Rightarrow$

$$\begin{aligned} &\chi_1(\epsilon, H) = \\ &\frac{48719}{40880}\epsilon^4 - \frac{11559}{10220}\epsilon^4 H - \frac{123}{365}\epsilon^3 - \frac{3123}{4088}\epsilon^2 \\ &- \frac{176}{125}\epsilon^4 H^2 - \frac{459}{292}\epsilon^3 H - \frac{8217}{10220}\epsilon^2 H - \frac{233}{2555}\epsilon \\ &+ \frac{3}{8176} + \frac{81}{511}\epsilon^3 H^2 - \frac{351}{10220}\epsilon H = 0. \end{aligned} \tag{30}$$

To verify the zero-stability of method (8), the parameter H is set to zero in equation (30), resulting in the corresponding first characteristic equation given by:

$$\frac{48719}{40880}\epsilon^4 - \frac{123}{365}\epsilon^3 - \frac{3123}{4088}\epsilon^2 - \frac{233}{2555}\epsilon + \frac{3}{8176} = 0. \tag{31}$$

The roots of equation (31) are obtained by solving for ϵ, resulting in:

$$\epsilon = 0.0038962201,\ \epsilon = 0.5863656155,$$
$$\epsilon = 0.1347661799,\ \epsilon = 1.$$

In accordance with Definition 1, the obtained roots ϵ indicate that method (8) is zero-stable, since none lie outside the unit circle and the root $\epsilon = 1$ has multiplicity one.

The absolute stability region corresponding to method (8) is obtained using the boundary locus approach. In this procedure, the substitution $\epsilon = e^{i\theta}$ is introduced into equation (30) to trace the stability boundary in the complex plane. The resulting stability region for method (8), generated using Maple 18 software, is illustrated in Figure 1.

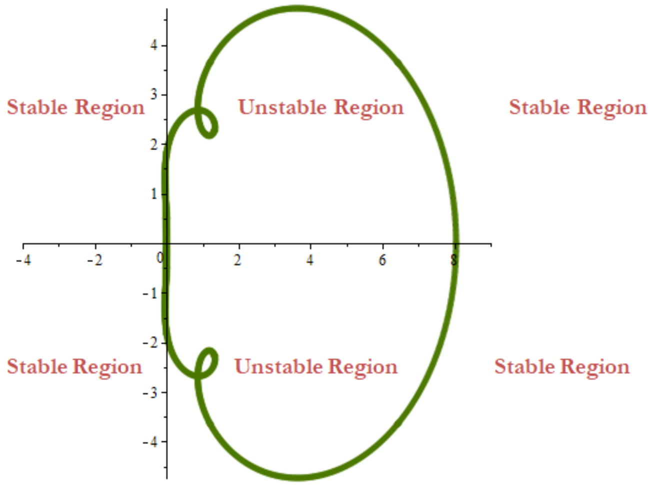

Figure 1. Stability region of the VSBBDF5 when $r = 1$.

Hence, the stability domain is defined by the exterior of the circular locus, indicating that the method is A-stable as it covers the whole left half of the complex plane.

ii. To examine the stability of method (9) for the step-size ratio $r = 2$, the scheme defined in equation (9) is first expressed as a system of linear difference equations. This representation allows the method to be written in matrix form, facilitating the analysis of its stability properties through the associated characteristic matrix equation.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_{n+1} \\ v_{n+2} \end{bmatrix} = \begin{bmatrix} -\frac{15}{18304} & \frac{21}{4576} \\ \frac{3}{493} & -\frac{772}{17255} \end{bmatrix}\begin{bmatrix} v_{n-3} \\ v_{n-2} \end{bmatrix} + \begin{bmatrix} \frac{35}{9152} & \frac{5145}{4576} \\ \frac{558}{3451} & -\frac{1812}{3451} \end{bmatrix}\begin{bmatrix} v_{n-1} \\ v_{n} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{3456}{2465} & 0 \end{bmatrix}\begin{bmatrix} v_{n+1} \\ v_{n+2} \end{bmatrix} + h\begin{bmatrix} 0 & \frac{315}{572} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_{n} \end{bmatrix} + h\begin{bmatrix} \frac{105}{143} & 0 \\ \frac{1152}{3451} & \frac{1536}{3451} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}. \tag{32}$$

This corresponds to the matrix equation:

$$\begin{bmatrix} 1 & 0 \\ -\frac{3456}{2465} & 1 \end{bmatrix}\begin{bmatrix} v_{n+1} \\ v_{n+2} \end{bmatrix} = \begin{bmatrix} -\frac{15}{18304} & \frac{21}{4576} \\ \frac{3}{493} & -\frac{772}{17255} \end{bmatrix}\begin{bmatrix} v_{n-3} \\ v_{n-2} \end{bmatrix} + \begin{bmatrix} \frac{35}{9152} & \frac{5145}{4576} \\ \frac{558}{3451} & -\frac{1812}{3451} \end{bmatrix}\begin{bmatrix} v_{n-1} \\ v_{n} \end{bmatrix} + h\begin{bmatrix} 0 & \frac{315}{572} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_{n} \end{bmatrix} + h\begin{bmatrix} \frac{105}{143} & 0 \\ \frac{1152}{3451} & \frac{1536}{3451} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}.$$

Substituting the Dahlquist linear test equation $v^{'} = \lambda v$ into (32), and simplifying, we obtain

$$\begin{bmatrix} \left(1 - \frac{105}{143}h\lambda\right) & 0 \\ -\frac{3456}{2465} - \frac{1152}{3451}h\lambda & \left(1 - \frac{1536}{3451}h\lambda\right) \end{bmatrix}\begin{bmatrix} v_{n+1} \\ v_{n+2} \end{bmatrix} = \begin{bmatrix} -\frac{15}{18304} & \frac{21}{4576} \\ \frac{3}{493} & -\frac{772}{17255} \end{bmatrix}\begin{bmatrix} v_{n-3} \\ v_{n-2} \end{bmatrix} + \begin{bmatrix} \frac{35}{9152} & \left(\frac{5145}{4576} - \frac{315}{572}h\lambda\right) \\ \frac{558}{3451} & -\frac{1812}{3451} \end{bmatrix}\begin{bmatrix} v_{n-1} \\ v_{n} \end{bmatrix}. \tag{33}$$

Let $H = \lambda h$ in equation (33). This leads to

$$\begin{bmatrix} \left(1 - \frac{105}{143}H\right) & 0 \\ -\frac{3456}{2465} - \frac{1152}{3451}H & \left(1 - \frac{1536}{3451}H\right) \end{bmatrix}\begin{bmatrix} v_{n+1} \\ v_{n+2} \end{bmatrix} = \begin{bmatrix} -\frac{15}{18304} & \frac{21}{4576} \\ \frac{3}{493} & -\frac{772}{17255} \end{bmatrix}\begin{bmatrix} v_{n-3} \\ v_{n-2} \end{bmatrix} + \begin{bmatrix} \frac{35}{9152} & \left(\frac{5145}{4576} - \frac{315}{572}H\right) \\ \frac{558}{3451} & -\frac{1812}{3451} \end{bmatrix}\begin{bmatrix} v_{n-1} \\ v_{n} \end{bmatrix}. \tag{34}$$

An equivalent representation of the matrix equation is given by:

$$DV_m = EV_{m-1} + FV_{m-2}, \qquad n = 2m. \tag{35}$$

where

$$D = \begin{bmatrix} 1 - \frac{105}{143}H & 0 \\ -\frac{3456}{2465} - \frac{1152}{3451}H & 1 - \frac{1536}{3451}H \end{bmatrix},$$

$$E = \begin{bmatrix} \frac{35}{9152} & \frac{5145}{4576} - \frac{315}{572}H \\ \frac{558}{3451} & -\frac{1812}{3451} \end{bmatrix},$$

$$F = \begin{bmatrix} -\frac{15}{18304} & \frac{21}{4576} \\ \frac{3}{493} & -\frac{772}{17255} \end{bmatrix}.$$

The stability polynomial $\chi_2(\epsilon, H)$ associated with method (9) is obtained by computing the determinant of the matrix expression

$$D\epsilon^2 - E\epsilon - F.$$

which gives:

$$\begin{aligned}\chi_2(\epsilon, H) = {} & \frac{83540}{70499}\epsilon^4 - \frac{35514}{29029}\epsilon^4 H - \frac{4081339}{3947944}\epsilon^3 \\ & - \frac{25819}{179452}\epsilon^2 + \frac{23040}{70499}\epsilon^4 H^2 - \frac{55032}{70499}\epsilon^3 H \\ & - \frac{119157}{986986}\epsilon^2 H - \frac{57835}{7895888}\epsilon + \frac{69}{7895888} \\ & + \frac{12960}{70499}\epsilon^3 H^2 - \frac{945}{281996}\epsilon H = 0.\end{aligned} \tag{36}$$

For the purpose of zero-stability analysis, equation (32) is evaluated at $H = 0$, thereby yielding the corresponding first characteristic polynomial as follows:

$$\frac{83540}{70499}\epsilon^4-\frac{4081339}{3947944}\epsilon^3-\frac{25819}{179452}\epsilon^2-\frac{57835}{7895888}\epsilon+\frac{69}{7895888}=0. \quad (37)$$

By determining the solutions of equation (37) in terms of ϵ, the following values are obtained:

$$\epsilon = 0.00116611505, \quad \epsilon = 0.07952388658,$$
$$\epsilon = 0.07952388658, \quad \epsilon = 1.$$

Thus, method (9) is zero-stable, as it fulfills the requirements of the zero-stability definition.

To determine the absolute stability region of method (9), the boundary locus method is employed. Specifically, substituting $\epsilon = e^{i\theta}$ into equation (36) enables the characterization of the stability boundary in the complex plane. The corresponding stability region, generated using Maple 18, is illustrated in Figure 2.

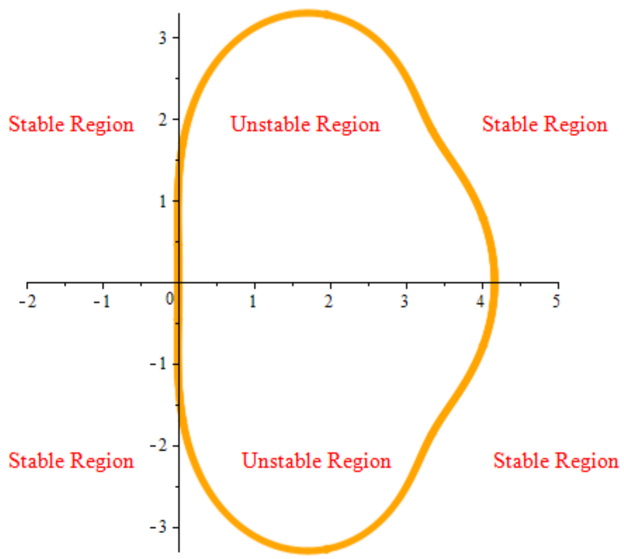

Figure 2. Stability region of the VSBBDF5 when $r = 2$.

From the plotted stability region, it is observed that the region extends to cover the entire left-half of the complex plane. This implies that all values of $H = \lambda h$ with $\Re e(\lambda) < 0$ lie within the stability domain of the method. Consequently, in accordance with the definition of A-stability, method (9) is said to be A-stable.

iii. Similarly, the same procedure is used for $r = \frac{5}{6}$, the stability polynomial of (10) is obtained:

$$\begin{aligned}\chi_{\frac{5}{6}}(\epsilon, H) = {} & \frac{4228801}{3558232}\epsilon^4 - \frac{3410473}{3113453}\epsilon^4 H + \frac{1998238653}{15567265000}\epsilon^3 \\ & - \frac{110658524237}{97295406250}\epsilon^2 + \frac{592416}{22223895}\epsilon^4 H^2 \\ & - \frac{2180786940}{1111947500}\epsilon^3 H - \frac{508477953}{3891816250}\epsilon^2 H \\ & - \frac{8766801028}{48647703125}\epsilon + \frac{35928684}{48647703125} \\ & + \frac{333234}{22223895}\epsilon^3 H^2 - \frac{945}{281996}\epsilon H = 0.\end{aligned} \quad (38)$$

Setting $H = 0$ in equation (38) simplifies the expression to the first characteristic polynomial used to assess zero-stability:

$$\begin{aligned}&\frac{4228801}{3558232}\epsilon^4 + \frac{1998238653}{15567265000}\epsilon^3 - \frac{110658524237}{97295406250}\epsilon^2 \\ &- \frac{8766801028}{48647703125}\epsilon + \frac{35928684}{48647703125} = 0.\end{aligned} \quad (39)$$

Equation (39) is solved with respect to ϵ, resulting in the following set of roots:

$$\epsilon = 0.0039974619, \ \epsilon = 0.9480239443,$$
$$\epsilon = 0.1639805539, \ \epsilon = 1$$

Thus, it follows from the definition of zero-stability that method (10) is zero-stable.

The absolute stability region for method (10) is evaluated via the boundary locus method. This involves substituting $\epsilon = e^{i\theta}$ into equation (38), which enables the characterization of the stability boundary in the complex plane. The corresponding stability region, obtained with the aid of Maple 18, is illustrated in Figure 3.

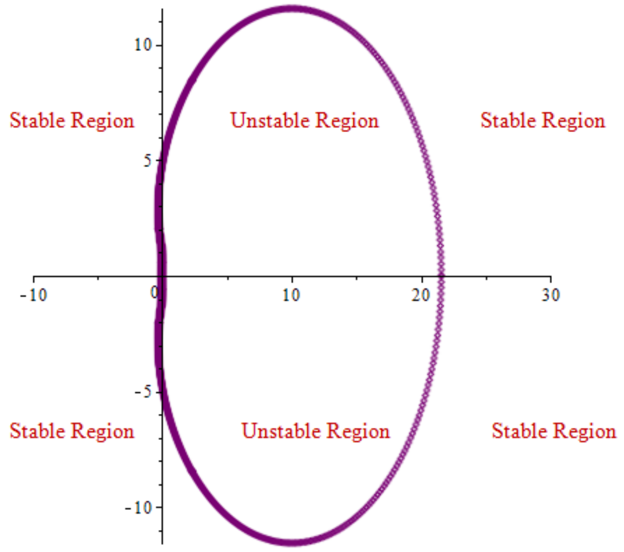

Figure 3. Stability region of the VSBBDF5 when $r = \frac{5}{6}$.

From the graphical representation, it is evident that the stability region encompasses the entire left half of the complex plane. Therefore, the method is A-stable, since all values satisfying $\Re e(\lambda) < 0$ lie within the region of absolute stability.

6 Implementation of the Method

The implementation of the method is carried out using Newton's iterative scheme, which proceeds as outlined below. Equations (8)-(10) can be reformulated in the following general form:

$$\left.\begin{aligned} v_{n+1} &= \beta_1 v_{n+2} + \sigma_1 h f_n + \sigma_2 h f_{n+1} + \tau_1, \\ v_{n+2} &= \beta_2 v_{n+1} + \sigma_3 h f_{n+1} + \sigma_4 h f_{n+2} + \tau_2. \end{aligned}\right\} \tag{40}$$

where τ_1 and τ_2 correspond to the backward values, and equation (40) can be cast into the following matrix representation:

$$\left(I - \Gamma_1\right) V = h\left(\Upsilon_1 F_1 + \Upsilon_2 F_2\right) + \Omega. \tag{41}$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{bmatrix}, \quad \Upsilon_1 = \begin{bmatrix} 0 & \sigma_1 \\ 0 & 0 \end{bmatrix},$$
$$\Upsilon_2 = \begin{bmatrix} \sigma_2 & 0 \\ \sigma_3 & \sigma_4 \end{bmatrix}, \quad V = \begin{bmatrix} v_{n+1} \\ v_{n+2} \end{bmatrix}, \quad F_1 = \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix},$$
$$F_2 = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}, \quad \Omega = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}.$$

Let

$$F = \left(I - \Gamma_1\right) V - h\left(\Upsilon_1 F_1 + \Upsilon_2 F_2\right) - \Omega = 0. \tag{42}$$

The Newton iteration associated with the VSBBDF5 method can be written in the following form:

$$V_{n+1,n+2}^{(i+1)} = V_{n+1,n+2}^{(i)} - \left[F_j'\left(v_{n+1,n+2}^{(i)}\right)\right]^{-1} F_j\left(v_{n+1,n+2}^{(i)}\right). \tag{43}$$

An equivalent representation of equation (43) is given by:

$$V_{n+1,n+2}^{(i+1)} - V_{n+1,n+2}^{(i)} = -\frac{\left(I - \Gamma_1\right)\left(V_{n+1,n+2}^{(i)}\right) - h\Upsilon_1 F_1 - h\Upsilon_2 F_2 - \Omega}{\left(I - \Gamma_1\right) - h\Upsilon_1 \frac{\partial F_1}{\partial V}\left(V_{n+1,n+2}^{(i)}\right) - h\Upsilon_2 \frac{\partial F_2}{\partial V}\left(V_{n+1,n+2}^{(i)}\right)}. \tag{44}$$

In order to compare accuracy, the maximum error is determined using the developed scheme. If v_i and $v\left(u_i\right)$ denote the computed and exact solutions of $v' = f(u, v)$, $v(a) = v_0$, respectively, then the absolute error is defined as:

$$\left(\text{error}_i\right)_t = \left|\left(v_i\right)_t - \left(v\left(u_i\right)\right)_t\right|. \tag{45}$$

The maximum error is defined by:

$$E_{\max} = \max_{1 \le i \le T}\left(\max_{1 \le t \le N}\left(\text{error}_i\right)_t\right). \tag{46}$$

where T denotes the total number of time steps and N represents the number of equations in the system. Let $V_{n+1}^{(i+1)}$ denote the $(i+1)th$ iterate. The iterative error is then given by

$$E_{1,2}^{(i+1)} = V_{n+1,n+2}^{(i+1)} - V_{n+1,n+2}^{(i)}. \tag{47}$$

Equation (47) can be reformulated as

$$E_{1,2}^{(i+1)} = \bar{A}^{-1}\bar{B}. \tag{48}$$

which is equivalently expressed as the linear system

$$\bar{A}E_{1,2}^{(i+1)} = \bar{B}. \tag{49}$$

where

$$\bar{A} = \left((I-\Gamma_1) - h\Gamma_1 \frac{\partial F_1}{\partial V}\left(V^{(i)}_{n+1,n+2}\right)\right) - h\Gamma_2 \frac{\partial F_2}{\partial V}\left(V^{(i)}_{n+1,n+2}\right),$$

and

$$\bar{B} = \left((I-\Gamma_1)\left(V^{(i)}_{n+1,n+2}\right) - h\Gamma_1 F_1 - h\Gamma_2 F_2 - \Omega\right).$$

Consequently, Newton's iterative method is employed to solve the system given in equation (49) for different values of the step-size ratio r.

$$\overline{A} = \begin{pmatrix} 1-\sigma_2 h \dfrac{\partial f_{n+1}}{\partial v_{n+1}} & -\beta_1 \\ -\beta_2 - \sigma_3 h \dfrac{\partial f_{n+1}}{\partial v_{n+1}} & 1-\sigma_4 h \dfrac{\partial f_{n+2}}{\partial v_{n+2}} \end{pmatrix},$$

$$\overline{B} = \begin{pmatrix} -v^i_{n+1} + \beta_1 v^i_{n+2} + \sigma_1 h f^i_n + \sigma_2 h f^i_{n+1} + \tau_1 \\ -v^i_{n+2} + \beta_2 v^i_{n+1} + \sigma_3 h f^i_{n+1} + \sigma_4 h f^i_{n+2} + \tau_2 \end{pmatrix}.$$

Specifically, when $r=1$

$$\overline{A} = \begin{pmatrix} 1-\dfrac{24}{35} h \dfrac{\partial f_{n+1}}{\partial v_{n+1}} & \dfrac{39}{350} \\ -\dfrac{1005}{584} - \dfrac{45}{146} h \dfrac{\partial f_{n+1}}{\partial v_{n+1}} & 1-\dfrac{30}{73} h \dfrac{\partial f_{n+2}}{\partial v_{n+2}} \end{pmatrix},$$

$$\overline{B} = \begin{pmatrix} -v^i_{n+1} - \dfrac{39}{350} v^i_{n+2} + \dfrac{18}{35} h f^i_n + \dfrac{24}{35} h f^i_{n+1} \\ -\dfrac{3}{175} v_{n-3} + \dfrac{1}{10} v_{n-2} - \dfrac{6}{35} v_{n-1} + \dfrac{6}{5} v_n \\ -v^i_{n+2} + \dfrac{1005}{584} v^i_{n+1} + \dfrac{45}{146} h f^i_{n+1} + \dfrac{30}{73} h f^i_{n+2} \\ +\dfrac{39}{584} v_{n-3} - \dfrac{30}{73} v_{n-2} + \dfrac{155}{146} v_{n-1} - \dfrac{105}{73} v_n \end{pmatrix}$$

When $r=2$

$$\overline{A} = \begin{pmatrix} 1-\dfrac{105}{143} h \dfrac{\partial f_{n+1}}{\partial v_{n+1}} & \dfrac{2415}{18304} \\ -\dfrac{3456}{2465} - \dfrac{1152}{3451} h \dfrac{\partial f_{n+1}}{\partial v_{n+1}} & 1-\dfrac{1536}{3451} h \dfrac{\partial f_{n+2}}{\partial v_{n+2}} \end{pmatrix}.$$

$$\overline{B} = \begin{pmatrix} -v^i_{n+1} - \dfrac{2415}{18304} v^i_{n+2} + \dfrac{315}{572} h f^i_n + \dfrac{105}{143} h f^i_{n+1} \\ -\dfrac{15}{18304} v_{n-3} + \dfrac{21}{4576} v_{n-2} + \dfrac{35}{9152} v_{n-1} + \dfrac{5145}{4576} v_n \\ -v^i_{n+2} + \dfrac{3456}{2465} v^i_{n+1} + \dfrac{1152}{3451} h f^i_{n+1} + \dfrac{1536}{3451} h f^i_{n+2} \\ +\dfrac{3}{493} v_{n-3} - \dfrac{772}{17255} v_{n-2} + \dfrac{558}{3451} v_{n-1} - \dfrac{1812}{3451} v_n \end{pmatrix}.$$

When $r=\frac{5}{6}$

$$\overline{A} = \begin{pmatrix} 1-\dfrac{1232}{1861} h \dfrac{\partial f_{n+1}}{\partial v_{n+1}} & \dfrac{29246}{284733} \\ -\dfrac{49113}{26768} - \dfrac{5049}{16730} h \dfrac{\partial f_{n+1}}{\partial v_{n+1}} & 1-\dfrac{3366}{8365} h \dfrac{\partial f_{n+2}}{\partial v_{n+2}} \end{pmatrix}.$$

$$\overline{B} = \begin{pmatrix} -v^i_{n+1} - \dfrac{29246}{284733} v^i_{n+2} + \dfrac{924}{1861} h f^i_n + \dfrac{1232}{1861} h f_{n+1} \\ -\dfrac{71104}{2093625} v_{n-3} + \dfrac{44793}{232625} v_{n-2} \\ -\dfrac{1342656}{3954625} v_{n-1} + \dfrac{298606}{232625} v_n \\ -v^i_{n+2} + \dfrac{49113}{26768} v^i_{n+1} + \dfrac{5049}{16730} h f^i_{n+1} + \dfrac{3366}{8365} h f^i_{n+2} \\ +\dfrac{131648}{1045625} v_{n-3} - \dfrac{12306249}{16730000} v_{n-2} \\ +\dfrac{1819584}{1045625} v_{n-1} - \dfrac{2054943}{1045625} v_n \end{pmatrix}.$$

A computer program developed in the C programming language is used to implement the proposed VSBBDF5 method.

6.1 Local truncation error (LTE) and Step Size Selection

To control the progression of the integration process, three step-size selection strategies are utilized. At the beginning of the computation, a suitable initial step size is chosen. The associated local truncation error is then estimated using

$$LTE_r = \left| v^{(p+1)}_{n+2} - v^{(p)}_{n+2} \right|. \tag{50}$$

where $v^{(p+1)}_{n+2}$ denotes the approximation obtained from the method of order $p+1$, while $v^{(p)}_{n+2}$ represents the solution computed using the method of order p. The evaluated local truncation errors provide an indication of the solution accuracy and are used to determine whether the current step should be accepted or if the step size needs to be modified in order to satisfy the prescribed error tolerance.

For the VSBBDF5 Method, the LTEs are given by

i. When $r=1$

$$\left.\begin{aligned}
v^{(5)}_{n+2} &= \frac{39}{584}v_{n-3} - \frac{30}{73}v_{n-2} + \frac{155}{146}v_{n-1} - \frac{105}{73}v_n \\
&\quad + \frac{1005}{584}v_{n+1} + \frac{30}{73}hf_{n+2} + \frac{45}{146}hf_{n+1}, \\
v^{(4)}_{n+2} &= -\frac{9}{109}v_{n-2} + \frac{46}{109}v_{n-1} - \frac{90}{109}v_n \\
&\quad + \frac{162}{109}v_{n+1} + \frac{48}{109}hf_{n+2} + \frac{36}{109}hf_{n+1}, \\
\mathrm{LTE}_1 &= \frac{39}{584}v_{n-3} - \frac{2613}{7957}v_{n-2} + \frac{10179}{15914}v_{n-1} - \frac{4875}{7957}v_n \\
&\quad + \frac{14937}{63656}v_{n+1} - \frac{234}{7957}hf_{n+2} - \frac{351}{159144}hf_{n+1}.
\end{aligned}\right\} \tag{51}$$

ii. When $r = 2$

$$\left.\begin{aligned}
v^{(5)}_{n+2} &= \frac{81}{11711}v_{n-3} - \frac{428}{8365}v_{n-2} + \frac{314}{1673}v_{n-1} - \frac{1116}{1673}v_n \\
&\quad + \frac{89216}{58555}v_{n+1} + \frac{768}{1673}hf_{n+2} + \frac{384}{1673}hf_{n+1}, \\
v^{(4)}_{n+2} &= -\frac{13}{995}v_{n-2} + \frac{19}{199}v_{n-1} - \frac{99}{199}v_n \\
&\quad + \frac{1408}{995}v_{n+1} + \frac{96}{199}hf_{n+2} + \frac{48}{199}hf_{n+1}, \\
\mathrm{LTE}_2 &= \frac{81}{11711}v_{n-3} - \frac{63423}{1664635}v_{n-2} + \frac{1284596}{10320737}v_{n-1} \\
&\quad - \frac{5079437}{10320737}v_n + \frac{9969721}{361225795}v_{n+1} + \frac{5085776}{10320737}hf_{n+2} \\
&\quad - \frac{2783944}{10320737}hf_{n+1}.
\end{aligned}\right\} \tag{52}$$

iii. When $r = \frac{5}{6}$

$$\left.\begin{aligned}
v^{(5)}_{n+2} &= \frac{131648}{1045625}v_{n-3} - \frac{12306249}{16730000}v_{n-2} + \frac{1819584}{1045625}v_{n-1} \\
&\quad - \frac{2054943}{1045625}v_n + \frac{49113}{26768}v_{n+1} + \frac{3366}{8365}hf_{n+2} + \frac{5049}{16730}hf_{n+1}, \\
v^{(4)}_{n+2} &= -\frac{15759}{114800}v_{n-2} + \frac{4608}{7175}v_{n-1} - \frac{22627}{21525}v_n \\
&\quad + \frac{3043}{1968}v_{n+1} + \frac{374}{861}hf_{n+2} + \frac{187}{574}hf_{n+1}, \\
\mathrm{LTE}_{5/6} &= \frac{131648}{1045625}v_{n-3} - \frac{51299523}{85741250}v_{n-2} + \frac{47070144}{42870625}v_{n-1} \\
&\quad - \frac{117561664}{128611875}v_n + \frac{118745}{411558}v_{n+1} - \frac{32912}{1028895}hf_{n+2} \\
&\quad - \frac{8228}{85741250}hf_{n+1}.
\end{aligned}\right\} \tag{53}$$

The step-size control procedure can be summarized as follows:

1. If the computed local truncation error satisfies the prescribed tolerance, that is, $LTE \leq TOL$, the step is accepted and the current block step size is retained (corresponding to $r = 1$). A new step size is then estimated using

$$h_{new} = c \times h_{old} \times \left(\frac{TOL}{LTE}\right)^{1/p},$$

where p is the order of the method and $c = 0.7$ is the safety factor.

2. To prevent an excessive increase in the step size, a restriction is imposed such that if

$$h_{new} > 1.2 \times h_{old},$$

then the step size is limited to

$$h = 1.2 \times h_{old},$$

which corresponds to the case $r = \frac{5}{6}$.

3. On the other hand, if the local truncation error exceeds the prescribed tolerance, that is, $LTE > TOL$, the step is rejected and the step size is reduced according to

$$h = \frac{1}{2}h_{old},$$

which corresponds to the case $r = 2$. The computation is then repeated using the reduced step size.

The step-size control strategy is summarized in Figure 4.

7 Test problems and Numerical results

To assess the performance and computational efficiency of the fully implicit fifth-order Variable Step-Size Superclass Block Backward Differentiation Formula (VSBBDF5), the method is applied to a set of carefully selected stiff systems of initial value problems, as described below:

Problem 1: Consider a stiff cosine reaction problem, adapted from [22], governed by the differential equation:

$$v' = -2\pi \sin(2\pi u) - \frac{1}{\psi}\left(v - \cos(2\pi u)\right),$$
$$v(0) = 1,\ 0 \leq u \leq 10.$$

The degree of stiffness increases markedly as the parameter ψ approaches zero. In the present study, the value $\psi = 10^{-3}$ is employed to ensure a highly stiff regime. The exact solution of the problem is given by

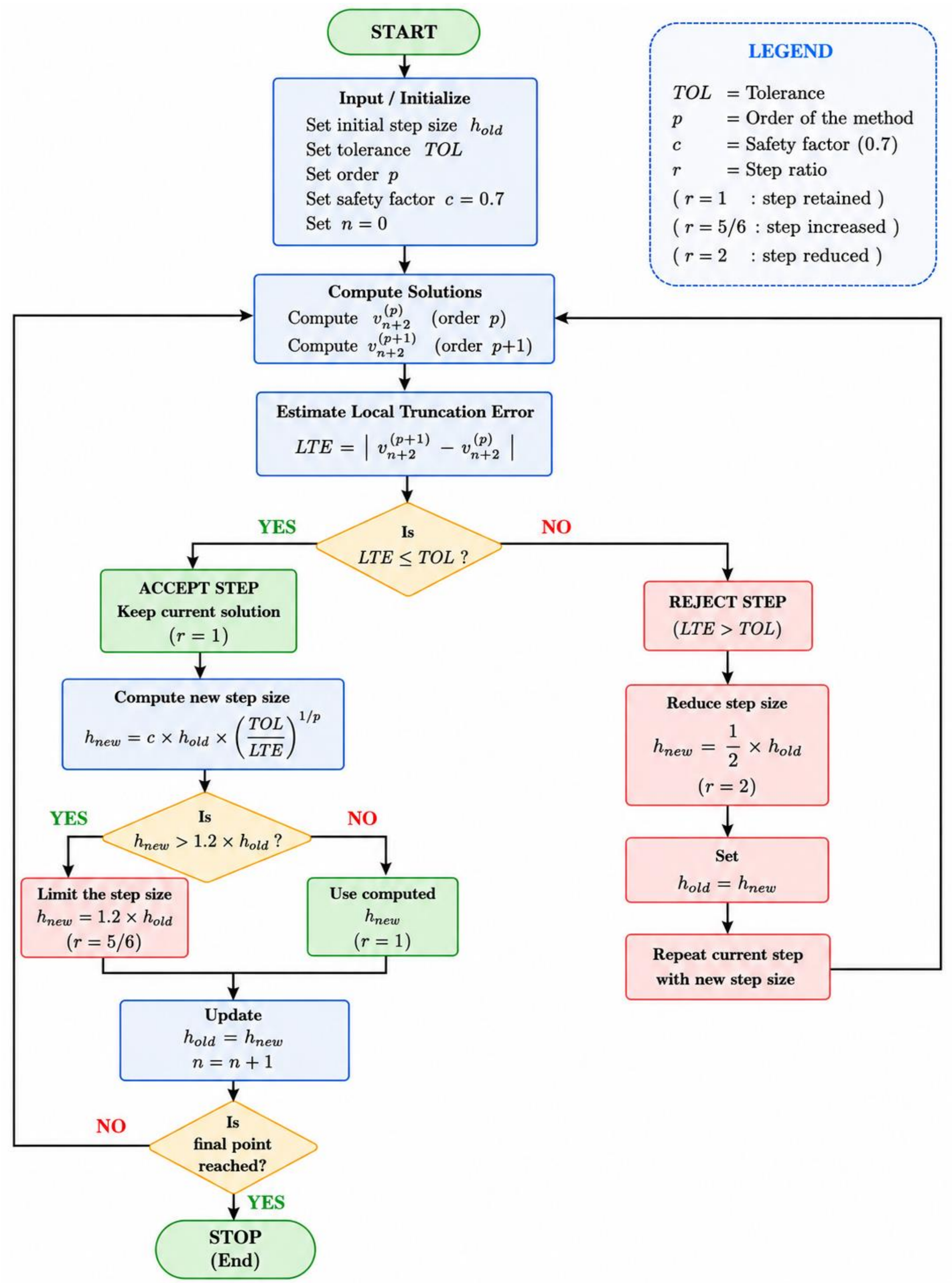

Figure 4. The step-size control strategy algorithm flow chart.

$$v(u) = \cos(2\pi u).$$

Problem 2: The well-known Lambert's Highly Stiff Oscillatory System, adopted from [22], is given by

$$\begin{cases} v_1' = -21v_1 + 19v_2 - 20v_3, & v_1(0) = 1, \\ v_2' = 19v_1 - 21v_2 + 20v_3, & v_2(0) = 0, \\ v_3' = 40v_1 - 40v_2 - 40v_3, & v_3(0) = -1, \end{cases}$$

where $0 \leq u \leq 20$.

The exact solutions are

$$\begin{aligned} v_1(u) &= \frac{1}{2}\left[e^{-2u} + e^{-40u}\left(\cos 40u + \sin 40u\right)\right], \\ v_2(u) &= \frac{1}{2}\left[e^{-2u} - e^{-40u}\left(\cos 40u + \sin 40u\right)\right], \\ v_3(u) &= 2e^{-40u}\left(-\frac{1}{2}\cos 40u + \frac{1}{2}\sin 40u\right). \end{aligned}$$

The corresponding eigenvalues of the Jacobian matrix are $\lambda_1 = -2$, $\lambda_2 = -40 + 40i$ and $\lambda_3 = -40 + 40i$.

The presence of complex conjugate eigenvalues indicates oscillatory behavior, while the large negative real parts imply stiffness; hence, the problem is highly stiff and oscillatory.

Problem 3: This stiff non-normal oscillator problem, obtained from [22], is defined by

$$v''(u) + 2\psi v'(u) + \psi^2 v(u) = \psi^2 f(u), \quad 0 \leq u \leq 20.$$

with the initial conditions

$$\begin{aligned} v(0) &= \frac{\psi^2(\psi - 1)}{(\psi^2 + 1)^2}, \\ v'(0) &= \frac{2\psi^2}{(\psi^2 + 1)^2}. \end{aligned}$$

which can be transformed into the following linear system

$$\begin{aligned} v_1' &= v_2, \\ v_2' &= \psi^2 \cos(u) - 2\psi v_2 - \psi^2 v_1, \\ v_1(0) &= \frac{\psi^2(\psi - 1)}{(\psi^2 + 1)^2}, \\ v_2(0) &= \frac{2\psi^2}{(\psi^2 + 1)^2}, \\ & 0 \leq u \leq 20. \end{aligned}$$

where ψ is the stiffness parameter. The system has a Jordan form, with double eigenvalues $-\psi$ and its general solution contains terms of the form $e^{\psi u}$ and $ue^{\psi u}$. In our experiment, we chose $\psi = 10^3$.

The test problems were solved using the built-in MATLAB solvers ODE15s and ODE23s, together with the fourth-order Variable Step-Size Superclass of Block Backward Differentiation Formula (VSBBDF4) [22], and the proposed fully implicit fifth-order Variable Step-Size Superclass of Block Backward Differentiation Formula (VSBBDF5).

To assess and compare the performance of these methods, both accuracy and computational efficiency were examined. Specifically, the maximum absolute errors and corresponding CPU times were computed for a range of tolerance values (TOL) for each test problem. In addition, the total number of integration steps and function evaluations required by each method were recorded to provide a comprehensive measure of their computational cost. The notations used in the following tables are defined as shown in Table 3.

Table 3. Notation and description of symbols used in the numerical results.

SYMBOL	DESCRIPTION
TOL	Tolerance value Used
METHOD	Method used
TS	Total number of steps
FE	Number of Function Evaluation
CPU TIME	Computation Time
MAXE	Maximum absolute error
ODE15s	Variable-Step, Variable-Order Implicit Solver
ODE23s	Variable-Step Modified Rosenbrock Method
VSBBDF4	Fourth Order Variable Step-Size Block Backward Differentiation Formula [22]
VSBBDF5	Proposed Fully implicit fifth-order Variable Step-Size Superclass of Block Backward Differentiation Formula

The numerical test problems considered in this study are standard benchmark examples obtained from the literature. However, the performance results for the reference methods namely ODE15s, ODE23s, and VSBBDF4 are independently computed within the present work rather than extracted from their original published sources.

In doing so, the parameter configurations, step-size strategies, and computational conditions are carefully implemented to remain consistent with those reported in the literature. The proposed method is then applied to the same set of benchmark problems to enable a direct and meaningful comparison of performance.

This approach ensures that all results are generated under a uniform computational framework, thereby eliminating discrepancies arising from differences in implementation and providing a fair, consistent, and reliable evaluation of the proposed method. The numerical results for the three test problems are presented in Tables 4–6, respectively.

Table 4. Numerical Result for Problem 1.

TOL	METHOD	TS	FE	MAXE	CPU TIME
10^{-2}	ODE15s	118	156	2.31000E-003	4.10000E-003
	ODE23s	152	198	4.76000E-003	3.60000E-003
	VSBBDF4	66	104	5.16437E-005	7.14360E-004
	VSBBDF5	58	94	2.83614E-005	6.19402E-005
10^{-4}	ODE15s	214	298	4.87000E-005	6.80000E-003
	ODE23s	356	472	8.92000E-005	8.90000E-003
	VSBBDF4	116	186	1.54385E-006	2.26100E-003
	VSBBDF5	102	161	3.72109E-007	6.42196E-004
10^{-6}	ODE15s	392	521	6.12000E-007	1.25000E-002
	ODE23s	684	901	1.14000E-006	1.97000E-002
	VSBBDF4	402	347	3.11839E-008	5.18310E-003
	VSBBDF5	363	247	1.30750E-008	1.64410E-003

Table 5. Numerical Result for Problem 2.

TOL	METHOD	TS	FE	MAXE	CPU TIME
10^{-2}	ODE15s	142	188	3.84000E-003	6.50000E-003
	ODE23s	198	254	6.91000E-003	7.90000E-003
	VSBBDF4	48	97	5.56533E-005	9.38140E-005
	VSBBDF5	41	83	5.17241E-006	6.42185E-005
10^{-4}	ODE15s	268	351	7.12000E-005	1.12000E-002
	ODE23s	412	538	1.36000E-004	1.68000E-002
	VSBBDF4	73	216	6.47609E-007	7.73260E-004
	VSBBDF5	66	206	2.47204E-008	4.64739E-005
10^{-6}	ODE15s	486	624	9.45000E-007	1.98000E-002
	ODE23s	801	1042	2.08000E-006	3.45000E-002
	VSBBDF4	142	382	6.61553E-009	5.25100E-003
	VSBBDF5	136	371	4.72451E-009	2.97410E-004

Table 6. Numerical Result for Problem 3.

TOL	METHOD	TS	FE	MAXE	CPU TIME
10^{-2}	ODE15s	176	228	5.62000E-003	8.10000E-003
	ODE23s	268	346	9.85000E-003	1.04000E-002
	VSBBDF4	83	151	5.16437E-005	3.61400E-004
	VSBBDF5	71	96	5.59146E-006	4.06247E-005
10^{-4}	ODE15s	324	412	9.74000E-005	1.46000E-002
	ODE23s	536	698	1.87000E-004	2.27000E-002
	VSBBDF4	152	298	1.54385E-006	5.19240E-004
	VSBBDF5	126	217	1.82610E-007	1.27824E-004
10^{-6}	ODE15s	598	741	1.21000E-006	2.69000E-002
	ODE23s	1042	1358	2.94000E-006	4.63000E-002
	VSBBDF4	190	512	2.45230E-007	5.82100E-003
	VSBBDF5	174	398	4.84257E-008	7.46325E-004

Figures 5, 6 and 7 show the variation of the maximum absolute error with respect to the prescribed tolerance.

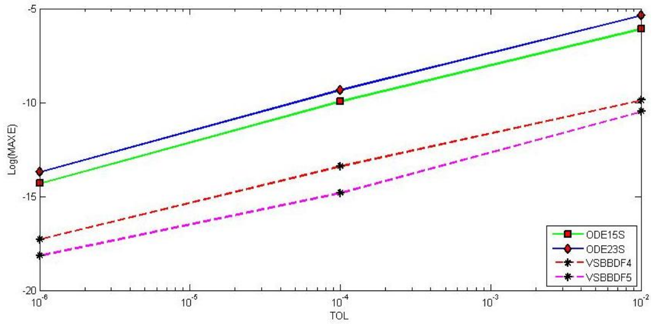

Figure 5. Accuracy curve for Problem 1.

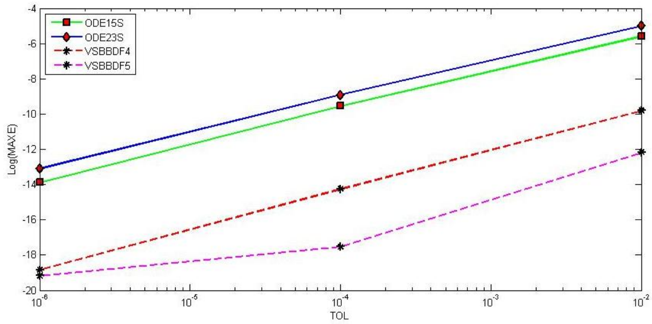

Figure 6. Accuracy curve for Problem 2.

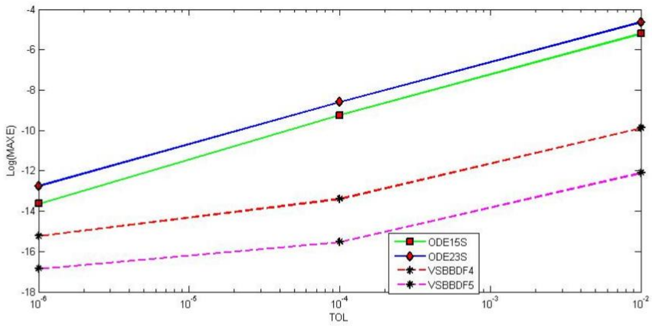

Figure 7. Accuracy curve for Problem 3.

8 Discussion of Results

The performance of the proposed fifth-order Variable Step-Size Superclass Block Backward Differentiation Formula (VSBBDF5) is examined in comparison with established stiff solvers, namely ODE15s and ODE23s, as well as the fourth-order block method (VSBBDF4). The assessment is based on key performance indicators, including accuracy, computational cost, number of integration steps, and function evaluations, across varying tolerance levels.

Across all experiments, the methods exhibit the expected behavior in which tighter tolerance requirements lead to improved accuracy at the expense of increased computational effort. However, the rate of this increase differs significantly among the methods, revealing clear distinctions in efficiency and scalability. While all methods respond correctly to changes in tolerance, their ability to balance accuracy and computational cost varies considerably.

In terms of accuracy, the proposed VSBBDF5 method consistently delivers the most accurate results. It produces smaller errors than both the classical solvers and the lower-order block method at all levels of tolerance. The VSBBDF4 method also demonstrates improved accuracy compared to the classical approaches, confirming the effectiveness of block-based formulations. In contrast, ODE15s and ODE23s yield comparatively larger errors, indicating that they are less efficient in capturing the dynamics of the stiff systems considered.

From the perspective of computational efficiency, the VSBBDF5 method consistently requires less CPU time than the other methods. This indicates that it achieves high accuracy with reduced computational cost. The VSBBDF4 method performs better than the classical solvers, further highlighting the advantages of block methods. Among the classical solvers, ODE23s generally exhibits the least efficiency, while ODE15s performs relatively better but still remains less competitive than the block-based approaches.

An examination of the number of integration steps and function evaluations provides further insight into the efficiency of the methods. The VSBBDF5 method consistently requires fewer steps and fewer function evaluations, demonstrating its ability to achieve greater accuracy per step. This efficiency is a direct consequence of its higher-order formulation and adaptive step-size strategy. The VSBBDF4 method also shows improved performance in this regard, although it does not match the efficiency of the fifth-order method. On the other hand, the classical solvers require more frequent function evaluations, which contributes to their higher computational cost and reflects their general-purpose design.

Another important aspect of the results is scalability. As the tolerance becomes more stringent, the increase in computational effort for the VSBBDF5 method is more gradual compared to the other methods. This indicates strong scalability and robustness, making it particularly suitable for problems that require high levels of accuracy. The VSBBDF4 method demonstrates moderate scalability, while the classical solvers show a more rapid increase in computational demand, especially in terms of function evaluations.

The overall superior performance of the VSBBDF5 method can be attributed to several key features. Its higher-order accuracy allows it to achieve precise results with fewer integration steps. The variable step-size mechanism enables efficient adaptation to regions of rapid and slow variation in the solution. In addition, the block formulation enhances computational efficiency by computing multiple solution values simultaneously, thereby reducing redundant operations. These characteristics collectively contribute to its improved accuracy and reduced computational cost.

In contrast, the classical solvers, such as ODE15s and ODE23s, are designed as general-purpose methods and do not fully exploit the structure of stiff systems. As a result, they require more computational effort to achieve comparable accuracy.

Overall, the results demonstrate that the proposed VSBBDF5 method achieves the best balance between accuracy and efficiency among the methods considered. It consistently outperforms both the classical solvers and the lower-order block method across all performance metrics. Its ability to deliver highly accurate results with reduced computational effort, combined with its strong scalability, makes it a reliable and effective approach for solving stiff ordinary differential equations.

9 Conclusion

This study presented a fifth-order Variable Step-Size Superclass Block Backward Differentiation Formula (VSBBDF5) for the numerical solution of stiff systems of ordinary differential equations. The method was developed within the framework of block multistep techniques, incorporating a variable step-size strategy and a stability control mechanism to enhance both accuracy and efficiency.

From the theoretical viewpoint, the proposed method satisfies the fundamental requirements for convergence. In particular, it is zero-stable, ensuring that small perturbations in the initial data or intermediate computations do not lead to the growth of errors as the integration proceeds. In addition, the method is shown to possess A-stability, which guarantees that it can effectively handle stiff problems without imposing restrictive step-size limitations. These stability properties provide a strong mathematical foundation for the reliability of the method when applied to stiff systems.

The performance of the proposed method was evaluated against established stiff solvers, namely

ODE15s and ODE23s, as well as a lower-order block method. The numerical results demonstrate that VSBBDF5 consistently provides superior accuracy while requiring less computational effort. In particular, the method achieves smaller errors with fewer integration steps and reduced function evaluations, leading to lower overall CPU time. These outcomes confirm the effectiveness of combining higher-order accuracy with adaptive step-size control and block formulation.

Furthermore, the method exhibits strong scalability as the tolerance is refined, maintaining efficiency even under stringent accuracy requirements. This behavior highlights its robustness and suitability for solving highly stiff problems, where traditional general-purpose solvers often incur significant computational overhead.

The improved performance of VSBBDF5 can be attributed to its key features, including its higher-order formulation, efficient handling of stiffness through implicit integration, and the ability to compute multiple solution points simultaneously. These characteristics enable the method to strike an optimal balance between accuracy and computational cost.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

AI Use Statement

The authors declare that no generative AI was used in the preparation of this manuscript.

Ethical Approval and Consent to Participate

Not applicable.

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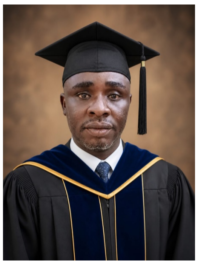

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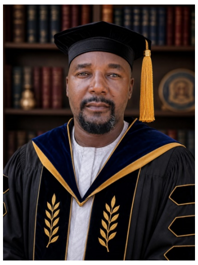

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