



An Algebraic Framework Toward the Solution of the Abel Differential Equation

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Abstract

The Abel differential equation of the first kind has no general closed-form solution, and most recent results on its rational solutions give upper bounds on their number or describe qualitative properties. This paper describes a direct algebraic procedure for finding rational solutions when the coefficients are polynomials. Writing the unknown as $y = P/Q$ with coprime polynomials P and Q , then applying the quotient rule and clearing denominators, one obtains the polynomial identity $(P'Q - PQ')Q = f_0Q^3 + f_1PQ^2 + f_2P^2Q + f_3P^3$. For prescribed degrees of P and Q , comparing coefficients gives a finite system of polynomial equations in the unknown coefficients, and rational solutions of those degrees correspond exactly to solutions of this system; an inconsistent system rules them out. The procedure is illustrated on $y' = -2y^2 + xy^3$, where it recovers $y = 1/x$ and shows that this is the only non-zero solution of linear-fractional form. The same reduction is extended to equations $y' = \sum_{k=0}^N f_k(x)y^k$ with necessary degree conditions. The method is suited to computer algebra systems and complements the existing bound-based results.

Keywords: Abel differential equation, rational solutions, polynomial identity, algebraic compatibility, nonlinear differential equations.

1 Introduction

Many nonlinear ordinary differential equations cannot be integrated in closed form, and a common alternative is to recast them as algebraic problems. Rational solutions of algebraic ordinary differential equations, for example, can be computed by algorithms from computer algebra [1], and a coefficient-comparison approach to polynomial solutions of first-order equations with polynomial right-hand sides has been developed by Behloul and Cheng [2]. Among first-order equations, the Abel equation of the first kind,

$$\frac{dy}{dx} = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3, \quad f_3 \neq 0, \quad (1)$$

is a standard test case. It appears in the study of limit cycles of planar polynomial systems and Hilbert's 16th problem [3] and in models from mathematical physics, such as dissipative cosmologies [4]. Equation (1) rarely admits closed-form solutions; the known integrable cases are collected in handbooks and classification studies [5–7].



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Recent work by Bravo, Calderón, Ojeda and co-authors has bounded the number of rational solutions and rational limit cycles of Abel equations with polynomial or trigonometric coefficients [8–11]; these results are reviewed in Section 2. They tell us how many rational solutions can exist, but not how to compute them for a given equation. Algorithms for rational solutions do exist for first-order algebraic ODEs [1, 12]; they target rational *general* solutions and rely on parametrising algebraic curves. Here we take a more elementary route aimed at *particular* rational solutions of (1) with polynomial coefficients, extending the coefficient-comparison strategy of [2] from *polynomial* solutions to *rational* solutions of the form $y = P/Q$.

The unknown is written as a quotient $y = P/Q$ of polynomials. The quotient rule and multiplication by Q^3 turn (1) into a polynomial identity, and comparison of coefficients reduces it to a finite system of polynomial equations in the coefficients of P and Q . This undetermined-coefficient reduction is elementary, but stating it carefully gives an exact characterisation (Theorems 1 and 2), a simple non-existence test (Corollary 2), and an extension to equations of any degree in y (Theorem 4).

The aim of this work is therefore to characterise rational solutions of the Abel equation of the first kind through a finite algebraic system and to give a procedure that computes them without integration.

The paper is organised as follows. Section 2 reviews existing approaches. Section 3 describes the method. Section 4 states and proves the main results, gives an example and the generalisation, and discusses the method. Section 5 concludes.

2 Related Work

2.1 Existing approaches to rational solutions

Bravo et al. [8] studied rational solutions of the polynomial Abel equation

$$x' = A(t)x^3 + B(t)x^2, \quad A, B \in \mathbb{C}[t], \quad (2)$$

by combining algebraic geometry with Darboux first integrals. They proved that such an equation has at most two rational solutions when $\deg A$ is even or $\deg B$ is large enough, gave a binomial bound otherwise, and showed that having many rational solutions forces the existence of a Darboux first integral. The result is structural: it bounds the solutions but does not construct them.

Bravo, Calderón and Ojeda [9] then studied rational limit cycles of (2) when A and B are real trigonometric polynomials, showing that their number is at most $\deg A + 1$. Calderón [10] added a linear term, $x' = A(t)x^3 + B(t)x^2 + C(t)x$, and bounded both the rational solutions (polynomial coefficients) and the rational limit cycles (trigonometric coefficients). Most recently, Calderón and Ojeda [11] considered three-term generalised Abel equations $x' = A_3(t)x^{n_3} + A_2(t)x^{n_2} + A_1(t)x^{n_1}$, proved that every non-constant rational solution has the form $x = 1/p(t)$, and used the Newton-Puiseux polygon at infinity to bound the possible denominator degrees and the number of solutions. Since polynomial solutions form only a restricted subclass of all possible closed-form solutions, the search for rational solutions of the form $y = P/Q$ with $Q \not\equiv 1$ provides a strictly broader and more flexible framework.

2.2 Limitations of existing approaches

The works above concentrate on upper bounds, invariant algebraic curves, Darboux first integrals and limit cycles. For a given equation they do not supply a routine that returns the rational solutions themselves, and the arguments are tailored to each class of equations considered. The method below addresses this narrower, computational question.

3 Methodology

Instead of studying qualitative properties or integrability conditions, we represent the unknown function as

$$y(x) = \frac{P(x)}{Q(x)}, \quad P, Q \in \mathbb{R}[x], \quad (3)$$

$Q(x) \neq 0$ on an interval $I \subset \mathbb{R}$.

Applying the quotient rule, substituting into (1) and multiplying by Q^3 clears all denominators and gives identity (4) below. When $f_0, \dots, f_3$ are polynomials, both sides of (4) are polynomials in x ; comparing coefficients of equal powers then yields a finite algebraic system whose unknowns are the coefficients of P and Q . Under this assumption, the differential problem becomes a finite-dimensional algebraic one.

The full procedure is summarised in Algorithm 1 (Section 4.3). It does not start from invariant curves, Darboux first integrals, Newton-Puiseux polygons or a qualitative analysis; it only uses the polynomial structure of (1).

4 Results and Discussion

4.1 Algebraic characterisation of rational solutions

Consider the Abel equation (1), where f_0, f_1, f_2, f_3 are continuous on an interval $I \subset \mathbb{R}$.

Theorem 1 (Algebraic characterisation). *Let $y = P/Q$ as in (3), with $\gcd(P, Q) = 1$. Then y is a solution of (1) on I if and only if*

$$(P'Q - PQ')Q = f_0Q^3 + f_1PQ^2 + f_2P^2Q + f_3P^3 \quad \text{on } I. \quad (4)$$

Proof. By the quotient rule,

$$\frac{dy}{dx} = \frac{P'Q - PQ'}{Q^2}. \quad (5)$$

Substituting (3) and (5) into (1) and multiplying by Q^3 gives (4). Conversely, if (4) holds on I , then, since $Q \neq 0$ on I , dividing by Q^3 and using (5) recovers (1). $\square$

Remark 1. The coprimality condition is not used in the proof; it fixes the representation of y up to a common constant factor (see Section 4.2).

4.2 Finite algebraic compatibility criterion

Theorem 1 holds for arbitrary continuous coefficients. To obtain a finite coefficient system, we now assume that $f_0, f_1, f_2, f_3 \in \mathbb{R}[x]$, so that (4) is a polynomial identity. Write

$$P(x) = \sum_{i=0}^m a_i x^i, \quad Q(x) = \sum_{j=0}^n b_j x^j, \quad (6)$$

with $a_i, b_j \in \mathbb{R}$, $m, n \in \{0, 1, 2, \dots\}$, and the normalisation $b_n = 1$. Substituting (6) into (4) gives a polynomial identity

$$\sum_{k=0}^K A_k x^k = \sum_{k=0}^K B_k x^k, \quad (7)$$

where K is the larger of the degrees of the two sides and every A_k, B_k is a polynomial expression in $a_0, \dots, a_m, b_0, \dots, b_{n-1}$ and the coefficients of the f_i .

Theorem 2 (Finite compatibility criterion). *Assume $f_0, f_1, f_2, f_3 \in \mathbb{R}[x]$ and fix m, n . Let P, Q be as in (6) with $\gcd(P, Q) = 1$ and $Q \neq 0$ on I . Then $y = P/Q$ solves (1) on I if and only if*

$$A_k = B_k, \quad k = 0, \dots, K. \quad (8)$$

Proof. Equation (7) is an identity between two polynomials, and two polynomials agree on an interval if and only if their coefficients coincide. Hence (4) holds on I exactly when (8) holds, and the claim follows from Theorem 1. $\square$

Corollary 1. *For prescribed degrees (m, n) , finding the rational solutions of (1) with $\deg P \leq m$ and $\deg Q = n$ is equivalent to solving the finite polynomial system (8).*

Proof. This is Theorem 2 applied to the fixed degrees (m, n) . $\square$

Corollary 2. *If system (8) is inconsistent, then (1) has no rational solution with $\deg P \leq m$ and $\deg Q = n$.*

Proof. If such a solution existed, Theorem 2 would make (8) consistent. $\square$

4.3 Constructive algorithm

Theorems 1 and 2 lead directly to Algorithm 1. No integration is needed: once the degrees are fixed, everything reduces to solving a polynomial system.

Algorithm 1: Rational solutions of the Abel equation for prescribed degrees

Data: polynomial coefficients $f_0, \dots, f_3 \in \mathbb{R}[x]$;
degrees m, n

Result: all rational solutions $y = P/Q$ with
 $\deg P \leq m, \deg Q = n$

write P and Q as in (6) with $b_n = 1$;

substitute into (4), i.e. multiply the equation by Q^3 ;

expand both sides as polynomials in x ;

equate coefficients of equal powers to obtain (8);

solve (8) for $a_0, \dots, a_m, b_0, \dots, b_{n-1}$;

if (8) is inconsistent **then**

 report that no solution of these degrees exists
 (Corollary 2);

else

 discard solutions with $\gcd(P, Q) \neq 1$; form

$y = P/Q$ and verify it by substitution into (1);

end

4.4 Application to linear-fractional solutions

To show how the method works, take $m = n = 1$:

$$P(x) = a_0 + a_1 x, \quad Q(x) = b_0 + b_1 x, \quad (9)$$

where a_0, a_1, b_0, b_1 are real constants and $Q(x) \neq 0$ on I . The quotient rule gives

$$\frac{dy}{dx} = \frac{a_1 b_0 - a_0 b_1}{(b_0 + b_1 x)^2}. \quad (10)$$

Substituting (9) and (10) into (1) yields

$$\frac{a_1 b_0 - a_0 b_1}{(b_0 + b_1 x)^2} = f_0 + f_1 \frac{P}{Q} + f_2 \frac{P^2}{Q^2} + f_3 \frac{P^3}{Q^3}, \quad (11)$$

and multiplication by $Q^3(x)$ gives the polynomial identity

$$(a_1 b_0 - a_0 b_1) Q = f_0 Q^3 + f_1 P Q^2 + f_2 P^2 Q + f_3 P^3. \quad (12)$$

Example 1. Consider

$$\frac{dy}{dx} = -2y^2 + x y^3, \quad (13)$$

that is, $f_0 = f_1 = 0$, $f_2 = -2$, $f_3 = x$, on $I = (0, \infty)$. With (9), identity (12) becomes a polynomial of degree four in x . Equating its coefficients to zero gives

$$\begin{aligned} x^4: & a_1^3 = 0, \\ x^3: & a_1^2(3a_0 - 2b_1) = 0, \\ x^2: & a_1(3a_0^2 - 4a_0 b_1 - 2a_1 b_0) = 0, \\ x^1: & a_0^3 - 2a_0^2 b_1 - 4a_0 a_1 b_0 + a_0 b_1^2 - a_1 b_0 b_1 = 0, \\ x^0: & b_0(2a_0^2 - a_0 b_1 + a_1 b_0) = 0. \end{aligned}$$

The first equation forces $a_1 = 0$; the x^1 equation then reads $a_0(a_0 - b_1)^2 = 0$. If $a_0 = 0$ then $y \equiv 0$. Otherwise $b_1 = a_0$, and the x^0 equation gives $a_0^2 b_0 = 0$, so $b_0 = 0$. Hence the only non-zero solution of the form (9) is

$$y(x) = \frac{a_0}{a_0 x} = \frac{1}{x}$$

(with the normalisation $b_1 = 1$ of Section 4.2, $a_0 = 1$ and $b_0 = 0$), which indeed satisfies (13): $y' = -1/x^2$ and $-2y^2 + xy^3 = -2/x^2 + 1/x^2 = -1/x^2$.

4.5 Structural properties of the algebraic system

The next result holds for any choice of degrees.

Theorem 3. For any prescribed degrees $m, n \in \{0, 1, 2, \dots\}$, the algebraic system generated by (4) is finite.

Proof. With m and n fixed, every product in (4) is a polynomial of bounded degree, so only finitely many powers of x occur. Comparing coefficients therefore gives finitely many equations in the finitely many unknowns $a_0, \dots, a_m, b_0, \dots, b_{n-1}$. $\square$

Remark 2. The derivation of (4) uses only the representation $y = P/Q$ and the quotient rule; neither ingredient depends on the degrees of P and Q . Raising the degrees increases the number of unknowns and equations but leaves the procedure unchanged.

Theorem 3 and Remark 2 allow the same construction to be carried over to more general polynomial equations, as done in Section 4.6.

4.6 Generalisation to polynomial differential equations

The construction is not specific to the Abel equation; it depends only on the polynomial dependence of the right-hand side on y . Consider

$$\frac{dy}{dx} = \sum_{k=0}^N f_k(x) y^k, \quad N \geq 1, \quad (14)$$

where each f_k is a polynomial in x , and let $y = P/Q$ with $P, Q \in \mathbb{R}[x]$ and $Q \neq 0$ on I . Substitution gives

$$\frac{P'(x)Q(x) - P(x)Q'(x)}{Q(x)^2} = \sum_{k=0}^N f_k(x) \left(\frac{P(x)}{Q(x)} \right)^k. \quad (15)$$

Multiplying by $Q(x)^M$, where $M = \max\{N, 2\}$, removes all denominators and gives

$$(P'Q - PQ')Q^{M-2} = \sum_{k=0}^N f_k P^k Q^{M-k}. \quad (16)$$

Theorem 4. Let (14) have polynomial coefficients f_k and degree $N \geq 1$ in y . For $P, Q \in \mathbb{R}[x]$ with $Q \neq 0$ on I , set $M = \max\{N, 2\}$. Then $y = P/Q$ solves (14) on I if and only if P and Q satisfy (16).

Proof. Substitute $y = P/Q$ into (14), apply the quotient rule and multiply by Q^M ; since $k \leq N \leq M$ and $M \geq 2$, all exponents in (16) are non-negative. Because $Q \neq 0$ on I , the steps can be reversed, which proves the equivalence. $\square$

Degree restrictions. Let $m = \deg P$, $n = \deg Q$, and $d_k = \deg f_k$ for the non-zero f_k . The left-hand side of (16) has degree at most $\deg(P'Q - PQ') + (M - 2)n \leq m + (M - 1)n - 1$ when $P'Q - PQ' \neq 0$. The k -th non-zero term on the right has degree at most $d_k + km + (M - k)n$. If both sides are non-zero, their actual degrees must agree after possible cancellation of leading terms; if the left-hand side vanishes identically, so must the right-hand side. These are necessary conditions on (m, n) , not sufficient ones, and they are not available if some f_k is not a polynomial.

Corollary 3. *The Abel equation (1) is the case $N = 3$ of (14), for which $M = 3$ and (16) reduces to (4). The method therefore applies in the same way to every equation of the form (14).*

4.7 Discussion

The method replaces the search for rational solutions of (1) by a compatibility question for a finite algebraic system. After identity (4) has been written down, the differential equation itself is no longer used: what remains is a system of polynomial equations in the coefficients of P and Q . The algorithm is the same for every choice of degrees; only the size of the system grows, and with it the cost of solving it.

Every step consists of polynomial manipulations, so the procedure is easy to implement in computer algebra systems such as Maple, Mathematica, SageMath or SymPy, and the resulting systems can be solved with elimination or Gröbner-basis techniques [13]. Section 4.6 shows that the same reduction applies to first-order equations of any degree N in the dependent variable.

The approach also has clear limits. The degrees (m, n) must be chosen in advance, and the degree restrictions above narrow the choice but do not bound it in general. Combining the present procedure with the bounds of [8, 10, 11], or with the denominator form $x = 1/p(t)$ established in [11], could remove this limitation. Solving the polynomial system can also become expensive as the degrees grow.

5 Conclusion

We have described an algebraic method for computing rational solutions of the Abel equation of the first kind with polynomial coefficients. Writing $y = P/Q$ turns the equation into the polynomial identity (4), and for prescribed degrees the rational solutions correspond exactly to the solutions of a finite coefficient system; inconsistency of that system excludes solutions of those degrees. The example $y' = -2y^2 + xy^3$ shows that the procedure recovers the solution $y = 1/x$ and excludes any other non-zero linear-fractional solution. The same reduction holds for first-order equations of any degree in y . Future work could combine the method with known degree bounds so that the degree search terminates, extend it to algebraic (non-rational) solutions, and apply it to Abel equations arising in physical models.

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