RESEARCH ARTICLE

Fixed/Predefined-Time Projective Synchronization of Delayed Discontinuous Fuzzy Neural Networks via Adaptive Aperiodically Switching Strategy

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Abstract

In this article, the issues of fixed-time projective synchronization (FTPS) and predefined-time projective synchronization (PTPS) in fuzzy neural networks (FNNs) with discontinuous activations and mixed-time delays are addressed by utilizing an adaptive aperiodically switching strategy. First of all, using the tool of Lyapunov function theory, the fixed-time stabilization (FS) in such FNNs is examined. Next, by developing suitable adaptive aperiodically switching strategy controllers, novel criteria for achieving FTPS and PTPS are established within such FNNs. Unlike recent works, in this paper, aperiodically switching control and adaptive control are employed to synchronize fuzzy neural networks (FNNs) within fixed and predefined time. Furthermore, depending on the selection of different projective factors, the results of projective synchronization in this paper can include results such as complete synchronization, anti-synchronization and fixed/predefined-time synchronization. Ultimately, illustrative simulations are conducted to support the efficacy of outcomes gained in this study.

Keywords: aperiodically switching control, adaptive control, fixed-time projective synchronization, predefined-time projective synchronization.

1 Introduction

Fuzzy neural networks (FNNs) amalgamate fuzzy theory with neural networks (NNs), thereby combining the advantages of both approaches [1, 2]. These systems possess key capabilities such as learning, association, recognition, and information processing. Due to these properties, FNNs have found wide practical applications, including image processing [3], pattern recognition [4], parallel processing [5], image encryption [6] and so on. In recent years, numerous noteworthy and intriguing findings have been accomplished, such as [7–11].

Synchronization, an important dynamic behavior of NNs, has garnered considerable attention from numerous researchers. Recently, numerous synchronization methodologies tailored to FNNs have been investigated, such as exponential synchronization [12], finite-time synchronization [13], projective synchronization [14], anti-synchronization [15], and general decay synchronization [16] among others. Among these approaches, fixed-time

Submitted: 02 July 2025
Accepted: 03 August 2025
Published: 28 August 2025

Vol. 1, **No.** 1, 2025.
10.62762/JNDA.2025.649261

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Citation
Gui, S., & Wang, Z. (2025). Fixed/Predefined-Time Projective Synchronization of Delayed Discontinuous Fuzzy Neural Networks via Adaptive Aperiodically Switching Strategy. *Journal of Nonlinear Dynamics and Applications*, 1(1), 10–25.

synchronization(FTS) has drawn significant scholarly attention due to its independence from system initial conditions. Recently, substantial research efforts have been devoted to investigating fixed-time synchronization in FNNs, see [17–19]. Kong *et al.* [17] researched FTS of the discontinuous second-order FNNs, which features uncertain parameters. Zheng *et al.* [18] concentrated on FTS for discontinuous fuzzy competitive NNs. Zheng *et al.* [19] explored the FTS of memristive delayed FNNs.

However, the value of system parameters and control strategies can dictate the settling time (ST) of FTS. In real-world applications, obtaining accurate parameters becomes particularly challenging when a chaotic system is subjected to perturbations and external noise. Therefore, Sánchez-Torres *et al.* [20**?**]. Then, predefined-time synchronization (PTS) was studied. It has the advantage of being able to synchronize at any predefined-time and be completely independent of the initial states, controller parameters and system. There is no doubt that such a stellar performance has attracted more and more attention, see [22–24] and the references therein. In [22], the authors researched FTS/PTS of FNNs with random perturbations. In [23], Han *et al.* explored discontinuous fuzzy cellular NNs for FTS/PTS problems. In [24], the FTS and PTS problems of time-varying delayed fuzzy memristive neural networks were explored.

It is essential to point out that the above studies do not link fixed-time and predefined-time with projective synchronization. On the one hand, projective synchronization control facilitates quicker communication rates and leverages its attributes for secure communication [25, 26]. Projective synchronization, on the other hand, is more versatile as complete synchronization [47], anti-synchronization [15], and stabilization [40] can be derived as particular instances when the projective factor is 1, -1, and 0, respectively [14, 27]. So we'll consider FTPS/PTPS for FNNs.

To attain synchronization within the drive-response system, a variety of control strategies have been introduced and widely explored, including quantized output control [18, 28], adaptive control [29, 30], intermittent control [31–33, 50], sliding mode control [21, 34], impulsive control [14, 35], pinning control [36], and so on. The aforementioned techniques can be classified into two categories: continuous control methods and discontinuous control methods. In comparison with the continuous control strategies, the discontinuous control methods are not only more practical, but also capable of reducing costs and the amount of transmitted information [37]. Among these methods, aperiodically intermittent control, as a type of discontinuous control strategy, has garnered significant interest from numerous researchers due to its distinct advantages. At present, the related researches on the synchronization of FNNs under the aperiodically semi-intermittent strategy mainly focus on finite/fixed-time synchronization [37–41, 48]. It is significant to observe that fixed/predefined-time projective synchronization of FNNs under aperiodically strategies has rarely been considered in previous studies.

Moreover, adaptive control refers to a type of control where the feedback gain varies over time and can be automatically adjusted according to the design's update rules, thereby reducing control costs to a certain extent. Therefore, this paper investigates FNNs by combining adaptive control with aperiodically switching control.

Informed by the insights gleaned from the foregoing analysis, this paper will center on achieving fixed-time projective synchronization (FTPS) and predefined-time projective synchronization (PTPS) in delayed FNNs via an adaptive aperiodically switching strategy. The principal contributions can be summarized as follows:

(1) Unlike literature [29, 30, 37, 39, 40], which only considered a single control strategy research system, this paper will simultaneously combine the aperiodically switching control with adaptive control to research FTPS/PTPS of FNNs.

(2) The outcomes of the projective synchronization we investigated will vary with the projective factor. When the projective factor is -1, 0, 1, it can obtain complete synchronization [47], stabilization [40] and anti-synchronization [15]. These results are all our special cases.

(3)Unlike these studies [13, 37, 38] that primarily focus on finite-time synchronization of FNNs, this article not only investigates FTS, which removes the impact of the system's initial value on the ST, but also explores PTS, thereby eliminating the influence of both system and controller parameters.

The other parts are: Necessary preliminaries and aas are offered in Section 1. Some criteria regarding FTPS and PTPS of FNNs are derived in Section 2. In Section 3, the simulation examples are showed. Conclusions are summarized in Section 4.

Notations: $\mathbf{R}^\gamma$ is γ-dimensional Euclidean space. For a given vector $\|\alpha\| = (\alpha_1, \alpha_2, \cdots, \alpha_\gamma)^T \in \mathbf{R}^\gamma$, $\|\alpha\|_1 = \sum_l^\gamma |\alpha_l|$ denotes $1-$ norm. $\vartheta = \max_{\beta\in\gamma}\{a_\beta, \varepsilon_\beta\}$. $\mathcal{C}([-\vartheta, 0], \mathbf{R}^\gamma)$ means the space of all continuous functions $f : [-\vartheta, 0] \to \mathbf{R}^\gamma$. The left and right derivative of $\Gamma(\cdot)$ at point Ξ are expressed as $\Gamma(\Xi^-)$, $\Gamma(\Xi^+)$, respectively. $\mathrm{K}[\Im]$ is the convex closure of set $\Im$. $\mathcal{M} = \{1, 2, 3, ...\gamma\}$, $\mathcal{L} = \{1, 2, 3, ...\jmath\}$, $\aleph = \{0, 1, 2, ...\}$.

2 Preliminaries

2.1 Model description

The drive system of discontinuous FNNs with mixed-time delays is:

$$\begin{aligned}\frac{\mathrm{d}x_l(t)}{\mathrm{d}t} =& - b_l x_l(t) + \sum_{\beta=1}^{\gamma} c_{l\beta} f_\beta(x_\beta(t)) + \sum_{\beta=1}^{\gamma} d_{l\beta} \\ &\times f_\beta(x_\beta(t - a_\beta(t))) + \sum_{\beta=1}^{\gamma} \varsigma_{l\beta} \\ &\times \int_{t-\varepsilon_\beta(t)}^{t} f_\beta(x_\beta(s))\mathrm{d}s + \bigwedge_{\beta=1}^{\gamma} \rho_{l\beta} \\ &\times f_\beta(x_\beta(t - \zeta_\beta(t))) + \bigvee_{\beta=1}^{\gamma} \varrho_{l\beta} \\ &\times f_\beta(x_\beta(t - \zeta_\beta(t))) + \mathcal{W}_l,\ l \in \gamma, t \geqslant 0,\end{aligned} \tag{1}$$

here $x_l(t)$ is the l-th neuron state. $b_l > 0$ is the neuron self-inhibition, $c_{l\beta}, d_{l\beta}, \varsigma_{l\beta}$ denote the connect weights among neurons, $f_\beta(\cdot)$ is discontinuous feedback function, time delays $\zeta_\beta(t), \varepsilon_\beta(t), a_\beta(t)$ meet $0 < a_\beta(t) \leqslant a_\beta, 0 \leqslant \varepsilon_\beta(t) \leqslant \varepsilon_\beta, 0 \leqslant \zeta_\beta(t) \leqslant \zeta_\beta$, $\rho_{l\beta}$ and $\varrho_{l\beta}$ represent elements of fuzzy feedback MIN template and fuzzy feedback MAX template, respectively. $\bigvee, \bigwedge$ denote the fuzzy OR and AND operations. $\mathcal{W}_l$ is the external input. The starting conditions are $x_l(s) = \Phi_l(s), l \in \mathcal{Z}, s \in [-\vartheta, 0]$ and $\Phi_l(s) \in \mathcal{C}([-\vartheta, 0], \mathbf{R})$.

Here, the response system is:

$$\begin{aligned}\frac{\mathrm{d}y_l(t)}{\mathrm{d}t} =& - b_l y_l(t) + \sum_{\beta=1}^{\gamma} c_{l\beta} f_\beta(y_\beta(t)) + \sum_{\beta=1}^{\gamma} d_{l\beta} \\ &\times f_\beta(y_\beta(t - a_\beta(t))) + \sum_{\beta=1}^{\gamma} \varsigma_{l\beta} \\ &\times \int_{t-\varepsilon_\beta(t)}^{t} f_\beta(y_\beta(s))\mathrm{d}s + \bigwedge_{\beta=1}^{\gamma} \rho_{l\beta} \\ &\times f_\beta(y_\beta(t - \zeta_\beta(t))) + \bigvee_{\beta=1}^{\gamma} \varrho_{l\beta} \\ &\times f_\beta(y_\beta(t - \zeta_\beta(t))) + \mathcal{W}_l \\ &+ \mathcal{U}_l(t),\ l \in \gamma, t \geqslant 0,\end{aligned} \tag{2}$$

where $\mathcal{U}_l(t)$ denotes controllers. The starting values of FNNs (2) are $y_l(s) = \Psi_l(s), l \in \gamma, s \in [-\vartheta, 0]$ and $\Psi_l(s) \in \mathcal{C}([-\vartheta, 0], \mathbf{R})$.

2.2 Difinitions and lemmas

The following are some definitions, assumptions, and lemmas that will be used.

A 1: $f_\beta(\cdot) \in \mathcal{C}(\mathbf{R}\backslash\mathbb{B}_\beta, \mathbf{R})$, the set $\mathbb{B}_\beta$ contains a finite number of discontinuous points $\wp_\beta^i$. The right limit and left limit of $f_\beta(\cdot)$ are $f_\beta(\wp^{i+})$, $f_\beta(\wp^{i-})$, respectively, $\beta \in \mathcal{M}, i \in \mathcal{L}$.

A 2: For $\forall \wp_\beta \in K[f_\beta(\ddagger)], \forall \Im_\beta \in K[f_\beta(\dagger)]$, there exit nonnegative constants $\Delta_\beta, \mathscr{A}_\beta, \mathscr{B}_\beta$ such that $|f_\beta(\cdot)| \leq \Delta_\beta$, and

$$\sup_{\wp_\beta \in K[f_\beta(\ddagger)], \Im_\beta \in K[f_\beta(\dagger)]} |\wp_\beta - \Im_\beta| \leqslant \mathscr{A}_\beta |\ddagger - \dagger| + \mathscr{B}_\beta$$

where $\dagger, \ddagger \in \mathbf{R}$, and

$$\begin{aligned}K[f_\beta(\ddagger)] =& \Big[\min\Big\{f_\beta(\ddagger^-), f_\beta(\ddagger^+)\Big\}, \max\Big\{f_\beta(\ddagger^-), \\ & f_\beta(\ddagger^+)\Big\}\Big], \\ K[f_\beta(\dagger)] =& \Big[\min\Big\{f_\beta(\dagger^-), f_\beta(\dagger^+)\Big\}, \max\Big\{f_\beta(\dagger^-), \\ & f_\beta(\dagger^+)\Big\}\Big].\end{aligned}$$

By adopting differential inclusion theories [42], one has:

$$\begin{aligned}\frac{\mathrm{d}x_l(t)}{\mathrm{d}t} \in& - b_l x_l(t) + \sum_{\beta=1}^{\gamma} c_{l\beta} K[f_\beta(x_\beta(t))] + \sum_{\beta=1}^{\gamma} d_{l\beta} \\ &\times K[f_\beta(x_\beta(t - a_\beta(t)))] + \sum_{\beta=1}^{\gamma} \varsigma_{l\beta} \\ &\times \int_{t-\varepsilon_\beta(t)}^{t} K[f_\beta(x_\beta(s))]\mathrm{d}s + \bigwedge_{\beta=1}^{\gamma} \rho_{l\beta} \\ &\times K[f_\beta(x_\beta(t - \zeta_\beta(t)))] + \bigvee_{\beta=1}^{\gamma} \varrho_{l\beta} \\ &\times K[f_\beta(x_\beta(t - \zeta_\beta(t)))] + \mathcal{W}_l,\ l \in \mathcal{M}, t \geqslant 0.\end{aligned} \tag{3}$$

and

$$\begin{aligned}\frac{\mathrm{d}y_l(t)}{\mathrm{d}t} \in& - b_l y_l(t) + \sum_{\beta=1}^{\gamma} c_{l\beta} K[f_\beta(y_\beta(t))] + \sum_{\beta=1}^{\gamma} d_{l\beta} \\ &\times K[f_\beta(y_\beta(t - a_\beta(t)))] + \sum_{\beta=1}^{\gamma} \varsigma_{l\beta} \\ &\times \int_{t-\varepsilon_\beta(t)}^{t} K[f_\beta(y_\beta(s))]\mathrm{d}s + \bigwedge_{\beta=1}^{\gamma} \rho_{l\beta} \\ &\times K[f_\beta(y_\beta(t - \zeta_\beta(t)))] + \bigvee_{\beta=1}^{\gamma} \varrho_{l\beta} \\ &\times K[f_\beta(y_\beta(t - \zeta_\beta(t)))] + \mathcal{W}_l \\ &+ \mathcal{U}_l(t),\ l \in \mathcal{M}, t \geqslant 0.\end{aligned} \tag{4}$$

equilvalently, one gets $\varphi_\beta(t) \in K[f_\beta(x_\beta(t))], \psi_\beta(t) \in K[f_\beta(y_\beta(t))]$,

$$\begin{aligned}\frac{\mathrm{d}x_l(t)}{\mathrm{d}t} =& -b_l x_l(t) + \sum_{\beta=1}^{\gamma} c_{l\beta}\varphi_\beta(t) + \sum_{\beta=1}^{\gamma} d_{l\beta}\varphi_\beta(t-a_\beta(t)) \\ &+ \sum_{\beta=1}^{\gamma} \varsigma_{l\beta}\int_{t-\varepsilon_\beta(t)}^{t}\varphi_\beta(s)\mathrm{d}s + \bigwedge_{\beta=1}^{\gamma}\rho_{l\beta} \\ &\times \varphi_\beta(t-\zeta_\beta(t)) + \bigvee_{\beta=1}^{\gamma}\varrho_{l\beta} \\ &\times \varphi_\beta(t-\zeta_\beta(t)) + \mathcal{W}_l,\ l\in\mathcal{M}, t\geqslant 0. \qquad (5)\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}y_l(t)}{\mathrm{d}t} =& -b_l y_l(t) + \sum_{\beta=1}^{\gamma} c_{l\beta}\psi_\beta(t) + \sum_{\beta=1}^{\gamma} d_{l\beta}\psi_\beta(t-a_\beta(t)) \\ &+ \sum_{\beta=1}^{\gamma} \varsigma_{l\beta}\int_{t-\varepsilon_\beta(t)}^{t}\psi_\beta(s)\mathrm{d}s + \bigwedge_{\beta=1}^{\gamma}\rho_{l\beta} \\ &\times \psi_\beta(t-\zeta_\beta(t)) + \bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\psi_\beta(t-\zeta_\beta(t)) \\ &+ \mathcal{W}_l + \mathcal{U}_l(t),\ l\in\mathcal{M}, t\geqslant 0. \qquad (6)\end{aligned}$$

Definition 1. Function $\widetilde{f}(t) = (\widetilde{x}_1(t), \widetilde{x}_2(t), ..., \widetilde{x}_\gamma(t))^T$ is a Filippov solution of FNNs (1) with fundamental positions $x_l(s) = \Phi_l(s), l \in \mathcal{M}, s \in [-\vartheta, 0]$ and $\Phi_l(s) \in \mathcal{C}([-\vartheta, 0], \mathbf{R})$, for $P \in [0, +\infty)$ and P is compact-interva, the absolutely continuous function $f(t)$ meets (3) or (5).

Remark 1. Under Assumptions A1 and A2, employing differential inclusion theory [42], there are FNNs (1) with a local-solution $f(t) = (x_1(t), x_2(t), ..., x_\gamma(t))^T$ at least with starting values $x_l(s) = \Phi_l(s), l \in \mathcal{M}, s \in [-\vartheta, 0]$ and $\Phi_l(s) \in \mathcal{C}([-\vartheta, 0], \mathbf{R})$.

2.3 Error systems of FNNs (1) and (2)

We choose to represent the projective factor with θ, $\theta \in \mathbb{R}$. Then, the error state $e_l(t) = y_l(t) - \theta x_l(t)$, one gets

$$\begin{aligned}\frac{\mathrm{d}e_l(t)}{\mathrm{d}t} =& -b_l e_l(t) + \sum_{\beta=1}^{\gamma} c_{l\beta}[\psi_\beta(t) - \theta\varphi_\beta(t)] + \sum_{\beta=1}^{\gamma} d_{l\beta} \\ &\times [\psi_\beta(t-a_\beta(t)) - \theta\varphi_\beta(t-a_\beta(t))] \\ &+ \sum_{\beta=1}^{\gamma}\varsigma_{l\beta}\int_{t-\varepsilon_\beta(t)}^{t}[\psi_\beta(s) - \theta\varphi_\beta(s)]\mathrm{d}s + \bigwedge_{\beta=1}^{\gamma}\rho_{l\beta} \\ &\times \psi_\beta(t-\zeta_\beta(t)) - \bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\theta\varphi_\beta(t-\zeta_\beta(t)) \\ &+ \bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\psi_\beta(t-\zeta_\beta(t)) - \bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\theta\varphi_\beta \\ &\times (t-\zeta_\beta(t)) + (1-\theta)\mathcal{W}_l \\ &+ \mathcal{U}_l(t),\ l\in\mathcal{M}, t\geqslant 0, \qquad (7)\end{aligned}$$

Definition 2 [43]. The system (7) achieves fixed-time stability, meaning that systems (1) and (2) get FTPS. If there exists a constant $\mathbf{T}_{\max}$ and $\mathbf{T}_{\max} \geqslant \mathbf{T}(e(0)) > 0$ which is called ST, such that $\lim_{t\to\mathbf{T}_{\max}} \|e_l(t)\|_1 = 0$, $\|e_l(t)\|_1 \equiv 0$ hold for all $t \geqslant \mathbf{T}_{\max}$.

Definition 3 [44]. If systems (1), (2) can acquire FTPS, there exists a positive constant $\mathbf{T}_p$ and $\mathbf{T}(e(0)) \leqslant \mathbf{T}_p$ for $\forall e(0) \in \mathbf{R}$, then systems (7) achieves predefined-time stability, and systems (1), (2) can accomplish PTPS, where $\mathbf{T}_p$ is the predefined-time.

Definition 4 [38]. Let $\lambda = \limsup\limits_{m\to\infty} \frac{t_{m+1}-s_m}{t_{m+1}-t_m}$, where $m \in \aleph$.

Lemma 1 [45]. Let $\chi_1, \chi_2, \cdots, \chi_\gamma \geq 0, 0 < v_1 \leq 1, v_2 \geq 1$,and such that

$$\sum_{\beta=1}^{\gamma}\chi_\beta^{v_1} \geqslant \Big(\sum_{\beta=1}^{\gamma}\chi_\beta\Big)^{v_1}, \sum_{\beta=1}^{\gamma}\chi_\beta^{v_2} \geqslant \gamma^{1-v_2}\Big(\sum_{\beta=1}^{\gamma}\chi_\beta\Big)^{v_2}.$$

Lemma 2 [23]. Let $x_\beta, y_\beta, \rho_{l\beta}, \varrho_{l\beta} \in \mathbf{R}$, f_β is discontinuous function for $l, k = 1, 2, \dots \gamma$. Then

$$\Big| \bigwedge_{\beta=1}^{\gamma} \rho_{l\beta} f_\beta(x_\beta) - \bigwedge_{\beta=1}^{\gamma} \rho_{l\beta} f_\beta(y_\beta) \Big| \leqslant \sum_{\beta=1}^{\gamma} \Big|\rho_{l\beta}\Big| \Big|f_\beta(x_\beta) - f_\beta(y_\beta)\Big|,$$

$$\Big| \bigvee_{\beta=1}^{\gamma} \varrho_{l\beta} f_\beta(x_\beta) - \bigvee_{\beta=1}^{\gamma} \varrho_{l\beta} f_\beta(y_\beta) \Big| \leqslant \sum_{\beta=1}^{\gamma} \Big|\varrho_{l\beta}\Big| \Big|f_\beta(x_\beta) - f_\beta(y_\beta)\Big|.$$

Lemma 3 [40]. Suppose the radial unbounded regular function $\mathbb{V}(\cdot)$ defined in $\mathcal{C}(\mathbf{R}^\gamma, [0, +\infty))$, and $\widetilde{P(t)} = 0 \Leftrightarrow \mathbb{V}(\widetilde{P(t)}) = 0$, and $\mathbb{V}(\cdot)$ meets

$$\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t} \leqslant \begin{cases} \hbar\mathbb{V}(t) - \mathbb{k}\mathbb{V}^{\varpi}(t) - \omega, & t_m \leqslant t < s_m, \\ 0, & s_m \leqslant t < t_{m+1}, \end{cases} \tag{8}$$

$m \in \aleph$, then, system (1) and (2) can achieve FTPS, in which $\mathbb{k} > 0, \hbar < \mathbb{k}, \omega > 0, \varpi = \text{sign}(\mathbb{V}(t) - 1) + \tau, 1 < \tau < 2$, and the ST is

$$\mathbf{T}_{\max} = \begin{cases} \frac{1}{1-\lambda}\left(\frac{1}{(2-\tau)(\mathbb{k}+\omega-\hbar)} + \frac{1}{(\mathbb{k}-\hbar)\tau}\right), & \hbar > 0, \\ \frac{1}{1-\lambda}\left(\frac{1}{(2-\tau)(\mathbb{k}+\omega)} + \frac{1}{\mathbb{k}\tau}\right), & \hbar < 0. \end{cases} \tag{9}$$

Lemma 4. Suppose the radially-unbounded $\mathcal{C}$-regular function $\mathbb{V}(\cdot)$ defined in $\mathcal{C}(\mathbf{R}^\gamma, [0, +\infty))$, and $\widetilde{P(t)} = 0 \Leftrightarrow \mathbb{V}(\widetilde{P(t)}) = 0$, and $\mathbb{V}(\cdot)$ meets

$$\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t} \leqslant \begin{cases} \frac{\mathbf{T}_{\max}}{\mathbf{T}_p}[\hbar\mathbb{V}(t) - \mathbb{k}\mathbb{V}^{\varpi}(t) - \omega], t_m \leqslant t < s_m, \\ 0, \quad s_m \leqslant t < t_{m+1}, m \in \aleph, \end{cases} \tag{10}$$

system (1) and (2) can achieve PTPS at predefined-time $\mathbf{T}_p$.

Proof. 1) If $\hbar > 0$, according to $\mathbb{V}(t)$, It can be categorized into the following two scenarios.

When $\mathbb{V}(t) \in (0, 1)$, one knows $\mathbb{V}^{\varpi} = \mathbb{V}^{\tau-1}, \mathbb{V} \leqslant \mathbb{V}^{\tau-1}$ and $-\omega \leqslant -\omega\mathbb{V}^{\tau-1}$. Then, (10) changes into

$$\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t} \leqslant \begin{cases} -\frac{\mathbf{T}_{\max}}{\mathbf{T}_p}(\mathbb{k} + \omega - \hbar)\mathbb{V}^{\tau-1}, & t_m \leqslant t < s_m, \\ 0, & s_m \leqslant t < t_{m+1}, \end{cases} \tag{11}$$

For $t \in [0, s_0)$, let $Q_0(t) = Z(t) - N_0, Z(t) = \frac{\mathbf{T}_p}{\mathbf{T}_{\max}}\mathbb{V}^{2-\tau}(t) + (\omega + \mathbb{k} - \hbar)(2 - \tau)t, N_0 = \frac{\mathbf{T}_p}{\mathbf{T}_{\max}}\mathbb{V}^{2-\tau}(0)$. Through caculation, we get $Q_0(0) = 0, \dot{Q}_0(t) \leqslant 0$. We have $Z(t) \leqslant N_0$ for all $t \in [0, s_0)$.

For $t \in [s_0, t_1)$, let $Q_0^*(t) = Z(t) - (\mathbb{k} + \omega - \hbar)(2 - \tau)(t - s_0) - N_0$, owing to $Z(s_0) \leqslant N_0$, and through caculation, one derives $\dot{Q_0}^*(t) \leqslant 0$. We attain $Z(t) \leqslant (\mathbb{k} + \omega - \hbar)(2 - \tau)(t - s_0) + N_0$ for all $t \in [s_0, t_1)$.

Next, by using a mathematical induction method similar to that in [37], one can get

$$Z(t) \leqslant \begin{cases} (\mathbb{k} + \omega - \hbar)(2 - \tau)\sum_{i=1}^{m}(t_i - s_{i-1}) \\ +N_0, \ t_m \leqslant t < s_m, \\ (\mathbb{k} + \omega - \hbar)(2 - \tau)\left(\sum_{i=1}^{m}(t_i - s_{i-1}) + t - s_m\right) \\ +N_0, \ s_m \leqslant t < t_{m+1}, \end{cases} \tag{12}$$

From **Definition 4**, it gets $0 \leqslant \lambda < 1$, using a method similar to that in [37], one deduces

$$Z(t) \leqslant (\mathbb{k} + \omega - \hbar)(2 - \tau)\lambda t + N_0, t \in [0, +\infty).$$

Consequently, for $t \in [0, +\infty)$, one has

$$\frac{\mathbf{T}_p}{\mathbf{T}_{\max}}\mathbb{V}^{(2-\tau)}(t) \leqslant \frac{\mathbf{T}_p}{\mathbf{T}_{\max}}\mathbb{V}^{(2-\tau)}(0) - (\mathbb{k} + \omega - \hbar) \times (2 - \tau)(1 - \lambda)t. \tag{13}$$

Owing to $\mathbb{V}(t) \in (0, 1)$, from (13), one can acquire

$$\mathbf{T}_1(e(0)) \leqslant \frac{\mathbf{T}_p}{\mathbf{T}_{\max}} \frac{1}{(\mathbb{k} + \omega - \hbar)(2 - \tau)(1 - \lambda)}. \tag{14}$$

When $\mathbb{V}(t) \in (1, +\infty)$, one obtains $\mathbb{V}^{\varpi}(t) = \mathbb{V}^{\tau+1}(t), \mathbb{V}(t) \leqslant \mathbb{V}^{\tau+1}(t)$. Subsequently, (10) transforms into

$$\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t} \leqslant \begin{cases} -\frac{\mathbf{T}_{\max}}{\mathbf{T}_p}(\mathbb{k} - \hbar)\mathbb{V}^{\tau+1}, & t_m \leqslant t < s_m, \\ 0, & s_m \leqslant t < t_{m+1}, \end{cases} \tag{15}$$

For $t \in [0, s_0)$, let $\overline{Q}_0(t) = \overline{Z}(t) - \overline{N}_0, \overline{Z}(t) = \frac{\mathbf{T}_p}{\mathbf{T}_{\max}}\mathbb{V}^{-\tau}(t) - (\mathbb{k} - \hbar)\tau t, \overline{N}_0 = \frac{\mathbf{T}_p}{\mathbf{T}_{\max}}\mathbb{V}^{-\tau}(0)$. Through caculation, one can obtain $\overline{Q}_0(0) = 0, \dot{\overline{Q}}_0(t) \geqslant 0$. We have $\overline{Z}(t) \geqslant \overline{N}_0$, for all $t \in [0, s_0)$, .

For $t \in [s_0, t_1)$, let $\overline{Q}_0^*(t) = \overline{Z}(t) + (\mathbb{k} - \hbar)\tau(t - s_0) - \overline{N}_0, \overline{Q}_0^*(s_0) = \overline{Z}(s_0) - \overline{N}_0 \geqslant 0$, Through caculation, one can obtain $\dot{\overline{Q}}_0^*(t) \geqslant 0$. Then, for $\forall t \in [s_0, t_1), \overline{Z}(t) \geqslant \overline{N}_0 - (\mathbb{k} - \hbar)\tau(t - s_0)$.

Next, using the method similar to that above, one can obtain

$$\overline{Z}(t) \geqslant \begin{cases} \overline{N}_0 - (\Bbbk - \hbar)\tau \sum\limits_{i=1}^{m}(t_i - s_{i-1}), t_m \leqslant t < s_m, \\ \overline{N}_0 - (\Bbbk - \hbar)\tau \Big(\sum\limits_{i=1}^{m}(t_i - s_{i-1}) \\ +t - s_m\Big), s_m \leqslant t < t_{m+1}, \end{cases} \tag{16}$$

from **Definition 4**, it gets $0 \leqslant \lambda < 1$, using a method similar to that in [40], one deduces

$$\overline{Z}(t) \geqslant \overline{N}_0 - (\Bbbk - \hbar)\tau\lambda t, t \in [0, +\infty). \tag{17}$$

Consequently, for $t \geqslant 0$, one gets

$$\frac{\mathbf{T}_p}{\mathbf{T}_{\max}}\mathbb{V}^{-\tau}(t) \geqslant \frac{\mathbf{T}_p}{\mathbf{T}_{\max}}\mathbb{V}^{-\tau}(0) + (\Bbbk - \hbar)\tau(1-\lambda)t. \tag{18}$$

Owing to $\mathbb{V}(t) \in (1, +\infty)$, one has $\mathbb{V}^{-\tau}(t) < 1$. From (18), we can get

$$\mathbf{T}_2(e(0)) \leqslant \frac{\mathbf{T}_p}{\mathbf{T}_{\max}} \frac{1}{(\Bbbk - \hbar)\tau(1-\lambda)}. \tag{19}$$

From (9), (14) and (19), we know

$$\mathbf{T}(e(0)) = \mathbf{T}_1(e(0)) + \mathbf{T}_2(e(0)) \leqslant \mathbf{T}_p. \tag{20}$$

2) If $\hbar \leqslant 0$, there are the following two situations resulting from $\mathbb{V}(t)$.

When $\mathbb{V}(t) \in (0, 1)$, one gets $\mathbb{V}^{\varpi} = \mathbb{V}^{\tau-1}$ and $-\omega \leqslant -\omega\mathbb{V}^{\tau-1}$. Then, (10) changes into

$$\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t} \leqslant \begin{cases} -\frac{\mathbf{T}_{\max}}{\mathbf{T}_p}(\Bbbk + \omega)\mathbb{V}^{\tau-1}, & t_m \leqslant t < s_m, \\ 0, & s_m \leqslant t < t_{m+1}, \end{cases} \tag{21}$$

from (21) and using processes similar to those in 1), there are

$$\mathbf{T}_1(e(0)) \leqslant \frac{\mathbf{T}_p}{\mathbf{T}_{\max}} \frac{1}{(\Bbbk + \omega)(2-\tau)(1-\lambda)}. \tag{22}$$

When $\mathbb{V}(t) > 1$, one gets $\mathbb{V}^{\varpi} = \mathbb{V}^{\tau+1}$. Then, (11) changes into

$$\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t} \leqslant \begin{cases} -\frac{\mathbf{T}_{\max}}{\mathbf{T}_p}\Bbbk\mathbb{V}^{\tau+1}, & t_m \leqslant t < s_m, \\ 0, & s_m \leqslant t < t_{m+1}, m \in \aleph, \end{cases} \tag{23}$$

from (23) and using processes similar to those in 1), there are

$$\mathbf{T}_2(e(0)) \leqslant \frac{\mathbf{T}_p}{\mathbf{T}_{\max}} \frac{1}{\Bbbk\tau(1-\lambda)}. \tag{24}$$

From (9), (22) and (24), we know

$$\mathbf{T}(e(0)) = \mathbf{T}_1(e(0)) + \mathbf{T}_2(e(0)) \leqslant \mathbf{T}_p. \tag{25}$$

Now, from (20) and (25), we find **Lemma 4** is applicable. The proof is complete.

Remark 2. Unlike references [29, 30], which only consider adaptive control strategy and also unlike references [39, 40], which only consider aperiodically switching control strategy, this paper combines these two control strategies to study fixed/predefined-time projective synchronization, integrating the advantages of both.On the one hand, the aperiodically switching control strategy, as a discontinuous control scheme, has practical and economical advantages in real-world applications and is more convenient to implement. On the other hand, the adaptive control strategy has the advantage that the control gain can automatically adjust itself on the basis of some appropriate update rules.

3 Main results

3.1 FTPS of FNNs

The aperiodically switching control with adaptive updating law $\mathcal{U}_l(t)$ in FNNs (7) is given as follows:

$$\mathcal{U}_l(t) = \begin{cases} -h_l(t)e_l(t) - \Big(\xi_l|e_l(t)|^{\varpi} + I_l\Big)\mathrm{sign}(e_l(t)) \\ + \sum\limits_{\beta=1}^{\gamma} c_{l\beta}[\theta f_\beta(x_\beta(t)) - f_\beta(\theta(x_\beta(t)))] \\ +(\theta-1)\mathcal{W}_l,\ t_m \leqslant t < s_m, \\ -h_l(t)e_l(t) - I_l\mathrm{sign}(e_l(t)) + \sum\limits_{\beta=1}^{\gamma} c_{l\beta} \\ \times[\theta f_\beta(x_\beta(t)) - f_\beta(\theta(x_\beta(t)))] \\ +(\theta-1)\mathcal{W}_l,\ s_m \leqslant t < t_{m+1}, m \in \aleph, \end{cases} \tag{26}$$

in which $\varpi = \tau + \mathrm{sign}(\mathbb{V}(t) - 1), \xi_l, I_l$ are positive numbers, $1 < \tau < 2$. The adaptive updating law fulfills

$$\begin{aligned} \dot{h}_l(t) =& (h_l(t) - \eta_l)|e_l(t)|\mathrm{sign}(h_l(t) - \eta_l) - p_l \\ & \times \mathrm{sign}(h_l(t) - \eta_l)|h_l(t) - \eta_l| - q_l \\ & \times \mathrm{sign}(h_l(t) - \eta_l)|h_l(t) - \eta_l|^{\varpi}, \end{aligned} \tag{27}$$

where $m = 0, 1, 2, \ldots, \eta_l, p_l, q_l$ are positive constants.

In accordance with differential inclusion theories [42], error system (7) with controller (26) is

$$\begin{aligned}\frac{\mathrm{d}e_l(t)}{\mathrm{d}t} =& -(b_l+h_l(t))e_l(t)+\sum_{\beta=1}^{\gamma}c_{l\beta}[\psi_\beta(t)-\phi_\beta(t)]\\&+\sum_{\beta=1}^{\gamma}d_{l\beta}[\psi_\beta(t-a_\beta(t))-\theta\varphi_\beta(t-a_\beta(t))]\\&+\sum_{\beta=1}^{\gamma}\varsigma_{l\beta}\int_{t-\varepsilon_\beta(t)}^{t}[\psi_\beta(s)-\theta\varphi_\beta(s)]\mathrm{d}s+\bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\\&\times\psi_\beta(t-\zeta_\beta(t))-\bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\theta\varphi_\beta(t-\zeta_\beta(t))\\&+\bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\psi_\beta(t-\zeta_\beta(t))-\bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\theta\varphi_\beta\\&\times(t-\zeta_\beta(t))-\Big(\xi_l|e_l(t)|^{\varpi}+I_l\Big)\\&\times\mathrm{sign}(e_l(t)),t_m\leqslant t<s_m,\\\frac{\mathrm{d}e_l(t)}{\mathrm{d}t} =& -(b_l+h_l(t))e_l(t)+\sum_{\beta=1}^{\gamma}c_{l\beta}[\psi_\beta(t)-\phi_\beta(t)]\\&+\sum_{\beta=1}^{\gamma}d_{l\beta}[\psi_\beta(t-a_\beta(t))-\theta\varphi_\beta(t-a_\beta(t))]\\&+\sum_{\beta=1}^{\gamma}\varsigma_{l\beta}\int_{t-\varepsilon_\beta(t)}^{t}[\psi_\beta(s)-\theta\varphi_\beta(s)]\mathrm{d}s+\bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\\&\times\psi_\beta(t-\zeta_\beta(t))-\bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\theta\varphi_\beta(t-\zeta_\beta(t))\\&+\bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\psi_\beta(t-\zeta_\beta(t))-\bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\theta\varphi_\beta\\&\times(t-\zeta_\beta(t))-I_l\mathrm{sign}(e_l(t)),\ s_m\leqslant t<t_{m+1},\end{aligned}\tag{28}$$

in which $\varphi_\beta(t)\in K[f_\beta(x_\beta(t))],\psi_\beta(t)\in K[f_\beta(y_\beta(t))],\phi_\beta(t)\in K[f_\beta(\theta x_\beta(t))]$.

To get FTPS results, we make the following notations.

$$\hbar=\max_{1\leqslant l\leqslant\gamma}\{-(b_l+\eta_l-\sum_{\beta=1}^{\gamma}|c_{\beta l}|\mathscr{A}_l),-p_l\},\tag{29}$$

$$\Bbbk_1=\min_{1\leqslant l\leqslant\gamma}\{\xi_l,q_l\},\Bbbk=\Bbbk_1\cdot(2\gamma)^{-\tau},\tag{30}$$

$$\begin{aligned}\omega_l=&I_l-\sum_{\beta=1}^{\gamma}\Big[|c_{l\beta}|\mathscr{B}_\beta+\Big(|d_{l\beta}|+|\varsigma_{l\beta}|\varepsilon_\beta+|\rho_{l\beta}|\\&+|\varrho_{l\beta}|\Big)(1+|\theta|)\Delta_\beta\Big].\end{aligned}$$

$$b_l+\eta_l-\sum_{\beta=1}^{\gamma}|c_{\beta l}|\mathscr{A}_l>0,\omega_l>0\tag{31}$$

Theorem 1. Suppose that **A 1**, **A 2**, $\hbar<\Bbbk$ and (31) hold, then FNNs (1) and (2) achieve FTPS with control (26), and the ST is $\mathbf{T}_{\max}$.

Proof. The nonnegative function are constructced

$$\mathbb{V}(t)=\sum_{l=1}^{\gamma}(|e_l(t)|+|h_l(t)-\eta_l|).\tag{32}$$

For $t_m\leqslant t<s_m,m\in\aleph$, owing to $\mathbb{V}(t)$ is $\mathcal{C}-regular$ function [46], one gets

$$\begin{aligned}\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t}=&\sum_{l=1}^{\gamma}\Big(\epsilon_l(t)\cdot\frac{\mathrm{d}e_l(t)}{\mathrm{d}t}+\epsilon_l^*(t)\cdot\frac{\mathrm{d}h_l(t)}{\mathrm{d}t}\Big)\\=&\sum_{l=1}^{\gamma}\Big\{\epsilon_l(t)\Big[-(b_l+h_l(t))e_l(t)+\sum_{\beta=1}^{\gamma}c_{l\beta}\\&\times[\psi_\beta(t)-\phi_\beta(t)]+\sum_{\beta=1}^{\gamma}d_{l\beta}[\psi_\beta(t-a_\beta(t))\\&-\theta\varphi_\beta(t-a_\beta(t))]+\sum_{\beta=1}^{\gamma}\varsigma_{l\beta}\\&\times\int_{t-\varepsilon_\beta(t)}^{t}[\psi_\beta(s)-\theta\varphi_\beta(s)]\mathrm{d}s+\bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\\&\times\psi_\beta(t-\zeta_\beta(t))-\bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\theta\varphi_\beta(t-\zeta_\beta(t))\\&+\bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\psi_\beta(t-\zeta_\beta(t))-\bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\theta\\&\times\varphi_\beta(t-\zeta_\beta(t))-\Big(\xi_l|e_l(t)|^{\varpi}+I_l\Big)\mathrm{sign}(e_l(t))\Big]\\&+\epsilon_l^*(t)\cdot\Big[(h_l(t)-\eta_l)|e_l(t)|\mathrm{sign}(h_l(t)-\eta_l)\\&-p_l\mathrm{sign}(h_l(t)-\eta_l)|h_l(t)-\eta_l|\\&-q_l\mathrm{sign}(h_l(t)-\eta_l)|h_l(t)-\eta_l|^{\varpi}\Big]\Big\}.\end{aligned}\tag{33}$$

in which $\epsilon_l(t)\in K[\mathrm{sign}(e_l(t))],\epsilon_l^*(t)\in K[\mathrm{sign}(h_l(t)-\eta_l)]$.

From **A 1**,**A 2**,**Lemma 3.**we gets

$$\begin{aligned}&|\psi_\beta(t)-\phi_\beta(t)|\leqslant\mathscr{A}_\beta|e_\beta(t)|+\mathscr{B}_\beta,\\&\Big|\bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\times\psi_\beta(t-\zeta_\beta(t))-\bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\times\theta\varphi_\beta(t-\zeta_\beta(t))\Big|\\\leqslant&\sum_{\beta=1}^{\gamma}|\rho_{l\beta}||\psi_\beta(t-\zeta_\beta(t)-\theta\varphi_\beta(t-\zeta_\beta(t)|\\\leqslant&\sum_{\beta=1}^{\gamma}|\rho_{l\beta}|(1+|\theta|)\Delta_\beta\end{aligned}$$

$$
\begin{aligned}
&\Big|\bigvee_{\beta=1}^{\gamma} \varrho_{l\beta}\times\psi_\beta(t-\zeta_\beta(t)) - \bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\times\theta\varphi_\beta(t-\zeta_\beta(t))\Big| \\
\leqslant &\sum_{\beta=1}^{\gamma}|\varrho_{l\beta}||\psi_\beta(t-\zeta_\beta(t)-\theta\varphi_\beta(t-\zeta_\beta(t)| \\
\leqslant &\sum_{\beta=1}^{\gamma}|\varrho_{l\beta}|(1+|\theta|)\Delta_\beta.
\end{aligned} \tag{34}
$$

then,we get

$$
\begin{aligned}
\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t} \leqslant &\sum_{l=1}^{\gamma}\Big[-(b_l+h_l(t))|e_l(t)| + \sum_{\beta=1}^{\gamma}|c_{l\beta}|(\mathscr{A}_\beta|e_\beta(t)| \\
&+\mathscr{B}_\beta) + \sum_{\beta=1}^{\gamma}|d_{l\beta}|(1+|\theta|)\Delta_\beta + \sum_{\beta=1}^{\gamma}|\varsigma_{l\beta}|\varepsilon_\beta(t) \\
&\times(1+|\theta|)\Delta_\beta + \sum_{\beta=1}^{\gamma}|\rho_{l\beta}|(1+|\theta|)\Delta_\beta \\
&+\sum_{\beta=1}^{\gamma}|\varrho_{l\beta}|(1+|\theta|)\Delta_\beta - \Big(\xi_l|e_l(t)|^{\varpi}+I_l\Big) \\
&+(h_l(t)-\eta_l)|e_l| - p_l|h_l(t)-\eta_l| \\
&-q_l|h_l(t)-\eta_l|^{\varpi}\Big] \\
\leqslant &\sum_{l=1}^{\gamma}\Big\{-(b_l+\eta_l-\sum_{\beta=1}^{\gamma}|c_{\beta l}|\mathscr{A}_l)|e_l(t)| - p_l \\
&\times|h_l(t)-\eta_l| - \xi_l|e_l(t)|^{\varpi} - q_l|h_l(t)-\eta_l(t)|^{\varpi} \\
&-\Big[I_l - \sum_{\beta=1}^{\gamma}\Big(|c_{l\beta}|\mathscr{B}_\beta + (|d_{l\beta}|+|\varsigma_{l\beta}|\varepsilon_\beta \\
&+|\rho_{l\beta}|+|\varrho_{l\beta}|)(1+|\theta|)\Delta_\beta\Big)\Big]\Big\}
\end{aligned} \tag{35}
$$

Basing on **Lemma 1**, one gets

(1) When $\mathbb{V}(t)\in(0,1)$, then $\varpi\in(0,1)$

$$
\begin{aligned}
&-\sum_{l=1}^{\gamma}[\xi_l|e_l(t)|^{\varpi}+q_l|h_l(t)-\eta_l|^{\varpi}] \\
\leqslant &-\mathbb{k}_1\Big(\sum_{l=1}^{\gamma}|e_l(t)|^{\varpi}+|h_l-\eta_l(t)|^{\varpi}\Big) \\
\leqslant &-\mathbb{k}_1\Big(\sum_{l=1}^{\gamma}|e_l(t)|+|h_l(t)-\eta_l|\Big)^{\varpi} \\
= &-\mathbb{k}_1\mathbb{V}^{\varpi}(t),
\end{aligned} \tag{36}
$$

where $\mathbb{k}_1=\min\limits_{1\leqslant l\leqslant\gamma}\{\xi_l,q_l\}$.

(2) When $\mathbb{V}(t)\in(1,+\infty)$, then $\varpi\in(2,3)$

$$
\begin{aligned}
&-\sum_{l=1}^{\gamma}[\xi_l|e_l(t)|^{\varpi}+q_l|h_l-\eta_l(t)|^{\varpi}] \\
\leqslant &-\mathbb{k}_1\Big(\sum_{l=1}^{\gamma}|e_l(t)|^{\varpi}+|h_l-\eta_l(t)|^{\varpi}\Big) \\
\leqslant &-\mathbb{k}_1(2\gamma)^{-\tau}\Big(\sum_{l=1}^{\gamma}|e_l(t)|+|h_l(t)-\eta_l|\Big)^{\varpi} \\
= &-\mathbb{k}\mathbb{V}^{\varpi}(t).
\end{aligned} \tag{37}
$$

Therefore, from (30) and (36)-(37), one gets, for $t_m\leqslant t<s_m, m\in\aleph$,

$$
-\sum_{l=1}^{\gamma}[\xi_l|e_l(t)|^{\varpi}+q_l|h_l(t)-\eta_l|^{\varpi}]\leqslant -\mathbb{k}\mathbb{V}^{\varpi}(t), \tag{38}
$$

where $\mathbb{k}=\mathbb{k}_1\cdot(2\gamma)^{-\tau}$.
According to (33)-(38), one has

$$
\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t}\leqslant \hbar\mathbb{V}(t)-\mathbb{k}\mathbb{V}^{\varpi}(t)-\omega,\ t_m\leqslant t<s_m. \tag{39}
$$

Now, for $s_m\leqslant t<t_{m+1}, m\in\aleph$, from (28)-(32), we get

$$
\begin{aligned}
\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t} \leqslant &\sum_{l=1}^{\gamma}\Big[-(b_l+h_l)|e_l(t)| + \sum_{\beta=1}^{\gamma}|c_{l\beta}|(\mathscr{A}_\beta|e_\beta(t)| \\
&+\mathscr{B}_\beta) + \sum_{\beta=1}^{\gamma}|d_{l\beta}|(1+|\theta|)\Delta_\beta + \sum_{\beta=1}^{\gamma}|\varsigma_{l\beta}|\varepsilon_\beta(t) \\
&\times(1+|\theta|)\Delta_\beta + \sum_{\beta=1}^{\gamma}|\rho_{l\beta}|(1+|\theta|)\Delta_\beta \\
&+\sum_{\beta=1}^{\gamma}|\varrho_{l\beta}|(1+|\theta|)\Delta_\beta - I_l + (h_l(t)-\eta_l)|e_l(t)| \\
&-p_l|h_l(t)-\eta_l| - q_l|h_l(t)-\eta_l|^{\varpi}\Big] \\
\leqslant &\sum_{l=1}^{\gamma}\Big\{-(b_l+\eta_l-\sum_{\beta=1}^{\gamma}|c_{\beta l}|\mathscr{A}_l)|e_l(t)| - \Big[I_l \\
&-\sum_{\beta=1}^{\gamma}\Big(|c_{l\beta}|\mathscr{B}_\beta+(|d_{l\beta}|+|\varsigma_{l\beta}|\varepsilon_\beta+|\rho_{l\beta}|+|\varrho_{l\beta}|) \\
&\times(1+|\theta|)\Delta_\beta\Big)\Big] - p_l|h_l(t)-\eta_l| - q_l|h_l(t)-\eta_l|^{\varpi}\Big\} \\
\leqslant &0
\end{aligned} \tag{40}
$$

From (39) and (40), we can know (8) holds. Through **Lemma 3**, we can know (7) is fixed-time stable; in other words, FNNs (1) and (2) achieve FTPS under the controller (26) at ST $\mathbf{T}_{\max}$.

Corollary 1. Under **Theorem 1** with controller (26), we find
(1) when $\theta = 0$, FNNs (1) achieve the FS;
(2) when $\theta = 1$, FNNs (1) and (2) attain the FTS;
(3) when $\theta = -1$, FNNs (1) and (2) fulfil fixed-time anti-synchronization, $\mathbf{T}_{\max}$ is the ST.

3.2 PTPS of FNNs

To facilitate, let

$$\Xi_l = \sum_{\beta=1}^{\gamma}\Big[|c_{l\beta}|\mathscr{B}_\beta + \Big(|d_{l\beta}| + |\varsigma_{l\beta}|\varepsilon_\beta + |\rho_{l\beta}| + |\varrho_{l\beta}|\Big) \times (1+|\theta|)\Delta_\beta\Big]. \tag{41}$$

Now consider the following controller

$$\mathcal{U}_l^*(t) = \begin{cases} -h_l(t)e_l(t) - \hbar(1-\frac{\mathbf{T}_{\max}}{\mathbf{T}_p})(|e_l(t)| \\ +|h_l(t)-\eta_l|)\text{sign}(e_l(t)) - \frac{\mathbf{T}_{\max}}{\mathbf{T}_p}\Big(\xi_l|e_l(t)|^{\varpi} \\ +q_l|h_l(t)-\eta_l|^{\varpi} + \frac{\mathbf{T}_p}{\mathbf{T}_{\max}}\Xi_l + \omega_l\Big)\text{sign}(e_l(t)) \\ + \sum_{\beta=1}^{\gamma} c_{l\beta}[\theta f_\beta(x_\beta(t)) - f_\beta(\theta(x_\beta(t)))] \\ +(\theta-1)\mathcal{W}_l,\ t_m \leqslant t < s_m, \\ -h_l(t)e_l(t) - I_l\text{sign}(e_l(t)) + \sum_{\beta=1}^{\gamma} c_{l\beta} \\ \times[\theta f_\beta(x_\beta(t)) - f_\beta(\theta(x_\beta(t)))] \\ +(\theta-1)\mathcal{W}_l,\ s_m \leqslant t < t_{m+1}, m\in\aleph, \end{cases} \tag{42}$$

the adaptive updating law of $h_l(t)$ satisfies

$$\begin{aligned}\dot{h}_l(t) =&(h_l(t)-\eta_l)|e_l(t)|\text{sign}(h_l(t)-\eta_l) - p_l \\ &\times \text{sign}(h_l(t)-\eta_l)|h_l(t)-\eta_l| - q_l \\ &\times \text{sign}(h_l(t)-\eta_l)|h_l(t)-\eta_l|^{\varpi},\end{aligned}$$

The error system is

$$\begin{aligned}\frac{\mathrm{d}e_l(t)}{\mathrm{d}t} =& -(b_l+h_l(t))e_l(t) + \sum_{\beta=1}^{\gamma} c_{l\beta}[\psi_\beta(t) - \phi_\beta(t)] \\ &+ \sum_{\beta=1}^{\gamma} d_{l\beta}[\psi_\beta(t-a_\beta(t)) - \theta\varphi_\beta(t-a_\beta(t))] \\ &+ \sum_{\beta=1}^{\gamma}\varsigma_{l\beta}\int_{t-\varepsilon_\beta(t)}^{t}[\psi_\beta(s)-\theta\varphi_\beta(s)]\mathrm{d}s + \bigwedge_{\beta=1}^{\gamma}\rho_{l\beta} \\ &\times\psi_\beta(t-\zeta_\beta(t)) - \bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\theta\varphi_\beta(t-\zeta_\beta(t)) \\ &+ \bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\psi_\beta(t-\zeta_\beta(t)) - \bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\theta\varphi_\beta \\ &\times(t-\zeta_\beta(t)) - \hbar(1-\frac{\mathbf{T}_{\max}}{\mathbf{T}_p}) \\ &\times(|e_l(t)| + |h_l(t)-\eta_l|)\text{sign}(e_l(t)) \\ &- \frac{\mathbf{T}_{\max}}{\mathbf{T}_p}\Big(\xi_l|e_l(t)|^{\varpi} + q_l|h_l(t)-\eta_l|^{\varpi} \\ &+ \frac{\mathbf{T}_p}{\mathbf{T}_{\max}}\Xi_l + \omega_l\Big)\text{sign}(e_l(t)),\ t_m\leqslant t < s_m, \\ \frac{\mathrm{d}e_l(t)}{\mathrm{d}t} =& -(b_l+h_l(t))e_l(t) + \sum_{\beta=1}^{\gamma} c_{l\beta}[\psi_\beta(t) - \phi_\beta(t)] \\ &+ \sum_{\beta=1}^{\gamma} d_{l\beta}[\psi_\beta(t-a_\beta(t)) - \theta\varphi_\beta(t-a_\beta(t))] \\ &+ \sum_{\beta=1}^{\gamma}\varsigma_{l\beta}\int_{t-\varepsilon_\beta(t)}^{t}[\psi_\beta(s)-\theta\varphi_\beta(s)]\mathrm{d}s + \bigwedge_{\beta=1}^{\gamma}\rho_{l\beta} \\ &\times\psi_\beta(t-\zeta_\beta(t)) - \bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\theta\varphi_\beta(t-\zeta_\beta(t)) \\ &+ \bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\psi_\beta(t-\zeta_\beta(t)) - \bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\theta\varphi_\beta \\ &\times(t-\zeta_\beta(t)) - I_l\text{sign}(e_l(t)),\ s_m\leqslant t<t_{m+1},\end{aligned} \tag{43}$$

Theorem 2. Under **A 1**, **A 2** and (42)-(43), if $\hbar < \mathbb{k}$ and (31) hold, then, FNNs (1) and (2) get PTPS.

Proof. Designating a following nonnegative function

$$\mathbb{V}(t) = \sum_{l=1}^{\gamma}(|e_l(t)| + |h_l(t)-\eta_l|). \tag{44}$$

For $t_m \leqslant t < s_m, m\in\aleph$, owing to $\mathbb{V}(t)$ is $\mathcal{C}-regular$

function [46], one gets

$$\begin{aligned}\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t} &= \sum_{l=1}^{\gamma}\left(\epsilon_l(t)\cdot\frac{\mathrm{d}e_l(t)}{\mathrm{d}t}+\epsilon_l^*(t)\cdot\frac{\mathrm{d}h_l(t)}{\mathrm{d}t}\right)\\ &= \sum_{l=1}^{\gamma}\Big\{\epsilon_l(t)\Big[-(b_l+h_l(t))e_l(t)+\sum_{\beta=1}^{\gamma}c_{l\beta}[\psi_\beta(t)\\ &\quad-\phi_\beta(t)]+\sum_{\beta=1}^{\gamma}d_{l\beta}[\psi_\beta(t-a_\beta(t))-\theta\\ &\quad\times\varphi_\beta(t-a_\beta(t))]+\sum_{\beta=1}^{\gamma}\varsigma_{l\beta}\int_{t-\varepsilon_\beta(t)}^{t}[\psi_\beta(s)-\\ &\quad\theta\varphi_\beta(s)]\mathrm{d}s+\bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\psi_\beta(t-\zeta_\beta(t))-\bigwedge_{\beta=1}^{\gamma}\rho_{l\beta}\theta\\ &\quad\times\varphi_\beta(t-\zeta_\beta(t))+\bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\psi_\beta(t-\zeta_\beta(t))\\ &\quad-\bigvee_{\beta=1}^{\gamma}\varrho_{l\beta}\theta\varphi_\beta(t-\zeta_\beta(t))-\hbar(1-\frac{\mathbf{T}_{\max}}{\mathbf{T}_p})\\ &\quad\times(|e_l(t)|+|h_l(t)-\eta_l|)\mathrm{sign}(e_l(t))-\frac{\mathbf{T}_{\max}}{\mathbf{T}_p}\\ &\quad\times\left(\xi_l|e_l(t)|^{\varpi}+q_l|h_l(t)-\eta_l|^{\varpi}+\frac{\mathbf{T}_p}{\mathbf{T}_{\max}}\Xi_l+\omega_l\right)\\ &\quad\times\mathrm{sign}(e_l(t))\Big]+\epsilon_l^*(t)\times\Big[(h_l(t)-\eta_l)|e_l(t)|\\ &\quad\times\mathrm{sign}(h_l(t)-\eta_l)-p_l\mathrm{sign}(h_l(t)-\eta_l)\\ &\quad\times|h_l(t)-\eta_l|-q_l\mathrm{sign}(h_l(t)-\eta_l)\\ &\quad\times|h_l(t)-\eta_l(t)|^{\varpi}\Big]\Big\}.\end{aligned}\tag{45}$$

From **A 1**,**A 2**,**Lemma 4.**we gets

$$\begin{aligned}\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t} &\leqslant\sum_{l=1}^{\gamma}\Big[-(b_l+h_l(t))|e_l(t)|+\sum_{\beta=1}^{\gamma}|c_{l\beta}|(\mathscr{A}_\beta|e_\beta(t)|\\ &\quad+\mathscr{B}_\beta)+\sum_{\beta=1}^{\gamma}|d_{l\beta}|(1+|\theta|)\Delta_\beta+\sum_{\beta=1}^{\gamma}|\varsigma_{l\beta}|\varepsilon_\beta(t)\\ &\quad\times(1+|\theta|)\Delta_\beta+\sum_{\beta=1}^{\gamma}|\rho_{l\beta}|(1+|\theta|)\Delta_\beta+\\ &\quad\sum_{\beta=1}^{\gamma}|\varrho_{l\beta}|(1+|\theta|)\Delta_\beta-\hbar(1-\frac{\mathbf{T}_{\max}}{\mathbf{T}_p})(|e_l(t)|\\ &\quad+|h_l(t)-\eta_l|)-\frac{\mathbf{T}_{\max}}{\mathbf{T}_p}\Big(\xi_l|e_l(t)|^{\varpi}+q_l\\ &\quad\times|h_l(t)-\eta_l|^{\varpi}+\frac{\mathbf{T}_p}{\mathbf{T}_{\max}}\Xi_l+\omega_l\Big)\\ &\quad+(h_l(t)-\eta_l)|e_l(t)|-p_l|h_l(t)-\eta_l|\\ &\quad-q_l|h_l(t)-\eta_l|^{\varpi}\Big]\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t} &\leqslant\sum_{l=1}^{\gamma}\Big\{-(b_l+\eta_l-\sum_{\beta=1}^{\gamma}|c_{\beta l}|\mathscr{A}_l)|e_l(t)|-p_l\\ &\quad\times|h_l(t)-\eta_l|-\hbar(1-\frac{\mathbf{T}_{\max}}{\mathbf{T}_p})(|e_l(t)|\\ &\quad+|h_l(t)-\eta_l|)-\frac{\mathbf{T}_{\max}}{\mathbf{T}_p}\Big(\xi_l|e_l(t)|^{\varpi}\\ &\quad+q_l|h_l(t)-\eta_l|^{\varpi}\Big)-\frac{\mathbf{T}_{\max}}{\mathbf{T}_p}\omega_l\Big\}\end{aligned}\tag{46}$$

Applying a proof resembling to that of **Theorem 1**, we obtain that

$$\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t}\leqslant\frac{\mathbf{T}_{\max}}{\mathbf{T}_p}(\hbar\mathbb{V}(t)-\Bbbk\mathbb{V}^{\varpi}(t)-\omega),\ t_m\leqslant t<s_m.\tag{47}$$

for $s_m\leqslant t<t_{m+1}, m\in\aleph$, from (40), (42)-(44), employing a proof analogous to that of **Theorem 1**, we get

$$\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t}\leqslant 0\tag{48}$$

From (47) and (48), we determine that (10) holds. Through employing **Lemma 4**, (7) is predefined-time stable; thus, FNNs (1) and (2) obtain PTPS with the controller (42) at ST $\mathbf{T}_{\max}$. This ends the proof.

Corollary 2. Under **Theorem 2** and controller (42), we find
(1) when $\theta=0$, FNNs (1) achieve the predefined-time stabilization;
(2) when $\theta=1$, FNNs (1) and (2) attain the PTS;
(3) when $\theta=-1$, FNNs (1) and (2) fulfil predefined-time anti-synchronization, $\mathbf{T}_{\max}$ is the ST.

Remark 3. Currently, research concerning synchronization under the aperiodically intermittent strategy primarily concentates on finite-time synchronization [37, 38] and fixed-time synchronization [39, 40]. Unlike these articles, from the perspective of practical reality, the fixed-time/predefined-time projective synchronizaton we researched are better. On the one hand, the ST of finite-time has a connection with the initial conditions, on the other hand, the ST of FTPS is correlated with system parameters and controller gains. Moreover, the ST of PTPS is not affected by starting conditions or system parameters.

Remark 4. Although references [38, 41, 48] adopt adaptive aperidically switching synchronization, we study projective synchronization. When the projective factor is -1, 0, 1, we can obtain the result

Table 1. Comparisons between previous literature and this papers

Related works	FNNs	adaptive aperiodically switching controller	FTPS	PTPS
[7–19, 22, 24, 40]	√	×	×	×
[38, 41, 48]	×	√	×	×
[27, 39]	×	×	√	√
[49]	√	×	√	×
[44, 50]	×	×	√	√
[23]	√	×	√	√
this paper	√	√	√	√

of anti-synchronization, stabilization and complete synchronization. This indicates that our results are more extensive.

Remark 5. From Table 1, we can see the differences and connections between this article and other literature.

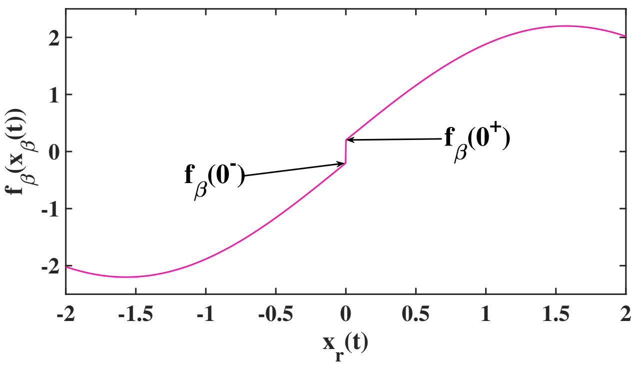

Figure 1. Discontinuous function $f_\beta(x_\beta(t))$ showed in (50).

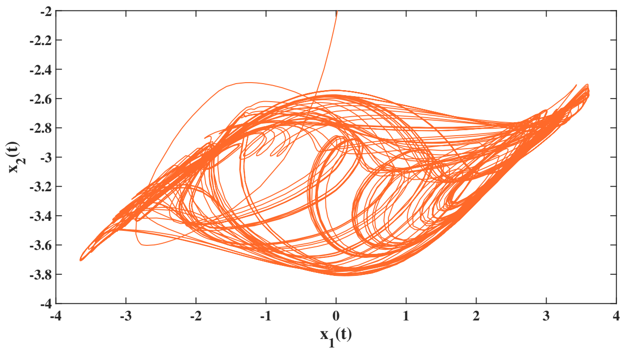

Figure 2. Phase trajectory of system (49)

4 Numerical simulations

In this section, two examples are given to show the validity of the theoretical results obtained.

Example 1. Examine a category of FNNs as follow

$$\begin{aligned}\frac{\mathrm{d}x_l(t)}{\mathrm{d}t} =& -b_l x_l(t) + \sum_{\beta=1}^{2} c_{l\beta} f_\beta(x_\beta(t)) + \sum_{\beta=1}^{2} d_{l\beta} \\ & \times f_\beta(x_\beta(t-a_\beta(t))) + \sum_{\beta=1}^{2} \varsigma_{l\beta} \\ & \times \int_{t-\varepsilon_\beta(t)}^{t} f_\beta(x_\beta(s))\mathrm{d}s + \bigwedge_{\beta=1}^{2} \rho_{l\beta} \\ & \times f_\beta(x_\beta(t-\zeta_\beta(t))) + \bigvee_{\beta=1}^{2} \varrho_{l\beta} \\ & \times f_\beta(x_\beta(t-\zeta_\beta(t))) + \mathcal{W}_l,\ l \in \mathcal{M}, t \geqslant 0,\end{aligned} \tag{49}$$

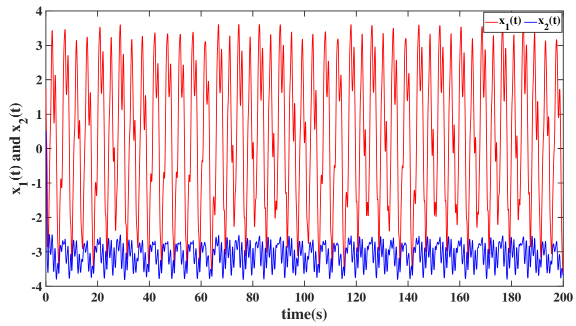

Figure 3. States $x_1(t)$ and $x_2(t)$ of FNNs (49)

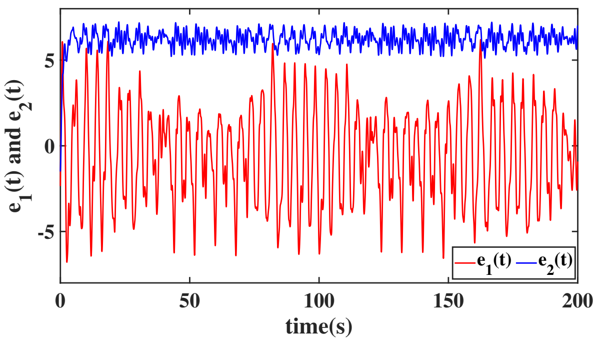

Figure 4. Error states $e_1(t), e_2(t)$ without control

in which

$$f_\beta(x_\beta(t)) = 2\sin(x_\beta(t)) + 0.2sign(x_\beta(t)). \tag{50}$$

Table 2. Parameter values of FNNs (49).

parameters	b_1	c_{11}	c_{12}	d_{11}	d_{12}	ς_{11}	ς_{12}	ρ_{11}	ρ_{12}	ϱ_{11}	ϱ_{12}
Values	1.4	1.2	-1.4	-1.6	-0.2	0.2	-0.1	-1.3	-0.4	-0.2	-1.2
parameters	b_2	c_{21}	c_{22}	d_{21}	d_{22}	ς_{21}	ς_{22}	ρ_{21}	ρ_{22}	ϱ_{21}	ϱ_{22}
Values	0.2	-0.5	2.5	0.1	-0.5	-0.1	0.2	-1.6	-1	-0.6	-1.1

where $a_\beta(t) = b_\beta(t) = \zeta_\beta(t) = \frac{exp(t)}{1+exp(t)}$, $\beta = 1, 2$. The other parameters are shown in Table 2. Figure 1 shows the graph of discontinuous function $f_\beta(x_\beta(t))$.

Under the starting values $x_1(s) = 1.75, x_2(s) = 0.5, \forall s \in [-1, 0)$, the trajectories $x_1(t)$ and $x_2(t)$ of FNNs are shown in Figures 2 and 3. Figure 4 shows error states $e_1(t), e_2(t)$ without control. Figure 5 and 6 show evolution of the adaptive updating law $h_l(t)$ with $\theta = 1, -1.2$.

Response system of FNNs (49) is

$$\begin{aligned}\frac{\mathrm{d}y_l(t)}{\mathrm{d}t} =& - b_l y_l(t) + \sum_{\beta=1}^{2} c_{l\beta} f_\beta(y_\beta(t)) + \sum_{\beta=1}^{2} d_{l\beta} \\ & \times f_\beta(y_\beta(t - a_\beta(t))) + \sum_{\beta=1}^{2} \varsigma_{l\beta} \\ & \times \int_{t-\varepsilon_\beta(t)}^{t} f_\beta(y_\beta(s))\mathrm{d}s + \bigwedge_{\beta=1}^{2} \rho_{l\beta} \\ & \times f_\beta(y_\beta(t - \zeta_\beta(t))) + \bigvee_{\beta=1}^{2} \varrho_{l\beta} \\ & \times f_\beta(y_\beta(t - \zeta_\beta(t))) + \mathcal{W}_l \\ & + \mathcal{U}_l(t),\ l \in \mathcal{M}, t \geqslant 0, \qquad (51)\end{aligned}$$

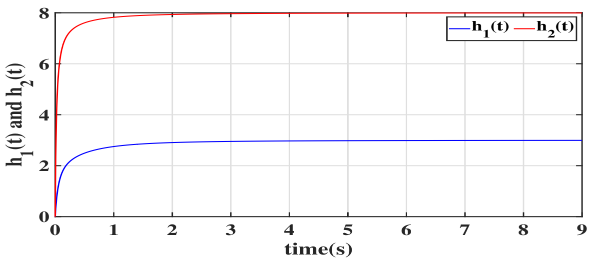

Figure 5. Evolution of the adaptive updating law $h_l(t)$ with $\theta = 1$

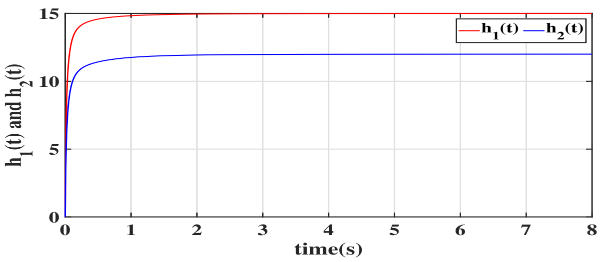

Figure 6. Evolution of the adaptive updating law $h_l(t)$ with $\theta = -1.2$

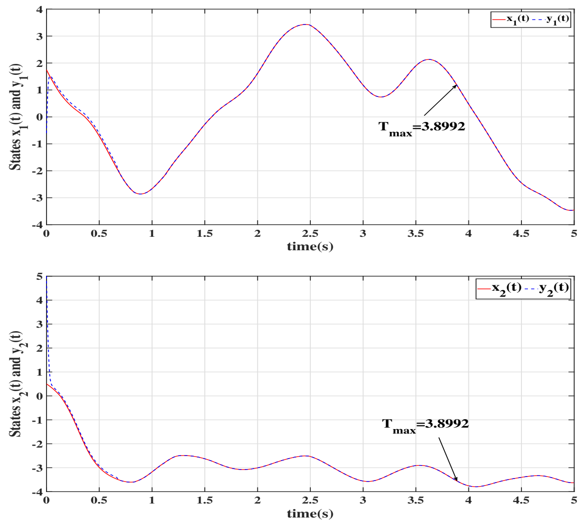

Figure 7. FTPS trajectories of states $x_1(t), y_1(t)$ and $x_2(t), y_2(t)$ with $\theta = 1$

here, The parameter values are consistent with those of FNNs (49). The starting values of (51) are $y_1(s) = -0.6, y_2(s) = 5, \forall s \in [-1, 0)$. Here, we take different projective factors $\theta = -1.2$, and $\theta = 1$.

4.1 FTPS of FNNs (49) and (51)

From FNNs (49), we get $\mathscr{A}_\beta = 2$, $\mathscr{B}_\beta = 0.2$,$\Delta_\beta = 2.2$, $a_\beta = \varepsilon_\beta = \zeta_\beta = 1$, $\mathcal{W}_l = 0.5, \beta = 1, 2$.

Case 1: Let $\theta = 1$, there, we choose $\xi_1 = \xi_2 = 4, \tau = 1.3, I_1 = 123.4, I_2 = 123.48, \eta_1 = 3, \eta_2 = 8, p_1 = 1, p_2 = 3, q_1 = q_2 = 3$. Then, we have $\Bbbk_1 = 2, \Bbbk = 0.4948, \hbar = -0.4$. We can get $\omega_1 = 100, \omega_2 = 100, \omega = 200$. The total operation time is 9 s, and the first-intermittent subintervals are

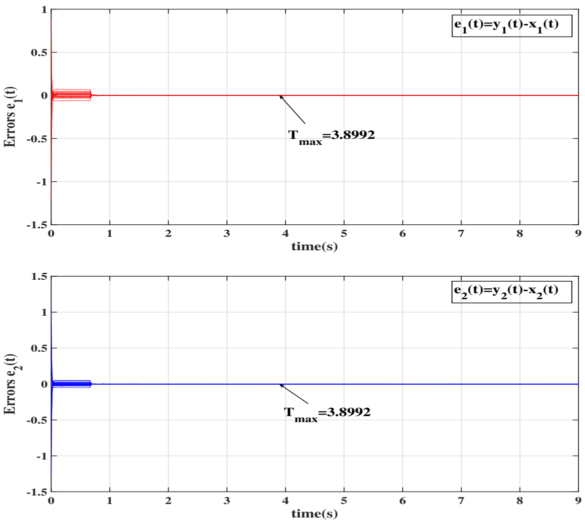

Figure 8. Error states $e_1(t), e_2(t)$ with control (26), and $\theta = 1$

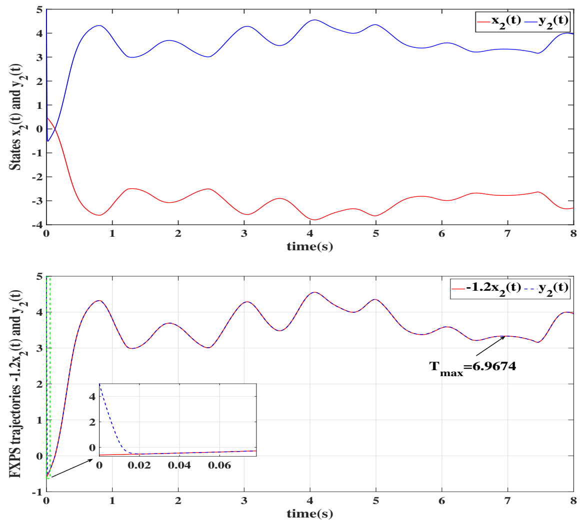

Figure 10. FTPS trajectories of states $x_2(t), y_2(t)$ with $\theta = -1.2$

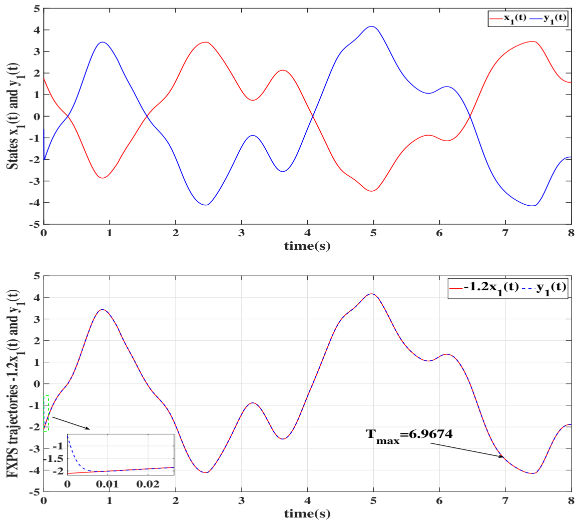

Figure 9. FTPS trajectories of states $x_1(t), y_1(t)$ with $\theta = -1.2$

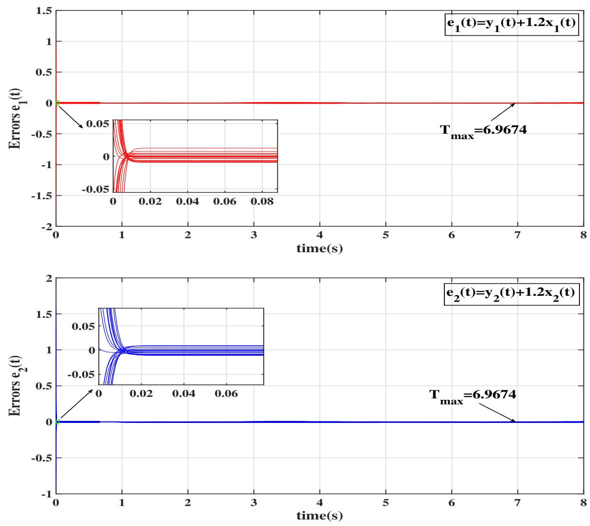

Figure 11. Error states $e_1(t), e_1(t)$ with control (26), and $\theta = -1.2$

$[0, 0.6), [1, 2), [3.6, 4.2), [5, 5.5), [6, 7.2), [8.8, 9)$, which also implies $\lambda = 8/13$.

Through simple calculations, we can determine that these parameters meet the conditions of **Theorem 1**. Therefore, the drive-response systems (49) and (51) can attain FTPS, the ST $\mathbf{T}_{\max} = 3.8992$. Then, 20 different initial values are randomly selected here. Figures 7 and 8 show the drive-response system can FTPS.

Case 2: Let $\theta = -1.2$, there, we choose $t_m = 2m, s_m = 2(m + 0.2), \lambda = 0.8, \xi_1 = \xi_2 = 4, \tau = 1.1, I_1 = 625.688, I_2 = 425.768, \eta_1 = 15, \eta_2 = 12, p_1 = 3, p_2 = 1, q_1 = q_2 = 3$. Then, we have $\Bbbk_1 = 3, \Bbbk = 0.6529, \hbar = -1$.we can get $\omega_1 = 600, \omega_2 = 400, \omega = 1000$. By calculation, one determines that the parameters meet the requirements of **Theorem 1**. So system (49) and (51) attain FTPS, the ST $\mathbf{T}_{\max} = 6.9674$. We select 20 starting values randomly. Figures 9, 10 and 11 show the drive-response system can FTPS.

4.2 PTPS of FNNs (49) and (51)

Now, we can select the parameters for the FTPS as provided above, contingent upon the variation in θ.

Let $\theta = 1$, we can gain $\Xi_1 = 23.4, \Xi_2 = 23.48$. By using controller (42), from Therorem 2, we choose predefined time $\mathbf{T}_p = 3 < \mathbf{T}_{\max} = 3.8992$. Then, FNNs (49) and (51) can derive PTPS. Figure 12 demonstrates the system (49) and (51) can realize PTPS with the controller (42) and $\theta = 1$.

Let $\theta = -1.2$, we can acquire $\Xi_1 = 25.688, \Xi_2 = 25.768$.

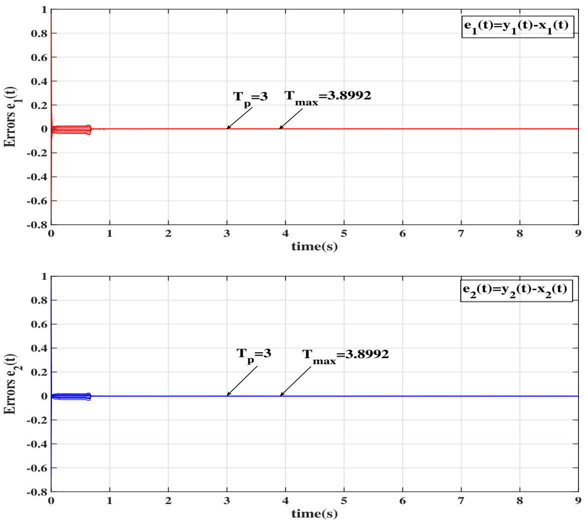

Figure 12. Error states $e_1(t), e_2(t)$ with $\theta = 1$ and controllers (42)

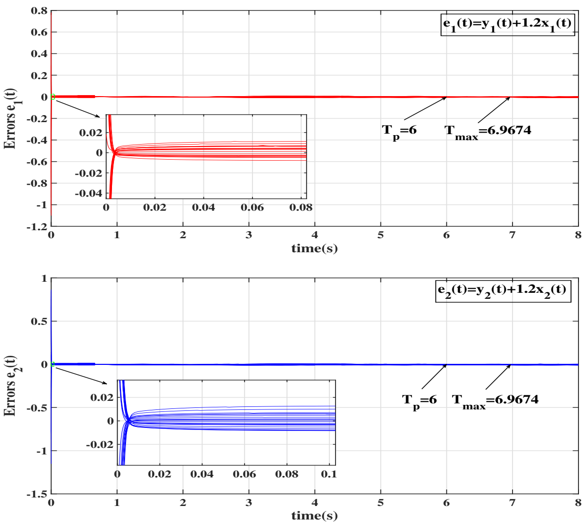

Figure 13. Error states $e_1(t), e_2(t)$ with $\theta = -1.2$ and controllers (42)

By using controller (42), from Therorem 2, we choose predefined time $\mathbf{T}_p = 6 < \mathbf{T}_{\max} = 6.9674$. Then FNNs (49) and (51) can accomplish PTPS. Figure 13 demonstrates the system (49) and (51) can realize PTPS with the controller (42) and $\theta = -1.2$.

5 Conclusions

Through utilizing the adaptive aperiodically switching strategy, this paper explores the FTPS/PTPS problem of FNNs. To attain the criteria of FTPS and PTPS, two aperiodically switching strategies equipped with adaptive updating laws are engineered, respectively. Different from the previous fixed/predefined-time synchronization [22, 23], complete synchronization [47] and anti-synchronization [15], the FTPS/PTPS attained in this paper are more general, as the above situations are our special cases. Ultimately, numerical simulations confirm the accuracy of the outcomes obtained. It should be emphasized that our parameters are real numbers. Indeed, FTPS and PTPS of FNNs with complex-valued fuzzy elements [11, 49] hold significant practical applications.

Data Availability Statement

Data will be made available on request.

Funding

This work was supported without any funding.

Conflicts of Interest

The authors declare no conflicts of interest.

Ethical Approval and Consent to Participate

Not applicable.

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