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Advances in the Mathematical Theory of WPAA Dynamics for Impulsive High Order Neural Systems in Clifford Algebras

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Abstract

The primary objective of this work is to establish the existence, uniqueness, and exponential stability of piecewise weighted pseudo–almost automorphic solutions for impulsive high-order Hopfield neural networks formulated within Clifford algebras. Using the Banach fixed-point principle together with a suitably adapted Gronwall–Bellman inequality, we derive novel and verifiable sufficient conditions that ensure these qualitative properties. The main contributions are as follows: (i) this study is the first to analyze weighted pseudo–almost automorphic (WPAA) dynamics for impulsive high-order Hopfield neural networks directly in the Clifford algebra setting, without reducing the model to real-valued components; (ii) it offers a unified framework that accommodates both first- and second-order synaptic interactions under impulsive perturbations and mixed delays; and (iii) the resulting conditions explicitly capture the geometric structure of Clifford-valued states, providing a broader and algebraically consistent formulation compared to real or quaternion-valued models. The theoretical findings are further supported by a numerical example demonstrating their applicability and effectiveness.

Keywords: impulsive systems, WPAA-functions, HOHNNs, Gronwall–Bellman inequality, exponential stability, clifford algebra.

1 Introduction and background

Recently, the theory of Artificial Neural Networks (ANNs) has received growing interest, as these models emulate key mechanisms of the human brain to perform complex computational tasks. Compared with classical numerical methods, ANNs have demonstrated remarkable efficiency in medical diagnosis, prognostic evaluation, signal and image processing, handwritten digit recognition, drug transport modeling and robotics [1–3]. Among these architectures, high-order neural networks constitute an important extension of standard models by incorporating higher-order interactions among neurons. Such structures offer improved approximation power, faster convergence, and greater robustness to disturbances (see [4–6], among others).

This growing interest has stimulated extensive

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research on the dynamical analysis of High-Order Hopfield Neural Networks (HOHNNs), particularly in the presence of various delay effects. Several works have investigated their stability and qualitative behaviors under different delay structures, including pseudo–almost periodic, almost automorphic, and anti-periodic regimes, as reported in [2, 5, 7, 8]. Collectively, these studies underscore that time delays constitute an inherent and unavoidable aspect of neural information processing, since real neuronal interactions are not instantaneous.

It is important to stress that the neural network models considered in the works cited above are predominantly real-valued or quaternion-valued. Yet, for problems involving large-scale spatial information or complex spatial transformations, Clifford-valued neural networks present a clear advantage, as their algebraic structure naturally accommodates multi-dimensional data in a compact and efficient manner. More fundamentally, Clifford algebra through its geometric product and multivector structure provides a natural language for representing high-order synaptic interactions. The geometric product for two vectors $\mathbf{u}$ and $\mathbf{v}$ is given by the displayed relation

$$\mathbf{u}\mathbf{v} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v},$$

which unifies the inner and outer products, allowing simultaneous capture of both magnitude alignment and oriented area|volume relationships between neuron states. This is especially relevant in high-order networks, where products of neuron outputs appear explicitly in Clifford algebra, such products are not mere scalar multiplications but multi-vector operations that preserve geometric and algebraic coherence across dimensions. Research on the dynamical behavior of such Clifford-valued architectures is still relatively sparse. Research on Clifford-valued neural networks has investigated equilibrium stability, anti-periodic solutions, and almost automorphic synchronization under various delays [7, 9, 16, 19]. Nevertheless, the literature remains limited, highlighting the need for further systematic studies in this area.

A wide range of dynamical systems are influenced by sudden variations, such as external shocks, harvesting events, or natural disasters. These abrupt perturbations typically manifest instantaneously and are therefore effectively captured through impulsive effects. Mathematically, impulsive differential equations provide an appropriate framework for describing such discontinuous dynamics and have been extensively applied in areas such as physics, population dynamics, ecology, biotechnology, industrial robotics, and artificial neural networks. Several studies have addressed impulsive phenomena in neural network models. For instance, [11] investigated piecewise asymptotically almost automorphic solutions for impulsive, HOHNNs with mixed delays. Likewise, [12] examined the existence of pseudo–almost periodic solutions in impulsive recurrent neural networks.

In the analysis of neural network dynamics, periodic, almost periodic, and pseudo almost periodic functions have long served as fundamental tools for characterizing the behavior of network outputs. More recently, the concept of pseudo–almost automorphy extending both almost periodicity and pseudo–almost periodicity—has gained considerable attention. This development has stimulated several generalizations, among which the class of weighted pseudo–almost automorphic (WPAA) functions, introduced by Blot et al. [13] in 2009, has emerged as particularly well suited for describing systems that deviate from strict periodicity.

A key question arises: how do Clifford-valued impulsive high-order Hopfield neural networks behave when all parameters are weighted pseudo–almost automorphic (WPAA)? To date, no study has addressed the existence, uniqueness, or stability of WPAA solutions for such networks with mixed delays in Clifford algebra. This paper fills this gap by establishing sufficient conditions for the existence, uniqueness, and exponential stability of WPAA solutions for these systems, governed by the following nonlinear differential equations: For $t \in \mathbb{R}$,

$$\begin{aligned}
\dot{x}_i(t) &= c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t-\varsigma_j(t))) \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) g_j(x_j(t-\sigma_j(t))) \\
&\times g_l(x_l(t-\upsilon_l(t))) \\
&+ \gamma_i(t),\ t \neq t_k, \qquad (1)\\
\Delta(x_i(t_k)) &= I_k(x(t_k)),\ k\in\mathbb{Z},\ t = t_k.
\end{aligned}$$

- The integer n denotes the total number of units constituting the neural network.
- $\mathcal{A}$ stands for a real Clifford algebra; its precise algebraic structure will be introduced in a subsequent section.

- For each $i = 1, \ldots, n$, the function $c_i(\cdot) \in \mathcal{A}$ describes the intrinsic rate at which the i^{th} unit relaxes to its equilibrium state when isolated from the network.
- The functions $a_{ij}(\cdot)$ and $b_{ijl}(\cdot)$, taking values in $\mathcal{A}$, represent respectively the first-order and second-order synaptic connection strengths between the units.
- The nonnegative functions $\varsigma_j(\cdot)$, $\sigma_j(\cdot)$, and $\upsilon_l(\cdot)$ model the transmission delays associated with the corresponding interactions.
- The signal processing within the network is governed by the activation functions $f_j(\cdot)$ and $g_j(\cdot)$,
- The external stimuli applied to the network are denoted by $\gamma_i(\cdot) \in \mathcal{A}$.
- Impulsive effects are represented by functions $I_k(\cdot) \in C(\mathbb{R}, \mathcal{A}^n)$ acting at prescribed instants t_k, where the jump of the state variable x_i at t_k is defined by
$$\Delta(x_i(t_k)) = x_i(t_k^+) - x_i(t_k^-).$$
- The sequence of impulse times $\{t_k\}$ is strictly increasing and satisfies $\lim_{k\to\infty} t_k = +\infty$.

Note that the product $g_j(x_j)g_l(x_l)$ in the high-order term is the geometric product in Clifford algebra, which inherently encodes both magnitude and directional relationships between neuron states, offering a richer representation of higher-order synaptic interactions than real or quaternion-valued products.

System (1) is considered together with the initial conditions specified as follows:

$$x_i(s) = \phi_i(s), \quad i = 1, \ldots, n, \quad s \in [-\zeta, 0], \tag{2}$$

such that $\phi(\cdot) \in PC\big([-\zeta, 0], \mathcal{A}^n\big)$, and

$$\zeta = \max\left\{ \max_{1\le j\le n} \sup_{t\in\mathbb{R}} \varsigma_j(t), \ \max_{1\le j\le n} \sup_{t\in\mathbb{R}} \sigma_j(t), \ \max_{1\le l\le n} \sup_{t\in\mathbb{R}} \upsilon_l(t) \right\}.$$

Throughout this work, we adapt the following notations:

$$f^* = \sup_{t\in\mathbb{R}} \|f(t)\|_{\mathcal{A}} \quad \text{or, if scalar, } f^* = \sup_{t\in\mathbb{R}} |f(t)|.$$

Our principal contributions of this paper:

- ▶ This paper provides a novel analysis of the existence, uniqueness, and exponential stability of impulsive HOHNNs with WPAA-coefficients.
- ▶ In the considered system, we account for the effects of both first-order and second-order interactions among neurons, providing a more comprehensive analysis of network dynamics.
- ▶ The class of WPAA-functions generalizes the notions of almost periodicity, almost automorphy, and pseudo almost automorphy. As such, our results extend and improve many existing findings in the literature, particularly those reported in [10].

The paper is structured as follows: Section 2 covers definitions, lemmas, and assumptions; Section 3 presents existence, uniqueness, and stability results; Section 4 gives a numerical example; and Section 5 concludes with remarks.

2 Mathematical Background

2.1 Real Clifford algebra

The real Clifford algebra over $\mathbb{R}^m$ is:

$$\mathcal{A} = \left\{ \sum_{A\subseteq\{1\cdots m\}} a_A e_A;\ a_A \in \mathbb{R} \right\},$$

where

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_\zeta}$$

with

$$A = \left\{ h_1 \cdots h_\zeta \right\}, \ 1 \le h_1 < h_2 < \cdots < h_\zeta \le m.$$

$\mathcal{A}$ equipped with m generators is defined as the Clifford algebra over the real number $\mathbb{R}$ with m multiplicative generators $e_1, \cdots, e_m$ satisfy the following relations

$$e_\emptyset = e_0 = 1, \ e_0^2 = 1$$

and

$$\begin{cases} e_0 e_i = e_i e_0 = e_i, & i = 1, 2, \cdots, m, \\ e_i e_j + e_j e_i = 0, & i \neq j, \ i, j \in \{1, \cdots, m\}, \\ e_i^2 = -1, & i = 1, 2, \cdots, m. \end{cases}$$

The product in $\mathcal{A}$ is the geometric product, which for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ (viewed as elements of $\mathcal{A}$) satisfies

$$\mathbf{u}\mathbf{v} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v},$$

where $\cdot$ is the symmetric inner product and $\wedge$ the antisymmetric outer product. For general multi-vectors, the geometric product is associative, bilinear, and invertible for nonzero vectors, providing a rich algebraic structure that unifies and extends real, complex, and quaternion algebras.
Let

$$\Omega = \Big\{\emptyset, 1, 2, \cdots, A, \cdots, 12, \cdots, m\Big\}.$$

we can see that

$$\mathcal{A} = \Big\{ \sum_{A \subseteq \{1 \cdots m\}} a_A e_A;\ a_A \in \mathbb{R} \Big\},$$

and $\dim \mathcal{A} = \sum_{k=0}^{m} \binom{m}{k} = 2^m$.
Henceforth, we define the norm on $\mathcal{A}^n$ by

- If $x = (x_1, \ldots, x_n) \in \mathcal{A}^n$, where each component

$$x_i = \sum_{A \subseteq \{1,\ldots,m\}} x_{iA} e_A,$$

the norm is extended as follows:

$$\|x\|_{\mathcal{A}} = \max_{1 \le i \le n} \left(\max_{A \subseteq \{1,\ldots,m\}} |x_{iA}| \right).$$

2.2 Piecewise weighted pseudo almost automorphic functions

Throughout this paper we adapt the following notations

- Let $J \subset \mathbb{R}$ and $\mathcal{A}^n$ be a vector space. The space $PC(J, \mathcal{A}^n)$ consists of piecewise continuous functions $x : J \to \mathcal{A}^n$ that may have first-kind discontinuities at a strictly increasing sequence $\{t_k\} \subset J$, with finite left and right limits. In typical applications to impulsive systems, functions are assumed left-continuous at each t_k, i.e.,

$$x(t_k^-) = x(t_k).$$

- $$B = \Big\{\{t_k\}_{k=-\infty}^{\infty} : t_k \in \mathbb{R},\ t_k < t_{k+1},\ \lim_{k \to \pm\infty} t_k = \pm\infty\Big\}$$

 represents the set of all sequences that are strictly increasing and unbounded in both directions.

Definition 1 *A bounded sequence $u = \{u_k\}_{k \in \mathbb{N}_0}$ is said to be almost automorphic if for every sequence of nonnegative integers $\{\sigma_m\}_{m \ge 1}$, there exists a subsequence $\{\tau_m\}_{m \ge 1}$ such that, for each $k \in \mathbb{N}_0$, the limit*

$$v_k := \lim_{m \to \infty} u_{k+\tau_m}$$

exists, and moreover, the limiting sequence satisfies

$$\lim_{m \to \infty} v_{k-\tau_m} = u_k, \qquad \text{for all } k \in \mathbb{N}_0.$$

The collection of all almost automorphic sequences from $\mathbb{N}_0$ into $\mathcal{A}$ is denoted by

$$\mathrm{AAS}(\mathbb{N}_0, \mathcal{A}).$$

Let U_s be the collection of sequences (weights) $\sigma : \mathbb{Z} \to (0, +\infty)$. For $\sigma \in U_s$ and $s \in \mathbb{Z},\ s > 0$, set

$$\mu_s(s, \sigma) = \sum_{n=-n}^{n} \sigma(n).$$

Denote

$$U_{s,\infty} := \Big\{\sigma \in U_s,\ \lim_{s \to \infty} \mu_s(s, \sigma) = \infty\Big\}.$$

Definition 2 *Let $\sigma \in U_{s,\infty}$. A sequence $w : \mathbb{N}_0 \mapsto \mathcal{A}$ is $\sigma - PAA_0$ if it is bounded and satisfies*

$$\lim_{s \to \infty} \frac{1}{\mu_s(s, \sigma)} \sum_{-s}^{s} \|w(n)\|_{\mathcal{A}} \sigma(n) = 0.$$

The collection of all such sequences is denoted by $PAA_0S(\mathbb{N}_0, \mathcal{A}, \sigma)$.

Definition 3 *Let $\sigma \in U_{s,\infty}$. A sequence $\omega : \mathbb{N}_0 \mapsto \mathcal{A}$ is weighted pseudo almost automorphic sequence if*

$$\omega = \omega^{aa} + \omega^{\sigma}$$

where $\omega^{aa} \in AAS(\mathbb{N}_0, \mathcal{A})$ and $\omega^{\sigma} \in PAA_0S(\mathbb{N}_0, \mathcal{A}, \sigma)$. Denote the set of all such sequences by $PAAS(\mathbb{N}_0, \mathcal{A}, \sigma)$.

Definition 4 ***Definition 5*** *Let $\phi \in \mathcal{PC}(\mathbb{R}, \mathcal{A})$ be a bounded piecewise continuous mapping. The function ϕ is said to be* almost automorphic *if the following conditions are satisfied:*

i) The family of discontinuity points $\{\theta_j\}_{j \in \mathbb{Z}}$ forms an almost automorphic sequence.

ii) For any sequence of real numbers $\{\rho'_m\}_{m \ge 1}$, there exists a subsequence $\{\rho_m\}_{m \ge 1}$ such that, for every $t \in \mathbb{R}$, the limit

$$\psi(t) := \lim_{m \to \infty} \phi(t + \rho_m)$$

exists, and the function ψ satisfies the reversibility condition

$$\lim_{m \to \infty} \psi(t - \rho_m) = \phi(t), \qquad \text{for all } t \in \mathbb{R}.$$

The set of all almost automorphic functions from $\mathbb{R}$ into $\mathcal{A}$ is denoted by $AA(\mathbb{R}, \mathcal{A})$.

Now, we introduce the concept of weighted pseudo almost automorphy. For more details, one can see papers [14]. Let

$$\mathbb{U} := \Big\{\rho : \mathbb{R} \to (0, +\infty) \;\Big|\; \rho \in L^1_{\text{loc}}(\mathbb{R}) \text{ and } \rho(t) > 0 \text{ for a.e. } t \in \mathbb{R}\Big\}.$$

If $\rho \in \mathbb{U}$, for $T > 0$:

$$\mu(T, \rho) := \int_{-T}^{T} \rho(x)dx.$$

Let

$$\mathbb{U}_\infty = \Big\{\rho \in \mathbb{U} : \lim_{T\to\infty} \mu(T, \rho) = \infty\Big\}.$$

Let $\rho \in \mathbb{U}_\infty$. The weighted ergodic space in Clifford algebra is

$$PAA_0(\mathbb{R}, \mathcal{A}^n, \rho) = \Big\{f \in BC(\mathbb{R}, \mathcal{A}^n) : \lim_{T\longrightarrow\infty} \frac{1}{\mu(T, \rho)} \int_{-T}^{T} \| f(t) \|_{\mathcal{A}} \; \rho(t)dt = 0\Big\}.$$

Definition 6 *Let $\rho \in \mathbb{U}_\infty$. A function $F \in PC(\mathbb{R}, \mathcal{A}^n)$ is weighted pseudo almost automorphic if*

$$F = F^{aa} + F^{\rho},$$

where $F^{aa} \in AA(\mathbb{R}, \mathcal{A}^n)$ and $F^{\rho} \in PAA_0(\mathbb{R}, \mathcal{A}^n, \rho)$. The set of all such functions is denoted by $WPAA(\mathbb{R}, \mathcal{A}^n, \rho)$.

Lemma 1 [14] *Fix $\rho \in \mathbb{U}_\infty$. The decomposition of WPAA function is unique.*

2.3 Main assumptions

Throughout this work, let $\rho : \mathbb{R} \to (0, +\infty)$, $\rho \in \mathbb{U}_\infty$ is continuous.

For each $\varsigma \in \mathbb{R}$, we assume that

$$\sup_{s\in\mathbb{R}} \left[\frac{\rho(s+\varsigma)}{p(s)}\right] < \infty, \; \sup_{T>0} \left[\frac{\rho(T+\varsigma, \rho)}{\mu(T, \rho))}\right] < \infty. \tag{3}$$

For the main results, we need the following basic assumptions.

Condition 1: For each $1 \le i \le n$,

$$M[c_i] = \lim_{T\to+\infty} \frac{1}{T} \int_{t}^{t+T} c_i(s)ds > 0,$$

and c_i is almost automorphic function with $0 < c_{i*} = \inf_{t\in\mathbb{R}}(c_i(t))$.

Condition 2: The functions a_{ij}, b_{ijl}, γ_i are piecewise WPAA and the sequence I_k is piecewise WPAA.

Condition 3: There exist positive constants numbers l^f, l^g, L^g, such that for all q_1, $q_2 \in \mathcal{A}$,

$$\|f_j(q_1) - f_j(q_2)\|_{\mathcal{A}} \le l^f_j \|q_1 - q_2\|_{\mathcal{A}},$$
$$\|g_j(q_1) - g_j(q_2)\|_{\mathcal{A}} \le l^g_j \|q_1 - q_2\|_{\mathcal{A}}, \|g_j(q_1)\|_{\mathcal{A}} \le L^g_j,$$

Condition 4: There exists a positive constant l^I such that:

$$\|I_k(q_1) - I_k(q_2)\|_{\mathcal{A}} \le L^I \|q_1 - q_2\|_{\mathcal{A}}, \; k \in \mathbb{N}_0, \; q_1, \; q_2 \in \mathcal{A}.$$

2.4 Technical lemmas

In this section, the following lemmas are fundamental.

Lemma 2 *Let $\pi, \pi_1, \pi_2 \in WPAA(\mathbb{R}, \mathcal{A}, \rho)$, $p \in \mathbb{R}$, and let $f : \mathcal{A} \to \mathcal{A}$ be a Lipschitz continuous function with Lipschitz constant $L^f > 0$. If $\varsigma \in AA(\mathbb{R}, \mathbb{R})$, then the following properties hold:*

1. *The shifted function $\pi(\cdot - p)$ belongs to $WPAA(\mathbb{R}, \mathcal{A}, \rho)$.*
2. *The pointwise product $\pi_1 \cdot \pi_2$ belongs to $WPAA(\mathbb{R}, \mathcal{A}, \rho)$.*
3. *For any $\phi \in WPAA(\mathbb{R}, \mathcal{A}, \rho)$, the time-shifted composition $t \mapsto \phi(t - \varsigma(t))$ belongs to $WPAA(\mathbb{R}, \mathcal{A}, \rho)$.*
4. *For any $\phi \in WPAA(\mathbb{R}, \mathcal{A}, \rho)$, the composed function $f \circ \phi$ belongs to $WPAA(\mathbb{R}, \mathcal{A}, \rho)$.*

Remark 2.1 *The proofs of these properties are similar to those given in [18].*

Remark 2.2 *We suppose that $f_j(0) = g_j(0) = 0$.*

Lemma 3 *Suppose that all the previously stated assumptions are satisfied.*
For each index i with $0 \le i \le n$, assume that

$$\sup_{T>0} \left\{ \int_{-T}^{T} e^{-c_{i*}(T+t)} \rho(t)\, dt \right\} < \infty. \tag{4}$$

Define a nonlinear operator Σ by

$$\Sigma : WPAA(\mathbb{R}, \mathcal{A}, \rho) \to \mathcal{A}^n,$$

where, for any $\phi \in WPAA(\mathbb{R}, \mathcal{A}, \rho)$,

$$(\Sigma\phi)_i(t) := \int_{-\infty}^{t} e^{-\int_s^t c_i(u)\, du} (\mathcal{G}_\phi)_i(s)\, ds, \tag{5}$$

and

$$(\mathcal{G}_\phi)_i(s) = \sum_{j=1}^{n} a_{ij}(s) f_j\big(\phi_j(s - \varsigma_j(s))\big) + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s)\, g_j\big(\phi_j(s - \sigma_j(s))\big)\, g_l\big(\phi_l(s - \upsilon_l(s))\big) + \gamma_i(s).$$

Then,

$$\Sigma\big(WPAA(\mathbb{R}, \mathcal{A}, \rho)\big) \subset WPAA(\mathbb{R}, \mathcal{A}, \rho).$$

Proof 1 *For $1 \le i \le n$, the function $(\mathcal{G}_\phi)_i(\cdot)$ is piecewise WPAA.*
By definition, $\mathcal{G}_i = \beth^1_i + \beth^2_i$ where $\beth^1_i(\cdot) \in AA(\mathbb{R}, \mathcal{A})$ and $\beth^2_i(\cdot) \in PAA_0(\mathbb{R}, \mathcal{A}, \rho)$.
Then,

$$(\Sigma_\phi)_i(t) \quad := \quad \Sigma_i \beth^1_i(t) + \Sigma_i \beth^2_i(t). \tag{6}$$

We observe that $M[c_i] > 0$. By applying the exponential dichotomy framework, it follows that

$$\int_{-\infty}^{t} e^{-\int_s^t c_i(u)\, du} (\beth^1_\phi)_i(s)\, ds \in AA(\mathbb{R}, \mathcal{A}) \tag{7}$$

provides a solution to the almost automorphic differential equation

$$\dot{y}(t) = -c_i(t)y(t) + (\beth^1_\phi)_i(t), \qquad 1 \le i, j \le n.$$

Now we deal with $\Sigma_i \daleth_i^2$.
We can see that

$$\begin{aligned} & \lim_{T \longrightarrow \infty} \frac{1}{\mu(T,\rho)} \int_{[-T,T]} \|\Sigma_i \daleth_i^2(t)\|_{\mathcal{A}} \rho(t) dt \\ \leq & \lim_{T \longrightarrow \infty} \frac{1}{\mu(T,\rho)} \int_{[-T,T]} \| \int\limits_{-\infty}^{t} e^{-(t-s)c_{i*}} \daleth_i^2(s) ds \|_{\mathcal{A}} \rho(t) dt. \end{aligned}$$

Let

$$\begin{aligned} \alpha^1 &= \lim_{T \longrightarrow \infty} \frac{1}{\mu(T,\rho)} \int_{[-T,T]} (\int\limits_{-T}^{t} e^{-(t-s)c_{i*}} \|\daleth_i^2(s)\|_{\mathcal{A}} ds) \rho(t) dt \\ \alpha^2 &= \lim_{T \longrightarrow \infty} \frac{1}{\mu(T,\rho)} \int_{[-T,T]} (\int\limits_{-\infty}^{-T} e^{-(t-s)c_{i*}} \|\daleth_i^2(s)\|_{\mathcal{A}} ds) \rho(t) dt \end{aligned}$$

Let $m = t - s$, then by Fubini's theorem, we obtain

$$\begin{aligned} \alpha^1 &= \lim_{T \longrightarrow \infty} \frac{1}{\mu(T,\rho)} \int_{[-T,T]} \Big(\int\limits_{0}^{t+T} e^{-mc_{i*}} \|\daleth_i^2(t-m)\|_{\mathcal{A}} dm \Big) \rho(t) dt \\ &\leq \lim_{T \longrightarrow \infty} \frac{1}{\mu(T,\rho)} \int_{[-T,T]} \Big(\int\limits_{0}^{+\infty} e^{-mc_{i*}} \|\daleth_i^2(t-m)\|_{\mathcal{A}} dm \Big) \rho(t) dt \\ &\leq \int\limits_{0}^{+\infty} e^{-mc_{i*}} \Big(\lim_{T \longrightarrow \infty} \frac{1}{\mu(T,\rho)} \int_{[-T,T]} \|\daleth_i^2(t-m)\|_{\mathcal{A}} \rho(t) dt \Big) dm \\ &= \int\limits_{0}^{+\infty} e^{-mc_{i*}} \Big(\lim_{T \longrightarrow \infty} \frac{1}{\mu(T,\rho)} \int_{[-T-m,T-m]} \|\daleth_i^2(t)\|_{\mathcal{A}} \rho(t+m) dt \Big) dm \\ &\leq \int\limits_{0}^{+\infty} e^{-mc_{i*}} \Big(\lim_{T \longrightarrow \infty} \frac{\mu(T+m,\rho)}{\mu(T,\rho)} \frac{\mu(T,\rho)}{\mu(T+m,\rho)} \\ &\times \int_{[-T-m,T-m]} \|\daleth_i^2(t)\|_{\mathcal{A}} \rho(t+m) dt \Big) dm. \end{aligned}$$

Then $\alpha^1 = 0$.
Also,

$$\begin{aligned} \alpha_2 &\leq \lim_{T \longrightarrow \infty} \frac{1}{\mu(T,\rho)} \int\limits_{-\infty}^{-T} e^{sc_{i*}} \|\daleth_i^2(s)\|_{\mathcal{A}} ds \int_{-T}^{T} e^{tc_{i*}} \rho(t) dt \\ &= \lim_{T \longrightarrow \infty} \frac{\|\daleth_i^2(s)\|_{\mathcal{A}}}{\mu(T,\rho) c_{i*}} \int_{-T}^{T} e^{-(T+t)c_{i*}} \rho(t) dt = 0. \end{aligned}$$

Combining with with equation 7 it leads to

$$\Sigma\big(WPAA(\mathbb{R},\mathcal{A},\rho)\big) \subset WPAA(\mathbb{R},\mathcal{A},\rho).$$

Lemma 4 *Let $\phi(\cdot) \in WPAA(\mathbb{R},\mathcal{A},\rho)$. We obtain*

$$\sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k(\phi_i(t_k)) \in WPAA(\mathbb{R},\mathcal{A},\rho). \tag{8}$$

Proof 2 *We have $I_k(\phi_i(t_k)) \in WPAA(\mathbb{R},\mathcal{A},\rho)$. It can be expressed as*

$$I_k(\phi_i(t_k)) = I_k^1(\phi_i(t_k)) + I_k^2(\phi_i(t_k)), \tag{9}$$

where $I_k^1(\phi_i(t_k)) \in AA(\mathbb{R},\mathcal{A})$ and $I_k^2(\phi_i(t_k)) \in PAA_0(\mathbb{R},\mathcal{A},\rho)$. Then

$$\begin{aligned} \sum_{t_k<t} e^{-\int\limits_{t_k}^{t} c_i(u)du} I_k(\phi_i(t_k)) &= \sum_{t_k<t} e^{-\int\limits_{t_k}^{t} c_i(u)du} I_k^1(\phi_i(t_k)) \\ &+ \sum_{t_k<t} e^{-\int\limits_{t_k}^{t} c_i(u)du} I_k^2(\phi_i(t_k)). \end{aligned}$$

For every real sequence $(t_n)_{n\in\mathbb{N}}$, there exists a subsequence $(t_{n_k})_{n_k\in\mathbb{N}}$ such that

$$\begin{aligned} \lim_{n_k \to +\infty} I_1^1(\phi_i(t_k + t_{n_k})) &= \hat{I}_k^1(\phi_i(t_k)), \\ \lim_{n_k \to +\infty} \hat{I}_k^1(\phi_i(t_k - t_{n_k})) &= I_k^1(\phi_i(t_k)). \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{t_k<t+t_{n_k}} e^{-\int\limits_{t_k}^{t+t_{n_k}} c_i(u)du} I_k^1(\phi_i(t_k)) \\ = & \sum_{t_k<t} e^{-\int\limits_{t_k+t_{n_k}}^{t+t_{n_k}} c_i(u)du} I_k^1(\phi_i(t_k + t_{n_k})), \end{aligned}$$

then

$$\begin{aligned} & \lim_{n_k \to +\infty} \sum_{t_k<t} e^{-\int\limits_{t_k+t_{n_k}}^{t+t_{n_k}} c_i(u)du} I_k^1(\phi_i(t_k + t_{n_k})) \\ = & \sum_{t_k<t} e^{-\int\limits_{t_k}^{t} c_i(u)du} \hat{I}_k^1(\phi_i(t_k)). \end{aligned}$$

Similarly

$$\begin{aligned} & \sum_{t_k<t-t_{n_k}} e^{-\int\limits_{t_k}^{t-t_{n_k}} c_i(u)du} I_k^1(\phi_i(t_k)) \\ = & \sum_{t_k<t} e^{-\int\limits_{t_k-t_{n_k}}^{t-t_{n_k}} c_i(u)du} \hat{I}_k^1(\phi_i(t_k - t_{n_k})), \end{aligned}$$

then

$$\begin{aligned} & \lim_{n_k \to +\infty} \sum_{t_k<t} e^{-\int\limits_{t_k-t_{n_k}}^{t-t_{n_k}} c_i(u)du} \hat{I}_k^1(\phi_i(t_k - t_{n_k})) \\ = & \sum_{t_k<t} e^{-\int\limits_{t_k}^{t} c_i(u)du} I_k^1(\phi_i(t_k)). \end{aligned}$$

Now, et

$$\xi(t) = e^{-\int\limits_{t_k}^{t} c_i(u)du} I_k^2(\phi_i(t_k)), \ t_k \leq t \leq t_{k+1}, \ k \in \mathbb{Z}.$$

Then

$$\begin{aligned} \lim_{t \to +\infty} \|\xi(t)\|_{\mathcal{A}} &= \lim_{t \to +\infty} \|e^{-\int\limits_{t_k}^{t} c_i(u)du} I_k^2(\phi_i(t_k))\|_{\mathcal{A}} \\ &\leq \lim_{t \to +\infty} e^{-(t-t_k)c_{i*}} \sup_{k\in\mathbb{Z}} \|I_k^2\|_{\mathcal{A}} = 0. \end{aligned}$$

Let

$$\xi_n(t) = e^{-\int\limits_{t_k-n}^{t} c_i(u)du} I_{k-n}^2, \ t_k \leq t \leq t_{k+1}, \ n \in \mathbb{N},$$

then $\xi_n \in PAA_0(\mathbb{R},\mathcal{A},\rho)$.
One has

$$\begin{aligned} \|\xi(t)\|_{\mathcal{A}} &= \|e^{-\int\limits_{t_k}^{t} c_i(u)du} I_k^2\|_{\mathcal{A}} \\ &\leq e^{-(t-t_k-n)a_{i*}} \sup_{k\in\mathbb{Z}} \|I_k^2\|_{\mathcal{A}} \\ &\leq e^{-(t-t_k)a_{i*}} e^{-a_{i*}n} \sup_{k\in\mathbb{Z}} \|I_k^2\|_{\mathcal{A}}. \end{aligned}$$

The series $\sum_{n=1}^{\infty}\xi_n$ is uniformly convergent on $\mathcal{A}$. Then $\sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k^2(\phi_i(t_k)) \in PAA_0(\mathbb{R},\mathcal{A},\rho)$.

3 Main Results

3.1 Existence and uniqueness

For $\phi = (\phi_1,\phi_2,\cdots,\phi_n)^T \in WPAA(\mathbb{R},\mathcal{A}^n,\rho)$, we define the norm of ϕ as

$$\| \phi \|_* := \max_{1\le i\le n} \sup_{t\in\mathbb{R}}\{\|\phi_i(t)\|_{\mathcal{A}}\}.$$

Lemma 5 *We suppose that Conditions 1-4 hold. For each $\phi(\cdot) \in WPAA(\mathbb{R},\mathcal{A},\mu)$, define the nonlinear operator Π as follows:*

$$(\Pi_\phi)_i(t) := (\Sigma_\phi)_i(t) + \sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k(\phi_i(t_k)).$$

Then, Π maps $WPAA(\mathbb{R},\mathcal{A},\mu)$ into itself.

Theorem 1 *Under assumptions (1)–(4) and in accordance with Lemma 5, suppose there exist nonnegative constants ρ^1 and ρ^2 such that:*

$$\rho^1 = \max_{1\le i\le n}\left\{\frac{1}{c_{i*}}(\sum_{j=1}^{n} a_{ij}^* l_j^f + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^* l_j^g L_l^g) + \frac{l^I}{1-e^{-c_{i*}}}\right\} < 1, \quad (10)$$

$$\rho^2 = \max_{1\le i\le n}\left\{\frac{1}{c_{i*}}\left(\sum_{j=1}^{n} a_{ij}^* l_j^f + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^* (l_j^g L_l^g + l_l^g L_j^g)\right) + \frac{l^I}{1-e^{-c_{i*}}}\right\} < 1. \quad (11)$$

Then, system (1) has a unique WPAA solution in the region

$$\urcorner = \left\{\phi \in WPAA(\mathbb{R},\mathcal{A}^n,\mu), \|\phi-\phi_0\|_* \le \frac{\rho^1}{1-\rho^1}\rho^0\right\}.$$

Proof 3 *Define the function $\phi_{0,i}(t) = (\phi_{0,1}(t),\ldots,\phi_{0,n}(t))^T$ by*

$$\phi_{0,i}(t) = \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_s^t c_1(u)du}\gamma_1(s)ds \\ \vdots \\ \int_{-\infty}^{t} e^{-\int_s^t c_n(u)du}\gamma_n(s)ds \end{pmatrix}.$$

Clearly, $\phi_{0,i} \in WPAA(\mathbb{R},\mathcal{A}^n,\mu)$. Moreover, using Condition 1 and the boundedness of γ_i, we have

$$\begin{aligned}\|\phi_{0,i}\|_* &\le \|\int_{-\infty}^{t} e^{-\int_s^t c_i(u)du}\gamma_i(s)ds\|_* \\ &\le \max_{1\le i\le n}\left(\frac{\gamma_i^*}{c_{i*}}\right) = \rho^0. \quad (12)\end{aligned}$$

Let $\phi \in \urcorner$. Then

$$\|\phi-\phi_0\|_* \le \frac{\rho^1}{1-\rho^1}\rho^0,$$

and consequently

$$\|\phi\|_* \le \|\phi-\phi_0\|_* + \|\phi_0\|_* \le \frac{\rho^1}{1-\rho^1}\rho^0 + \rho^0 = \frac{\rho^0}{1-\rho^1}.$$

Now, consider $(\Pi\phi)_i(t) - \phi_{0,i}(t)$. Using the definition of Π and ϕ_0, we have

$$\begin{aligned}(\Pi\phi)_i(t) - \phi_{0,i}(t) &= \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du}\Bigg[\sum_{j=1}^{n} a_{ij}(s) f_j(\phi_j(s-\zeta_j(s))) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s) g_j(\phi_j(s-\sigma_j(s))) \\ &\times g_l(\phi_l(s-\upsilon_l(s)))\Bigg]ds \\ &+ \sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du} I_k(\phi_i(t_k)).\end{aligned}$$

Taking norms and using the Lipschitz conditions (Condition 3) and the boundedness of g_j, we obtain

$$\begin{aligned}&\|(\Pi\phi)_i(t) - \phi_{0,i}(t)\|_{\mathcal{A}} \\ &\le \int_{-\infty}^{t} e^{-c_{i*}(t-s)}\Bigg(\sum_{j=1}^{n} a_{ij}^* l_j^f \|\phi_j(s-\zeta_j(s))\|_{\mathcal{A}} \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^* l_j^g L_l^g \|\phi_j(s-\sigma_j(s))\|_{\mathcal{A}}\Bigg)ds \\ &+ \sum_{t_k<t} e^{-c_{i*}(t-t_k)} l^I \|\phi_i(t_k)\|_{\mathcal{A}}.\end{aligned}$$

Since $\|\phi_j(\cdot)\|_{\mathcal{A}} \le \|\phi\|_*$ for all j, we get

$$\begin{aligned}&\|(\Pi\phi)_i(t) - \phi_{0,i}(t)\|_{\mathcal{A}} \\ &\le \|\phi\|_*\Bigg[\int_{-\infty}^{t} e^{-c_{i*}(t-s)}\left(\sum_{j=1}^{n} a_{ij}^* l_j^f + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^* l_j^g L_l^g\right)ds \\ &+ \sum_{t_k<t} e^{-c_{i*}(t-t_k)} l^I\Bigg].\end{aligned}$$

Computing the integrals and sums:

$$\int_{-\infty}^{t} e^{-c_{i*}(t-s)}ds = \frac{1}{c_{i*}},$$

and

$$\sum_{t_k<t} e^{-c_{i*}(t-t_k)} \le \frac{1}{1-e^{-c_{i*}}}.$$

Therefore,

$$\begin{aligned}&\|(\Pi\phi)_i(t) - \phi_{0,i}(t)\|_{\mathcal{A}} \\ &\le \|\phi\|_*\Bigg[\frac{1}{c_{i*}}\left(\sum_{j=1}^{n} a_{ij}^* l_j^f + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^* l_j^g L_l^g\right) \\ &+ \frac{l^I}{1-e^{-c_{i*}}}\Bigg].\end{aligned}$$

Then, we obtain

$$\begin{aligned}\|\Pi\phi-\phi_0\|_* &\le \|\phi\|_*\cdot\rho^1 \\ &\le \frac{\rho_0}{1-\rho^1}\cdot\rho^1 \\ &= \frac{\rho^1}{1-\rho^1}\rho^0.\end{aligned}$$

Thus, $\Pi\phi \in \daleth$, i.e., Π maps $\daleth$ into $\daleth$.

Let $\phi, \psi \in \daleth$. For each $i = 1, \dots, n$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} &(\Pi\phi)_i(t) - (\Pi\psi)_i(t) \\ &= \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} \Bigg[\sum_{j=1}^{n} a_{ij}(s)\big(f_j(\phi_j(s-\zeta_j(s))) \\ - \quad & f_j(\psi_j(s-\zeta_j(s)))\big) \\ + \quad & \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s)\big(g_j(\phi_j(s-\sigma_j(s)))g_l(\phi_l(s-\upsilon_l(s))) \\ - \quad & g_j(\psi_j(s-\sigma_j(s)))g_l(\psi_l(s-\upsilon_l(s)))\big)\Bigg] ds \\ + \quad & \sum_{t_k<t} e^{-\int_{t_k}^{t} c_i(u)du}\big(I_k(\phi_i(t_k)) - I_k(\psi_i(t_k))\big). \end{aligned}$$

Using the Lipschitz conditions and the triangle inequality, we estimate the norm:

$$\begin{aligned} &\|(\Pi\phi)_i(t) - (\Pi\psi)_i(t)\|_{\mathcal{A}} \\ &\le \int_{-\infty}^{t} e^{-c_{i*}(t-s)} \Bigg[\sum_{j=1}^{n} a_{ij}^{*} l_j^{f} \|\phi_j(s-\zeta_j(s)) - \psi_j(s-\zeta_j(s))\|_{\mathcal{A}} \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^{*}\Big(l_j^{g} L_l^{g} \|\phi_j(s-\sigma_j(s)) - \psi_j(s-\sigma_j(s))\|_{\mathcal{A}} \\ &\quad + l_l^{g} L_j^{g} \|\phi_l(s-\upsilon_l(s)) - \psi_l(s-\upsilon_l(s))\|_{\mathcal{A}}\Big)\Bigg] ds \\ &\quad + \sum_{t_k<t} e^{-c_{i*}(t-t_k)} l^{I} \|\phi_i(t_k) - \psi_i(t_k)\|_{\mathcal{A}}. \end{aligned}$$

Since $\|\phi_j(\cdot) - \psi_j(\cdot)\|_{\mathcal{A}} \le \|\phi - \psi\|_*$ for all j, we obtain

$$\begin{aligned} \|(\Pi\phi)_i(t) - (\Pi\psi)_i(t)\|_{\mathcal{A}} \le \|\phi - \psi\|_* \Bigg[\int_{-\infty}^{t} e^{-c_{i*}(t-s)} \Big(& \sum_{j=1}^{n} a_{ij}^{*} l_j^{f} \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^{*}(l_j^{g} L_l^{g} + l_l^{g} L_j^{g})\Big) ds \\ & + \sum_{t_k<t} e^{-c_{i*}(t-t_k)} l^{I} \Bigg]. \end{aligned}$$

Computing the integrals and sums as before yields

$$\begin{aligned} \|(\Pi\phi)_i(t) - (\Pi\psi)_i(t)\|_{\mathcal{A}} \quad \le \quad & \|\phi - \psi\|_* \Bigg[\frac{1}{c_{i*}} \Big(\sum_{j=1}^{n} a_{ij}^{*} l_j^{f} \\ + \quad & \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^{*}(l_j^{g} L_l^{g} + l_l^{g} L_j^{g})\Big) \\ + \quad & \frac{l^{I}}{1 - e^{-c_{i*}}} \Bigg]. \end{aligned}$$

Then, we get

$$\|\Pi\phi - \Pi\psi\|_* \le \rho^2 \|\phi - \psi\|_*.$$

Since $\rho^2 < 1$, Π is a contraction on $\daleth$.

Applying the Banach fixed-point theorem, we conclude that the operator Π possesses exactly one fixed point in $\daleth$.

This fixed point therefore represents the unique WPAA solution of system (1) within the the region $\daleth$.

3.2 Global exponential stability

Theorem 2 *Let the assumptions of Theorem 1 hold, so that system (1) possesses a unique WPAA solution $z^*(t)$. Suppose, in addition, that the following conditions are satisfied:*

$$\begin{aligned} &(i) \quad 1 + l^{I} < e, \\ &(ii) \quad \min_{1\le i\le n} \Bigg[c_{i*} - \sum_{j=1}^{n} a_{ij}^{*} l_j^{f} - \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^{*}(L_l^{g} l_j^{g} + L_j^{g} l_l^{g}) \\ &\quad - N \ln(1 + l^{I}) \Bigg] > 0, \end{aligned}$$

where N is the number of impulse times t_k within any interval of unit length. Then the unique WPAA solution of (1) is globally exponentially stable.

Proof 4 *Let*

- *$z^*(\cdot)$: The unique WPAA solution of (1) with initial condition $\phi^*(\cdot)$,*
- *$z(\cdot)$: Any other arbitrary solution of (1) with initial condition $\phi(\cdot)$.*

Let

$$Z(t) = z(t) - z^*(t).$$

For $t \ne t_k$, the error dynamics are :

$$\begin{aligned} \dot{Z}_i(t) &= -c_i(t) Z_i(t) \\ + \quad & \sum_{j=1}^{n} a_{ij}(t)\Big[f_j\big(z_j(t-\varsigma_j(t))\big) - f_j\big(z_j^*(t-\varsigma_j(t))\big)\Big] \\ + \quad & \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\Big[g_j\big(z_j(t-\sigma_j(t))\big)g_l\big(z_l(t-\upsilon_l(t))\big) \\ - \quad & g_j\big(z_j^*(t-\sigma_j(t))\big)g_l\big(z_l^*(t-\upsilon_l(t))\big)\Big]. \end{aligned}$$

At impulse instants $t = t_k$, we have:

$$\Delta Z_i(t_k) = I_k\big(z_i(t_k)\big) - I_k\big(z_i^*(t_k)\big).$$

Using the Lipschitz conditions, we obtain:

$$\|g_j(z_j)g_l(x_l) - g_j(z_j^*)g_l(x_l^*)\|_{\mathcal{A}} \quad \le \quad L_j^{g} l_l^{g} \|Z_l\|_{\mathcal{A}} + l_j^{g} L_l^{g} \|Z_j\|_{\mathcal{A}}.$$

For $t \ge 0$, the solution of the error equation can be written as:

$$\begin{aligned} Z_i(t) &= Z_i(0) e^{-\int_0^t c_i(u)du} \\ + \quad & \int_0^t e^{-\int_s^t c_i(u)du} \Bigg[\sum_{j=1}^{n} a_{ij}(s)\big(f_j(z_j(s-\varsigma_j(s))) \\ - \quad & f_j(z_j^*(s-\varsigma_j(s)))\big) \\ + \quad & \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s)\big(g_j(z_j(s-\sigma_j(s)))g_l(z_l(s-\upsilon_l(s))) \\ - \quad & g_j(z_j^*(s-\sigma_j(s)))g_l(z_l^*(s-\upsilon_l(s)))\big)\Bigg] ds \\ + \quad & \sum_{0<t_k<t} e^{-\int_{t_k}^{t} c_i(u)du}\big(I_k(z_i(t_k)) - I_k(z_i^*(t_k))\big). \end{aligned}$$

Let

$$\mathcal{Z} = \max_{1\le i\le n} \|Z_i(t)\|_{\mathcal{A}}.$$

Define:

$$\begin{aligned} c_* &= \min_i c_{i*}, \\ \alpha &= \max_i \sum_{j=1}^{n} a_{ij}^* l_j^f, \\ \beta &= \max_i \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^* (L_l^g l_j^g + L_j^g l_l^g). \end{aligned}$$

Then

$$\begin{aligned} \mathcal{Z}(t) &\le \mathcal{Z}(0)e^{-c_* t} + (\alpha+\beta)\int_0^t e^{-c_*(t-s)}\mathcal{Z}(s)\,ds \\ &+ l^I \sum_{0<t_k<t} e^{-c_*(t-t_k)}\mathcal{Z}(t_k). \end{aligned}$$

Set $v(t) = e^{c_* t}\mathcal{Z}(t)$, *So*

$$v(t) \le v(0) + (\alpha+\beta)\int_0^t v(s)\,ds + l^I \sum_{0<t_k<t} v(t_k).$$

Applying the generalized Gronwall inequality for impulsive systems (see [18]), we have:

$$v(t) \le v(0) \prod_{0<t_k<t} (1+l^I)\, e^{(\alpha+\beta)t}.$$

Consequently:

$$\mathcal{Z}(t) \le \mathcal{Z}(0) \prod_{0<t_k<t} (1+l^I)\, e^{-[c_*-(\alpha+\beta)]t}.$$

Let $N(t)$ *denote the number of impulses in* $[0,t)$. *By assumption,*

$$N(t) \le Nt$$

. *Hence:*

$$\prod_{0<t_k<t} (1+l^I) \le (1+l^I)^{Nt} = e^{N\ln(1+l^I)t}.$$

Therefore:

$$\mathcal{Z}(t) \le \mathcal{Z}(0)\, e^{\left[N\ln(1+l^I)-(c_*-(\alpha+\beta))\right]t}.$$

Condition (ii) of the theorem ensures that the exponent is negative:

$$N\ln(1+l^I) - \left[c_* - (\alpha+\beta)\right] < 0.$$

Thus, there exists $\lambda > 0$ *such that:*

$$\mathcal{Z}(t) \le \mathcal{Z}(0)e^{-\lambda t}, \quad \forall t \ge 0.$$

This proves that the unique WPAA solution $z^*(t)$ *is globally exponentially stable.*

Remark 3.1 *All our conditions are structurally tighter than those in real or quaternion-valued studies because they must control both the magnitude and oriented geometric relations encoded in multi-vector products, particularly in the second-order terms* $g_j(\cdot)g_l(\cdot)$. *The impulse terms also differ substantially from previous work.*

4 From Theory to Practice: Applications and Benchmarking Against Existing Work

4.1 An example and its numerical simulations

For $1 \le i \le 2$, we consider the following HOHNNs model in Clifford algebra:

$$\begin{cases} \dot{x}_i(t) &= c_i(t)x_j(t) + \sum_{j=1}^{2} a_{ij}(t) f_j(x_j(t-\varsigma_j(t))) \\ &+ \sum_{j=1}^{2}\sum_{l=1}^{2} b_{ijl}(t) g_j(x_j(t-\sigma_j(t))) \\ &\times g_l(x_l(t-\upsilon_l(t))) + \gamma_i(t),\ t \ne t_k, \\ \Delta(x_i(t_k)) &= I_k(x(t_k)),\ k\in\mathbb{Z},\ t\in\mathbb{R},\ t=t_k. \end{cases} \tag{13}$$

For $1 \le i \le 2$, and $\rho = e^t$, we take $f(x_i) = g(x_i)$ such that

$$\begin{aligned} f(x) &= \frac{\sqrt{2}}{40}\sin(\sqrt{2}x)e_0 + \frac{1}{90}|x|e_1 \\ &+ \frac{1}{70}\cos(x)e_2 + \frac{1}{60}\sin(x)e_{12} \\ & l_j^f = l_j^g = L_j^g = L^I = 1 \\ \varsigma_j(t) &= \sigma_j(t) = \upsilon_l(t) = 3 - \sin\frac{\sqrt{5}}{7}t, \\ c_1(t) &= 0.4 + 0.05\,\sin(\sqrt{2}\,t),\ c_2(t) = 0.3 + 0.03\,\cos(3\,t), \\ a_{11}(t) &= 0.03\sin\left(\frac{2\pi}{2+\sin t+\sin\sqrt{3}t}\right) \\ &+ 0.03\left(\sin\left(\frac{2\pi}{2+\sin t+\sin\sqrt{3}t}\right) + e^{-t}\right)e_1 \\ &+ \left(0.05\cos(t) + 0.2e^{-t}\right)e_2 \\ a_{22}(t) &= \left(0.05\sin(\sqrt{2}t) + 0.02e^{-t}\right)e_1 \\ &+ 0.05\cos\left(\frac{2\pi}{2+\sin t+\sin\sqrt{3}t}\right)e_2 \\ &+ 0.04\left(\cos(\sqrt{5}t) + e^{-t}\right)e_{12} \\ a_{12}(t) &= a_{21}(t) = 0. \\ \gamma_1(t) &= 0.2\sin\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right) \\ &+ \frac{0.1}{1+t} + 0.3\sin\left(\frac{2\pi}{2+\sin t+\sin\sqrt{3}t}\right)e_1 \\ &+ 0.3\left(\cos\left(\frac{2\pi}{2+\sin t+\sin\sqrt{3}t}\right) + e^{-t}\right)e_2 \\ &+ \left(0.5\cos(t) + 0.2e^{-t}\right)e_{12} \\ \gamma_2(t) &= \left(0.5\sin(\sqrt{2}t) + 0.02e^{-t}\right) \\ &+ 0.5\cos\left(\frac{2\pi}{2+\sin t+\sin\sqrt{3}t}\right)e_1 \\ &+ 0.4\left(\cos(\sqrt{2}t) + e^{-t}\right)e_2 \\ &+ 0.4\left(\sin(\sqrt{5}t) + e^{-t}\right)e_{12} \\ b_{111}(t) &= 0, \end{aligned}$$

$$
\begin{aligned}
b_{112}(t) &= \Big(0.03\cos\big(\frac{1}{2+\sin t+\sin(\sqrt{2}t)}\big)+\frac{0.01}{1+t^2}\Big)e_1 \\
&+ \Big(0.05+\frac{0.1}{1+t}\Big)e_{12}, \\
b_{121}(t) &= 0, \\
b_{122}(t) &= \Big(0.03\cos\big(\frac{1}{2+\cos t+\cos(\sqrt{5}t)}\big)+\frac{0.01}{1+t}\Big)e_1 \\
&+ \Big(0.05+\frac{0.1}{1+t}\Big)e_2. \\
b_{211}(t) &= \Big(0.03\sin\big(\frac{1}{2+\cos t+\sin(\sqrt{5}t)}\big)+\frac{0.01}{1+t}\Big)e_1, \\
b_{212}(t) &= 0, \\
b_{221}(t) &= 0, \\
b_{222}(t) &= 0.03\sin\big(\frac{1}{2+\cos t+\sin(\sqrt{2}t)}\big)e_2,
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta x^0(2k) &= -\frac{1}{40}x^0(2k)+\frac{1}{80}\sin(x^0(2k))+\frac{1}{20}, \\
\Delta x^0(2k) &= -\frac{1}{40}x^0(2k)+\frac{1}{80}\cos(x^0(2k))+\frac{1}{30}, \\
\Delta x^1(2k) &= -\frac{1}{30}x^1(2k)+\frac{1}{30}\sin(x^1(2k))+\frac{1}{20}, \\
\Delta x^1(2k) &= -\frac{1}{30}x^2(2k)+\frac{1}{30}\cos(x^1(2k))+\frac{1}{30}, \\
\Delta x^2(2k) &= -\frac{1}{40}x^2(2k)+\frac{1}{80}\sin(x^2(2k))+\frac{1}{20}, \\
\Delta x^2(2k) &= -\frac{1}{40}x^2(2k)+\frac{1}{80}\cos(x^2(2k))+\frac{1}{30}, \\
\Delta x^{12}(2k) &= -\frac{1}{40}x^{12}(2k)+\frac{1}{80}\sin(x^{12}(2k))+\frac{1}{20}, \\
\Delta x^{12}(2k) &= -\frac{1}{40}x^{12}(2k)+\frac{1}{80}\cos(x^{12}(2k))+\frac{1}{80},
\end{aligned}
$$

Under the conditions of Theorem 1, system (13) possesses a unique exponentially stable WPAA solution. Numerical simulations (Figures 1–4) confirm this stability and reveal local chaotic behavior in the WPAA dynamics.

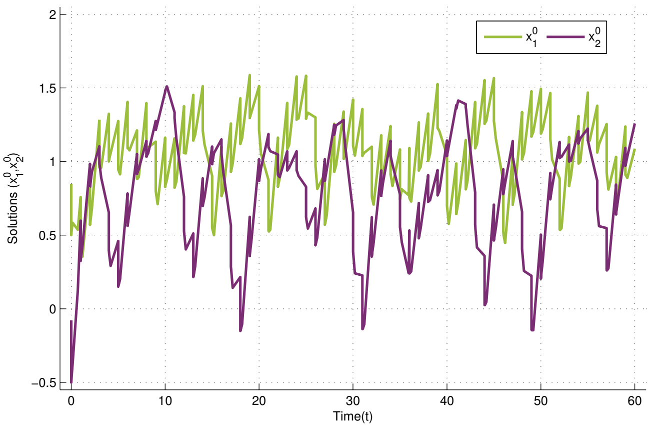

Figure 1. Curves of x_1^0 and x_2^0 of model (13).

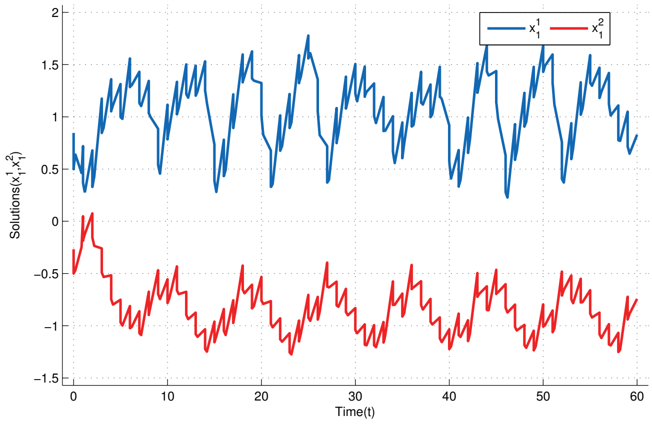

Figure 2. Curves of x_1^1 and x_2^1 of model (13).

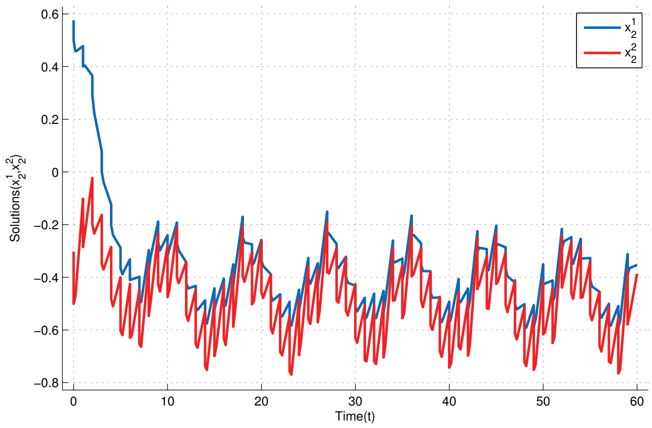

Figure 3. Curves of x_1^2 and x_2^2 of model (13).

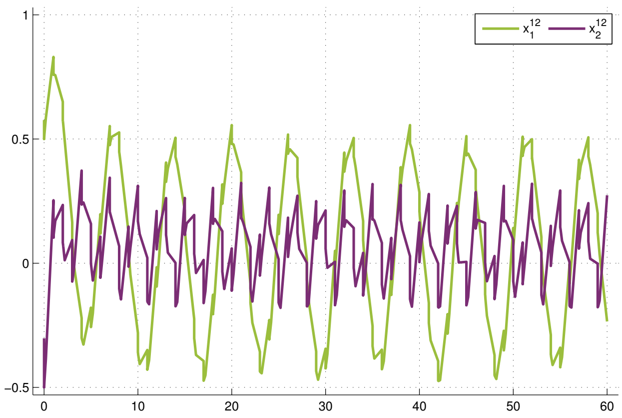

Figure 4. Curves of x_1^{12} and x_2^{12} of model (13).

Here, each figure displays the time evolution of one component of the Clifford-valued states for the two neurons. All trajectories remain bounded and exhibit irregular, non-periodic oscillations characteristic of weighted pseudo almost automorphic dynamics. The solutions do not diverge, confirming the exponential stability.

The observed complex patterns, including local chaotic like fluctuations, demonstrate the rich dynamical behavior that the Clifford-valued model can capture under the combined effects of high-order interactions, mixed delays, and impulses. These simulations validate the theoretical results and show the feasibility of the derived conditions.

4.2 Comparison with existing literature

- Our work differs substantially from recent studies such as Dong et al. in [15] analysis of piecewise pseudo almost periodic solutions for interval general BAM neural networks with mixed delays and impulsive perturbations. While their work employs real valued interval analysis in first-order BAM architectures, we investigate high-order Hopfield networks within the Clifford-algebraic framework, which naturally encodes geometric relationships via the geometric product without requiring decomposition. Furthermore, we consider weighted pseudo almost automorphic (WPAA) dynamics a broader function class than pseudo almost periodicity.
- Our study addresses a different line of inquiry compared to works such as Huo and Li's analysis of μ-almost periodic octonion-valued stochastic shunting inhibitory neural networks [17]. Their research examines stochastic dynamics and almost periodic behavior in distribution, whereas our focus is on deterministic Clifford-algebra–based systems governed by weighted pseudo almost automorphic (WPAA) solutions. Instead of using octonions which are non-associative and limited to eight dimensions, we employ Clifford algebra, an associative framework that can naturally scale to higher dimensions.
- Our work distinguishes itself from Clifford-valued Cohen–Grossberg networks [20] by establishing weighted pseudo almost automorphic solutions for non-decomposed Clifford-valued high-order Hopfield networks with mixed delays and impulsive perturbations, thereby extending and unifying these previous approaches within a broader algebraic and dynamical framework.

5 Conclusions

In this work, we have investigated impulsive HOHNNs in Clifford algebra with mixed delays. By employing the Banach fixed point theorem and a generalized Gronwall–Bellman inequality, we have established sufficient conditions for the existence, uniqueness, and exponential stability of weighted pseudo almost automorphic (WPAA) solutions. The key contributions and novel aspects of this paper are:

- First study of WPAA solutions in Clifford algebra: This is the first work to address the existence, uniqueness, and stability of WPAA solutions for impulsive HOHNNs within the Clifford-algebraic framework.
- Non-decomposed approach: All results are obtained without decomposing the Clifford-valued system into real-valued components.
- Unified modeling of high-order interactions: The model incorporates both first- and second-order synaptic connections, and the sufficient conditions explicitly account for the combined effects of continuous dynamics, time-varying delays, and impulsive jumps.
- Generalization of existing results: By working in the weighted pseudo almost automorphic setting, our results extend and unify earlier findings restricted to periodic, almost periodic, or almost automorphic solutions.
- Algebraic and geometric consistency: The conditions are formulated directly in terms of Clifford-algebraic norms and operations, maintaining the natural relationship between algebraic structure and dynamical behavior.

The theoretical findings are supported by a numerical example. This work opens a new direction for studying complex neural dynamics in hypercomplex algebras and provides a foundation for further research with discontinuous perturbations and high-order connections.

Data Availability Statement

No datasets were generated or analyzed during the current study.

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Conflicts of Interest

The authors declare no conflicts of interest.

AI Use Statement

The authors declare that no generative AI was used in the preparation of this manuscript.

Ethical Approval and Consent to Participate

Not applicable.

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