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RESEARCH ARTICLE

The TGCNMFE Method for the Generalized Nonlinear Time Fractional Fourth-Order Reaction Diffusion Equation

Yuejie Li[1] and **Zhendong Luo**[2,3,*]

[1] School of Mathematics and Computer Engineering, Ordos Institute of Technology, Ordos 017000, China
[2] School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China
[3] Academician Expert Workstation, Hunan Sany Polytechnic College, Changsha 410129, China

Abstract

Herein, we mainly focus on developing a new two-grid Crank-Nicolson (CN) mixed finite element (MFE) (TGCNMFE) method for the generalized nonlinear time fractional fourth-order reaction diffusion equation. To do so, by introducing an auxiliary function, the nonlinear time fractional fourth-order reaction diffusion equation is first split into two second-order nonlinear equations. Thereafter, a new time semi-discrete mixed CN (TSDMCN) scheme is constructed through discretizing the time derivative and time fractional derivative by the CN difference quotient, and the existence, steadiness, and errors of the TSDMCN solutions are analysed. Next, a new TGCNMFE method is developed through using two-grid MFE technique to discretize the spacial variables, and the existence, steadiness, and error estimations for the TGCNMFE solutions are discussed. Lastly, the correctness of theory results and the superiority of the TGCNMFE method are verified by some numerical experiments.

Keywords: numerical simulations, finite element method, finite difference scheme, and finite volume element method.

1 Introduction

Let $\Omega \subset R^d$ $(d = 2, 3)$ be a bounded and connected domain with the boundary $\partial\Omega$. For given time upper-limit t_e, we study the following generalized nonlinear time fractional fourth-order reaction diffusion equation (GNTFFORDE).

Problem 1 *Seek* $w : [0, t_e] \to C^4(\bar{\Omega})$ *from the following equation:*

$$\begin{cases} w_t(\boldsymbol{x},t) + D_t^{\alpha}\Delta w(\boldsymbol{x},t) \\ -\Delta w(\boldsymbol{x},t) + \Delta^2 w(\boldsymbol{x},t) \\ = f(w(\boldsymbol{x},t)) + g(\boldsymbol{x},t),\ (\boldsymbol{x},t) \in \Omega\times(0,t_e), \\ w(\boldsymbol{x},t) = \Delta w(\boldsymbol{x},t) = 0,\ (\boldsymbol{x},t)\in\partial\Omega\times(0,t_e), \\ w(\boldsymbol{x},0) = w_0(\boldsymbol{x}),\ \boldsymbol{x}\in\Omega, \end{cases} \tag{1}$$

in which $w_t(\boldsymbol{x},t) = \partial w(\boldsymbol{x},t)/\partial t$, $\boldsymbol{x} = (x_1, x_2, \cdots, x_d)$, $\Delta = \sum_{i=1}^{d} \partial^2/\partial x_i^2$ *is the Laplacian operator,* $f(w)$ *is*

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***Corresponding author:**
Zhendong Luo
zhdluol@ncepu.edu.cn

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a function with continuous second-order derivatives that satisfies $f(0) = 0$*, the source term* $g(\boldsymbol{x}, t)$ *and the initial function* $w_0(\boldsymbol{x})$ *are sufficiently smooth known functions, and* $D_t^\alpha \varphi$ $(0 < \alpha < 1)$ *stands for the* α*th-order Gerasimov–Caputo fractional derivative (see [3, 4]), which is denoted by*

$$D_t^\alpha w(\boldsymbol{x}, t) = \frac{\partial^\alpha w(\boldsymbol{x}, t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial w(\boldsymbol{x}, s)}{\partial s} \frac{\mathrm{d}s}{(t-s)^\alpha}, 0 \leqslant s \leqslant t \leqslant t_e. \quad (2)$$

Remark 1 *There are various selections for the nonlinear term* $f(w)$*.*

(i) If it is taken as $f(w) = \sin(w)$*, it is a time fractional fourth-order parabolic type sine-Gordon equation.*

(ii) If it is taken as $f(w) = (1 - w^2)$*, it is just the nonlinear time fractional fourth-order reaction diffusion equation in [1].*

Therefore, Problem 1 has wider applications than the above-mentioned partial differential equations (PDEs), named as so-called "GNTFFORDE".

The fractional derivative has been around for a long time. It was first proposed in the letter from Leibniz to L'Hospital in 1695 (see [2]). Later, it was popularized by Gerasimov [3] and Caputo [4] to propose the Gerasimov-Caputo fractional derivative [5]. It has been found that the fractional PDEs have very wide applications in fields such as physics, chemistry, and biology (see [6–10]). Peculiarly, the GNTFFORDE (Problem 1) is procured by adding a time fractional-order derivative and a generalized nonlinear term $f(w)$ to the standard fourth-order reaction-diffusion equation (see [11–13]), so its application scope is wider. It can be accustomed to describe other problems besides the interaction and proliferation for organisms (see [14]), the mesoscopic systems of phase transition for binary systems (see [15]), the marching waves for reaction diffusion system (see [16]), and the directing wave movement in nematic liquid crystals (see [17]), just like the standard fourth-order reaction-diffusion equation. Thereby, the research on the GNTFFORDE (i.e., Problem 1) has great significance.

However, owe to the complexity for the fractional PDEs, they generally can not be solved by analytical methods. It is the best selection to calculate their approximate solutions by numerical methods. The numerical methods for solving the fractional PDEs mostly include the finite difference (FD) scheme (see [18–22]), the finite element (FE) method (see [23–26]), the discontinuous Galerkin method (see [30, 31]), the collocation method (see [27–29]), and the meshless method (see [32]).

Whereas, the GNTFFORDE (i.e., Problem 1) with the nonlinear term $f(w)$ and the time fractional derivative as well as the fourth-order derivative term $\Delta^2 w$ is very difficult to calculate through the usual FE method.

Thereupon, the primary task for this article is to create a new TGCNMFE method to the GNTFFORDE. The TGCNMFE method has at the fewest the next three benefits. Firstly, by inducting an auxiliary function $\varphi = w - \Delta w$, the GNTFFORDE may be split into two second-order equations to facilitate the solution using lower-order FEs (such as linear or quadratic FEs) and procure the optimal order error estimation. Secondly, the TGCNMFE method is unconditionally stable, thus allowing for a longer time step in numerical calculations. Thirdly, the TGCNMFE method is made up of a few nonlinear equations on coarser meshes and the linear equations on sufficiently fine meshes, which can greatly simplify the computing process and enhance the calculating efficiency.

Although for the nonlinear time fractional fourth-order reaction diffusion equation with nonlinear term $f(w) = w(1 - w^2)$, a general two-grid MFE method without adopting CN technique was provided in [1], it is not a conditional stable scheme in time and had not provided the theoretical analysis of time semi-discrete solutions. Therefore, it completely differs from the TGCNMFE method of this paper. In another words, the TGCNMFE method of this paper is a new development.

The rest for this article is composed of the following four sections. In Section 2, we construct a new TSDMCN scheme for the GNTFFORDE and discuss the existence, stability, and error estimations of the TSDMCN solutions. In Section 3, we design a new TGCNMFE method for GNTFFORDE and analyze the existence, unconditional stability, and errors of the TGCNMFE solutions. In Section 4, we resort to some numerical experiments to confirm the procured theory results and the advantage of the TGCNMFE method. Lastly, we offer the major conclusions of this paper and the prospects for future research in Section 5.

2 A New TSDMCN Scheme

The Sobolev spaces and their norms adopted subsequently are classical (see [33–35]). Let $\mathbb{W} = H_0^1(\Omega)$ and $\varphi = w - \Delta w$. In order to facilitate theory analysis and without loss of generality, we assume that $g(\boldsymbol{x},t)=0$. Thereupon, with the Green formula, we may build the below weak form of Problem 1.

Problem 2 *$\forall t\in(0,\ t_e)$, seek $(w,\varphi)\in\mathbb{W}\times\mathbb{W}$ from the following system of equations:*

$$\begin{cases}(\upsilon,w_t)+(\upsilon,D_t^\alpha\varphi)+(\nabla\upsilon,\nabla\varphi)\\ \quad=(\upsilon,f(w)),\quad\forall\upsilon\in\mathbb{W},\\ (\vartheta,w)+(\nabla\vartheta,\nabla w)=(\vartheta,\varphi),\quad\forall\vartheta\in\mathbb{W},\\ w(\boldsymbol{x},0)=w_0(\boldsymbol{x}),\boldsymbol{x}\in\Omega,\end{cases}\tag{3}$$

herein $(\varphi,\vartheta)=\int_\Omega\varphi\cdot\vartheta\mathrm{d}\boldsymbol{x}$.

By using the proof method in [8] or the proof method as the next Theorem 1, the existence and stability for generalized solutions to Problem 2 may be proven.

In order to build the TGCNMFE method, we firstly set up a bran-new TSDMCN scheme. To do so, we suppose that $N>0$ is an integer, $\Delta t=t_e/N$ indicates the time step, and φ^n and w^n stand for the approximations to $\varphi(\boldsymbol{x},t)$ and $w(\boldsymbol{x},t)$ at $t_n=n\Delta t$ $(0\leqslant n\leqslant N)$ separately. Thus, when $\varphi=w-\Delta w$, i.e., w is fully smooth in time, the αth-order Gerasimov–Caputo fractional derivative is expanded as follows (see [18])

$$D_t^\alpha w(\boldsymbol{x},t_n)=\frac{\partial^\alpha w(\boldsymbol{x},t_n)}{\partial t^\alpha}=\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^n a_{n-k}(w^k-w^{k-1})+\varepsilon_0(\boldsymbol{x}),\tag{4}$$

in which $a_{n-k}=(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}>0$, $|\varepsilon_0|\leqslant C_\varphi\Delta t^{2-\alpha}$, and C_φ is a constant dependent on φ and t_e. The above-mentioned coefficients a_{n-k} meet

$$\begin{cases}1=a_0>a_1>a_2>\cdots>a_n>0,\\ a_n\to0\ (n\to\infty),\\ \sum_{j=1}^{n-1}(a_{j-1}-a_j)=a_0-a_{n-1}<1,\ \ 1\leqslant n\leqslant N,\\ \sum_{j=1}^{n-1}a_j\leqslant k^{1-\alpha},\ \ 1\leqslant n\leqslant N.\end{cases}\tag{5}$$

Using an implicit FD scheme to discretize time for the first equation in Problem 2 yields

$$\frac{1}{\Delta t}\left(w^n-w^{n-1},\upsilon\right)+(\nabla\varphi^n,\nabla\upsilon)+\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^n a_{n-k}(w^k-w^{k-1},\upsilon)=(f(w^n),\upsilon),\quad\forall\upsilon\in\mathbb{W}.\tag{6}$$

Using an explicit FD scheme to discretize time for the first equation in Problem 2 yields

$$\frac{1}{\Delta t}\left(w^n-w^{n-1},\upsilon\right)+(\nabla\varphi^{n-1},\nabla\upsilon)+\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^n a_{n-k}(w^k-w^{k-1},\upsilon)=(f(w^{n-1}),\upsilon),\ \forall\upsilon\in\mathbb{W}.\tag{7}$$

By adding (6) to (7), we obtain the following brand-new TSDMCN scheme, which is distinguished from the existed time semi-discrete schemes, including that in [1].

Problem 3 *Seek $\{(w^n,\varphi^n)\}\in\mathbb{W}\times\mathbb{W}$ $(1\leqslant n\leqslant N)$ by the following system of equations*

$$\frac{1}{\Delta t}(w^n-w^{n-1},\upsilon)+\frac{1}{2}(\nabla(\varphi^n+\varphi^{n-1}),\nabla\upsilon)+\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^n a_{n-k}(w^k-w^{k-1},\upsilon)=\frac{1}{2}(f(w^n)+f(w^{n-1}),\upsilon),\quad\forall\upsilon\in\mathbb{W},\ 1\leqslant n\leqslant N,\tag{8}$$

$$(\nabla w^n,\nabla\vartheta)+(w^n,\vartheta)=(\varphi^n,\vartheta),\ \forall\vartheta\in\mathbb{W},\ 0\leqslant n\leqslant N,\tag{9}$$

$$w^0=w_0(\boldsymbol{x}),\ \ \varphi^0=-\Delta w_0(\boldsymbol{x}),\ \boldsymbol{x}\in\Omega.\tag{10}$$

The next discrete Gronwall lemma (see [36, Lemma 3.1]) is often used in succeeding theoretical analysis.

Lemma 1 *Let $\{b_n\}$ be a nonnegative real number sequence, $\{c_n\}$ is a non-descending real number sequence, $\{\delta_n\}$ be also a nonnegative real number sequence, and they satisfy $b_n\leqslant e_n+\sum_{j=1}^{n-1}\delta_jb_j$ $(n\geqslant1)$, then they also satisfy $b_n\leqslant e_n\exp\left(\sum_{j=1}^{n-1}\delta_j\right)$ $(n\geqslant1)$.*

For Problem 3, we obtain the following results.

Theorem 1 *From Problem 3 we can find a unique solution set $\{(w^n,\varphi^n)\}_{n=1}^N\subset\mathbb{W}\times\mathbb{W}$ to meet the following boundedness (i.e., stability):*

$$\|\nabla w^n\|_0+\|\nabla\varphi^n\|_0\leqslant c\|w_0\|_1,\ 1\leqslant n\leqslant N.\tag{11}$$

where and subsequent $c>0$ is a constant independent of Δt. Furthermore, while $w_0(\boldsymbol{x})$ is smooth enough, the error estimations for the TSDCNM solutions $\{(w^n,\varphi^n)\}_{n=1}^N$ to Problem 3 are reckoned by the following inequalities

$$\|\nabla(w(t_n)-w^n)\|_0+\|\nabla(\varphi(t_n)-\varphi^n)\|_0\leqslant c\Delta t^{2-\alpha},\ \ 1\leqslant n\leqslant N,\tag{12}$$

where $w(t_n)=w(\boldsymbol{x},t_n)$ and $\varphi(t_n)=\varphi(\boldsymbol{x},t_n)$.

Proof. Theorem 1 is proven by the below three parts.

(1) *Prove the existence and uniqueness for the TSDCNM solutions.*

Taking $\vartheta = w^n - w^{n-1}$ in (9), we obtain

$$
\begin{aligned}
&(\varphi^n - \varphi^{n-1}, w^n - w^{n-1}) \\
&= (\nabla(w^n - w^{n-1}), \nabla(w^n - w^{n-1})) \\
&\quad + (w^n - w^{n-1}, w^n - w^{n-1}) \\
&= \|\nabla(w^n - w^{n-1})\|_0^2 + \|w^n - w^{n-1}\|_0^2. \quad (13) \\
&(\varphi^k - \varphi^{k-1}, w^n - w^{n-1}) \\
&= (\nabla(w^k - w^{k-1}), \nabla(w^n - w^{n-1})) \\
&\quad + (w^k - w^{k-1}, w^n - w^{n-1}) \\
&= \|\nabla(w^k - w^{k-1})\|_0^2 + \|w^k - w^{k-1}\|_0^2 \\
&+\|\nabla(w^n - w^{n-1})\|_0^2 + \|w^n - w^{n-1}\|_0^2. \quad (14)
\end{aligned}
$$

Taking $\vartheta = w^n$ in (9) and using the Hölder inequality, we obtain

$$
\begin{aligned}
&\|\nabla w^n\|_0^2 + \|w^n\|_0^2 = (\varphi^n, w^n) \\
&\leqslant \|\varphi^n\|_0 \|w^n\|_0 \\
&\leqslant \frac{1}{2}(\|\varphi^n\|_0^2 + \|w^n\|_0^2), \quad 1 \leqslant n \leqslant N. \quad (15)
\end{aligned}
$$

Thereupon, we procure

$$\|\nabla w^n\|_0^2 + \|w^n\|_0^2 \leqslant \|\varphi^n\|_0^2, \quad 1 \leqslant n \leqslant N. \quad (16)$$

Taking $\upsilon = \varphi^n - \varphi^{n-1}$ in (10), and using (13), (14), the Hölder and Cauchy inequalities, and differential mean value theorem (DMVT), we obtain

$$
\begin{aligned}
&\|\nabla(w^n - w^{n-1})\|_0^2 + \|w^n - w^{n-1}\|_0^2 \\
&+\frac{\Delta t}{2}(\|\nabla\varphi^n\|_0^2 - \|\nabla\varphi^{n-1}\|_0^2) \\
&= (\varphi^n - \varphi^{n-1}, w^n - w^{n-1}) \\
&+\frac{\Delta t}{2}(\nabla(\varphi^n + \varphi^{n-1}), \nabla(\varphi^n - \varphi^{n-1})) \\
&= \frac{\Delta t}{2}(f(w^n) + f(w^{n-1}), w^n - w^{n-1}) \\
&-\frac{\Delta t^{1-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n} a_{n-k}(\varphi^k - \varphi^{k-1}, w^n - w^{n-1}) \\
&\leqslant \frac{1}{2}(\|\nabla(w^n - w^{n-1})\|_0^2 + \|w^n - w^{n-1}\|_0^2) \\
&+c\Delta t^{2-2\alpha}\sum_{k=1}^{n-1} a_{n-k}\|\nabla(w^k - w^{k-1})\|_0^2 \\
&+c\Delta t^{2-2\alpha}\sum_{k=1}^{n-1} a_{n-k}\|w^k - w^{k-1}\|_0^2 \\
&+c\Delta t^2(\|w^n\|_0^2 + \|w^{n-1}\|_0^2), 1 \leqslant n \leqslant N. \quad (17)
\end{aligned}
$$

Simplifying (17) and using (16) yields

$$
\begin{aligned}
&\|w^n - w^{n-1}\|_0^2 + \|\nabla(w^n - w^{n-1})\|_0^2 \\
&+\Delta t(\|\nabla\varphi^n\|_0^2 - \|\nabla\varphi^{n-1}\|_0^2) \\
&\leqslant c\Delta t^{2-2\alpha}\sum_{k=1}^{n-1} a_{n-k}\|\nabla(w^k - w^{k-1})\|_0^2 \\
&+c\Delta t^{2-2\alpha}\sum_{k=1}^{n-1} a_{n-k}\|w^k - w^{k-1}\|_0^2 \\
&+c\Delta t^2(\|\varphi^n\|_0^2 + \|\varphi^{n-1}\|_0^2), 1 \leqslant n \leqslant N. \quad (18)
\end{aligned}
$$

Summating (18) from 1 until n ($n \leqslant N$), while $\triangle t$ is sufficiently small to meet $c\Delta t \leqslant 1/2$, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n}(\|\nabla(w^i - w^{i-1})\|_0^2 + \|w^i - w^{i-1}\|_0^2) \\
&+\Delta t\|\nabla\varphi^n\|_0^2 \\
&\leqslant c\Delta t^{2-2\alpha}\sum_{i=1}^{n-1}\sum_{k=1}^{i-1} a_{n-k}\|\nabla(w^k - w^{k-1})\|_0^2 \\
&+c\Delta t^{2-2\alpha}\sum_{i=1}^{n-1}\sum_{k=1}^{i-1} a_{n-k}\|w^k - w^{k-1}\|_0^2 \\
&+\Delta t\|\nabla\varphi^0\|_0^2 + c\Delta t^2\sum_{i=0}^{n-1}\|\varphi^i\|_0^2, 1 \leqslant n \leqslant N. \quad (19)
\end{aligned}
$$

Applying Lemma 1 to (19), and using (5), we procure

$$
\begin{aligned}
&\sum_{i=1}^{n}(\|\nabla(w^i - w^{i-1})\|_0^2 + \|w^i - w^{i-1}\|_0^2) \\
&+\Delta t\|\nabla\varphi^n\|_0^2 \\
&\leqslant \Delta t\|\nabla\varphi^0\|_0^2 \exp\left(c\Delta t^{2-2\alpha}\sum_{k=1}^{i-n} a_{n-k} + c\Delta t^2 n\right) \\
&\leqslant c\Delta t\|\nabla\varphi^0\|_0^2, 1 \leqslant n \leqslant N. \quad (20)
\end{aligned}
$$

Thereupon, by (20) and (16), we procure

$$\|\nabla w^n\|_0 + \|\nabla\varphi^n\|_0 \leqslant c\|\nabla\varphi^0\|_0, 1 \leqslant n \leqslant N. \quad (21)$$

Thus, if $\varphi^0 = 0$, then from (21) we can assert that $w^n = \varphi^n = 0$ $(1 \leqslant n \leqslant N)$. This means that Problem 3 exists at the fewest a solution set $\{(w^n, \varphi^n)\}_{n=1}^N$.

If Problem 3 has another solution set $\{(\tilde{w}^n, \tilde{\varphi}^n)\}_{n=1}^N$, it should meet the below system of equations:

$$\begin{cases} \frac{1}{\Delta t}\left(\tilde{w}^n-\tilde{w}^{n-1},\upsilon\right)+\frac{1}{2}\left(\nabla(\tilde{\varphi}^n+\tilde{\varphi}^{n-1}),\nabla\upsilon\right) \\ +\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n}a_{n-k}(\tilde{\varphi}^k-\tilde{\varphi}^{k-1},\upsilon) \\ =\frac{1}{2}\left(f(\tilde{w}^n)+f(\tilde{w}^{n-1}),\upsilon\right),\forall\upsilon\in\mathbb{W},1\leqslant n\leqslant N, \\ (\nabla\tilde{w}^n,\nabla\vartheta)+(\tilde{w}^n,\vartheta) \\ =(\tilde{\varphi}^n,\vartheta),\forall\vartheta\in\mathbb{W},0\leqslant n\leqslant N, \\ \tilde{w}^0=w_0(\boldsymbol{x}),\quad \tilde{\varphi}^0=-\Delta w_0(\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega. \end{cases} \tag{22}$$

Let $E^n=w^n-\tilde{w}^n$ and $e^n=\varphi^n-\tilde{\varphi}^n$. Subtracting (22) from (8)–(8) produces

$$\begin{cases} \frac{1}{\Delta t}\left(E^n-E^{n-1},\upsilon\right)+\frac{1}{2}\left(\nabla(e^n+e^{n-1}),\nabla\upsilon\right) \\ +\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n}a_{n-k}(E^k-E^{k-1},\upsilon) \\ =\frac{1}{2}\left(E^nf'(\xi_n)+E^{n-1}f'(\xi_{n-1}),\upsilon\right), \\ \forall\upsilon\in\mathbb{W},\quad 1\leqslant n\leqslant N, \\ (\nabla E^n,\nabla\vartheta)+(E^n,\vartheta)=(e^n,\vartheta), \\ \quad\forall\vartheta\in\mathbb{W},\quad 0\leqslant n\leqslant N, \\ E^0=0,\quad e^0=0,\quad \boldsymbol{x}\in\Omega, \end{cases} \tag{23}$$

here ξ_i lies between w^i and $\tilde{w}^i$ $(i=n,n-1)$.

Using the second equation of (23), the Hölder and Cauchy inequalities, we procure

$$\begin{aligned} &\|\nabla(E^n-E^{n-1})\|_0^2+\|E^n-E^{n-1}\|_0^2 \\ &=(\nabla(E^n-E^{n-1}),\nabla(E^n-E^{n-1})) \\ &+(E^n-E^{n-1},E^n-E^{n-1}) \\ &=(E^n-E^{n-1},e^n-e^{n-1}) \\ &\leqslant\frac{1}{2}(\|E^n-E^{n-1}\|_0^2+\|e^n-e^{n-1}\|_0^2), \end{aligned} \tag{24}$$

$$\begin{aligned} &\|\nabla(E^n-E^{n-1})\|_0^2+\|E^n-E^{n-1}\|_0^2 \\ &\leqslant\|e^n-e^{n-1}\|_0^2. \end{aligned} \tag{25}$$

$$\|\nabla E^n\|_0^2+\|E^n\|_0^2\leqslant\|e^n\|_0^2, \tag{26}$$

$$\begin{aligned} &\frac{2\Delta t^{1-\alpha}}{\Gamma(1-\alpha)}(E^k-E^{k-1},e^n-e^{n-1}) \\ &=\frac{2\Delta t^{1-\alpha}}{\Gamma(1-\alpha)}[(\nabla(E^k-E^{k-1}),\nabla(E^n-E^{n-1})) \\ &+(E^k-E^{k-1},E^n-E^{n-1})] \\ &\leqslant c\Delta t^{2-2\alpha}(\|\nabla(E^k-E^{k-1})\|_0^2+\|E^k-E^{k-1}\|_0^2) \\ &+\frac{1}{2}(\|\nabla(E^n-E^{n-1})\|_0^2+\|E^n-E^{n-1}\|_0^2), \end{aligned} \tag{27}$$

$$\begin{aligned} &\Delta tf'(\xi_n)(E^k,e^n-e^{n-1}) \\ &=\Delta tf'(\xi_n)[(\nabla E^k,\nabla(E^n-E^{n-1})) \\ &\quad+(E^k,E^n-E^{n-1})] \\ &\leqslant\frac{1}{4}(\|\nabla(E^k-E^{k-1})\|_0^2+\|E^k-E^{k-1}\|_0^2) \\ &+c\Delta t^2(\|\nabla E^n\|_0^2+\|E^n\|_0^2) \\ &\leqslant\frac{1}{4}(\|\nabla(E^k-E^{k-1})\|_0^2+\|E^k-E^{k-1}\|_0^2) \\ &+c\Delta t^2\|e^n\|_0^2, \end{aligned} \tag{28}$$

$$\begin{aligned} &\Delta tf'(\xi_{n-1})(E^{k-1},e^n-e^{n-1}) \\ &=\Delta tf'(\xi_{n-1})[(\nabla E^{k-1},\nabla(E^n-E^{n-1})) \\ &\quad+(E^{k-1},E^n-E^{n-1})] \\ &\leqslant\frac{1}{4}(\|\nabla(E^k-E^{k-1})\|_0^2+\|E^k-E^{k-1}\|_0^2) \\ &+c\Delta t^2(\|\nabla E^{n-1}\|_0^2+\|E^{n-1}\|_0^2) \\ &\leqslant\frac{1}{24}(\|\nabla(E^k-E^{k-1})\|_0^2+\|E^k-E^{k-1}\|_0^2) \\ &+c\Delta t^2\|e^{n-1}\|_0^2. \end{aligned} \tag{29}$$

Taking $\upsilon=e^n-e^{n-1}$ in the first equation of (23), by (24)–(29), the Hölder and Cauchy inequalities, and the DMVT, noting that $E^0=e^0=0$, we get

$$\begin{aligned} &2(\|\nabla(E^n-E^{n-1})\|_0^2+\|E^n-E^{n-1}\|_0^2) \\ &+\Delta t(\|\nabla e^n\|_0^2-\|\nabla e^{n-1})\|_0^2) \\ &=2\left(E^n-E^{n-1},e^n-e^{n-1}\right) \\ &+\Delta t\left(\nabla(e^n+e^{n-1}),\nabla(e^n-e^{n-1})\right) \\ &=\Delta t\left(E^nf'(\xi_n)+E^{n-1}f'(\xi_{n-1}),e^n-e^{n-1}\right) \\ &-\frac{2\Delta t^{1-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n}a_{n-k}(E^k-E^{k-1},e^n-e^{n-1}) \\ &\leqslant\|\nabla(E^n-E^{n-1})\|_0^2+\|E^n-E^{n-1}\|_0^2 \\ &+c\Delta t^2(\|e^n\|_0^2+\|e^{n-1}\|_0^2) \\ &+c\Delta t^{2-2\alpha}\sum_{k=1}^{n-1}a_{n-k}(\|\nabla(E^k-E^{k-1})\|_0^2 \\ &+\|E^k-E^{k-1}\|_0^2),\quad 1\leqslant n\leqslant N. \end{aligned} \tag{30}$$

It follows that

$$\begin{aligned} &\|\nabla(E^n-E^{n-1})\|_0^2+\|E^n-E^{n-1}\|_0^2 \\ &\quad+\Delta t(\|\nabla e^n\|_0^2-\|\nabla e^{n-1})\|_0^2) \\ &\leqslant c\Delta t^2(\|e^n\|_0^2+\|e^{n-1}\|_0^2) \\ &+c\Delta t^{2-2\alpha}\sum_{k=1}^{n-1}a_{n-k}(\|\nabla(E^k-E^{k-1})\|_0^2 \\ &+\|E^k-E^{k-1}\|_0^2),\quad 1\leqslant n\leqslant N. \end{aligned} \tag{31}$$

Summating (31) from 1 until n $(n\leqslant N)$, when $c\Delta t$ is sufficiently small to satisfy $c\Delta t\leqslant 0.5$, we procure

$$\sum_{i=1}^{n}(\|\nabla(E^i-E^{i-1})\|_0^2+\|E^i-E^{i-1}\|_0^2)$$

$$+\Delta t\|\nabla e^n\|_0^2 \leqslant c\Delta t^2\sum_{i=1}^{n-1}\|e^n\|_0^2$$
$$+c\Delta t^{2-2\alpha}\sum_{i=1}^{n}\sum_{k=1}^{i-1}a_{n-k}(\|\nabla(E^k-E^{k-1})\|_0^2$$
$$+\|E^k-E^{k-1}\|_0^2),\quad 1\leqslant n\leqslant N. \tag{32}$$

Applying Lemma 1 to (32) procures

$$\sum_{i=1}^{n}(\|\nabla(E^i-E^{i-1})\|_0^2+\|E^i-E^{i-1}\|_0^2)$$
$$+\Delta t\|\nabla e^n\|_0^2=0,\quad 1\leqslant n\leqslant N. \tag{33}$$

By (26) and (33), we obtain $e^n=E^n=0$, i.e, $w^n=\tilde{w}^n$ and $\varphi^n=\tilde{\varphi}^n$ $(1\leqslant n\leqslant N)$.

Thereupon, Problem 3 exists a sole solution set $\{(w^n,\varphi^n)\}_{n=1}^N$.

(2) *Analyze the stability of solutions* $\{(w^n,\varphi^n)\}_{n=1}^N$.

The above first step has procured that Problem 3 has a unique TSDMCN solution set $\{(w^n,\varphi^n)\}_{n=1}^N$. By (21), we conclude that the set of solutions $\{w^n,\varphi^n\}_{n=1}^N$ for Problem 3 is bounded, i.e., stable.

(3) *Reckon the errors for the TSDMCN solutions* $\{(w^n,\varphi^n)\}_{n=1}^N$.

Using Taylor's formula, we obtain

$$w'(t_{n-\frac{1}{2}})=\frac{w(t_n)-w(t_{n-1})}{\Delta t}$$
$$-\frac{\Delta t^2}{24}w'''(\xi_n),\quad t_{n-1}\leqslant\xi_n\leqslant t_{n+1}. \tag{34}$$
$$w(t_{n-\frac{1}{2}})=\frac{w(t_n)+w(t_{n-1})}{2}$$
$$-\frac{\Delta t^2}{16}w''(\varsigma_n),\quad t_{n-1}\leqslant\varsigma_n\leqslant t_{n-\frac{1}{2}}. \tag{35}$$
$$f(w(t_{n-\frac{1}{2}}))=\frac{f(w(t_n))+f(w(t_{n-1}))}{2}$$
$$-\Delta t^2R(\boldsymbol{x},t), \tag{36}$$

in which $R(\boldsymbol{x},t)$ is a bounded remainder.

Thereupon, subtracting (8)–(10) from (3) after taking $t=t_{n-\frac{1}{2}}$ and setting $\rho^n=w(\boldsymbol{x},t_n)-w^n$ and $\varrho^n=\varphi(\boldsymbol{x},t_n)-\varphi^n$, we obtain the following system of error equations.

$$\frac{1}{\Delta t}\left(\rho^n-\rho^{n-1},\upsilon\right)+\frac{1}{2}\left(\nabla(\varrho^n+\varrho^{n-1}),\nabla\upsilon\right)$$
$$=\frac{1}{2}\left(f'(\eta^n)\rho^n+f'(\eta^{n-1})\rho^{n-1},\upsilon\right)$$
$$-\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n}a_{n-k}(\rho^k-\rho^{k-1},\upsilon)$$
$$+\left(\varepsilon_0(\boldsymbol{x},t),\upsilon\right)+\frac{\Delta t^2}{24}\left(w'''(\xi_n),\upsilon\right)$$
$$+\frac{\Delta t^2}{16}\left(\varphi''(\varsigma_n),\upsilon\right)+\frac{\Delta t^2}{16}\left(\nabla\varphi''(\varsigma_n),\nabla\upsilon\right)$$
$$-\Delta t^2\left(R(\boldsymbol{x},t),\upsilon\right),1\leqslant n\leqslant N,\forall\upsilon\in\mathbb{W}, \tag{37}$$
$$(\nabla\rho^n,\nabla\vartheta)+(\rho^n,\vartheta)=(\varrho^n,\vartheta),$$
$$0\leqslant n\leqslant N,\quad\forall\vartheta\in\mathbb{W}, \tag{38}$$
$$\rho^0=\varrho^0=0. \tag{39}$$

By (38), the Hölder and Cauchy inequalities, we procure

$$\|\nabla(\rho^{n-1}-\rho^n)\|_0^2+\|\rho^{n-1}-\rho^n\|_0^2$$
$$=(\nabla(\rho^{n-1}-\rho^n),\nabla(\rho^{n-1}-\rho^n))$$
$$+(\rho^{n-1}-\rho^n,\rho^{n-1}-\rho^n)$$
$$=(\rho^{n-1}-\rho^n,\varrho^{n-1}-\varrho^n)$$
$$\leqslant\frac{1}{2}(\|\rho^{n-1}-\rho^n\|_0^2+\|\varrho^{n-1}-\varrho^n\|_0^2), \tag{40}$$
$$\|\nabla(\rho^{n-1}-\rho^n)\|_0^2+\|\rho^{n-1}-\rho^n\|_0^2$$
$$\leqslant\|\varrho^{n-1}-\varrho^n\|_0^2. \tag{41}$$
$$\|\nabla\rho^n\|_0^2+\|\rho^n\|_0^2\leqslant\|\varrho^n\|_0^2, \tag{42}$$
$$\frac{2\Delta t^{1-\alpha}}{\Gamma(1-\alpha)}(\rho^{k-1}-\rho^k,\varrho^{n-1}-\varrho^n)$$
$$=\frac{2\Delta t^{1-\alpha}}{\Gamma(1-\alpha)}[(\nabla(\rho^{k-1}-\rho^k),\nabla(\rho^{n-1}-\rho^n))$$
$$+(\rho^{k-1}-\rho^k,\rho^{n-1}-\rho^n)]$$
$$\leqslant c\Delta t^{2-2\alpha}(\|\nabla(\rho^{k-1}-\rho^k)\|_0^2+\|\rho^{k-1}-\rho^k\|_0^2)$$
$$+\frac{1}{2}(\|\nabla(\rho^{n-1}-\rho^n)\|_0^2+\|\rho^{n-1}-\rho^n\|_0^2), \tag{43}$$
$$\Delta tf'(\eta_n)(\rho^k,\varrho^n-\varrho^{n-1})$$
$$=\Delta tf'(\eta_n)[(\nabla\rho^k,\nabla(\rho^n-\rho^{n-1}))$$
$$+(\rho^k,\rho^n-\rho^{n-1})]$$
$$\leqslant\frac{1}{4}(\|\nabla(\rho^k-\rho^{k-1})\|_0^2+\|\rho^k-\rho^{k-1}\|_0^2)$$
$$+c\Delta t^2(\|\nabla\rho^n\|_0^2+\|\rho^n\|_0^2)$$
$$\leqslant\frac{1}{4}(\|\nabla(\rho^{k-1}-\rho^k)\|_0^2+\|\rho^{k-1}-\rho^k\|_0^2)$$
$$+c\Delta t^2\|\varrho^n\|_0^2, \tag{44}$$
$$\Delta tf'(\eta_{n-1})(\rho^{k-1},\varrho^n-\varrho^{n-1})$$
$$=\Delta tf'(\xi_n)[(\nabla\rho^{k-1},\nabla(\rho^n-\rho^{n-1}))$$
$$+(\rho^{k-1},\rho^n-\rho^{n-1})]$$
$$\leqslant\frac{1}{4}(\|\nabla(\rho^{k-1}-\rho^k)\|_0^2+\|\rho^{k-1}-\rho^k\|_0^2)$$
$$+c\Delta t^2(\|\nabla\rho^{n-1}\|_0^2+\|\rho^{n-1}\|_0^2)$$
$$\leqslant\frac{1}{24}(\|\nabla(\rho^{k-1}-\rho^k)\|_0^2+\|\rho^{k-1}-\rho^k\|_0^2)$$

$$+c\Delta t^2\|\varrho^{n-1}\|_0^2. \tag{45}$$

Taking $\upsilon=\varrho^n-\varrho^{n-1}$ in (37), by (40)–(45), the Hölder and Cauchy inequalities, the Green formula as well as the DMVT, we procure

$$\begin{aligned}
&2(\|\nabla(\rho^n-\rho^{n-1})\|_0^2+\|\rho^n-\rho^{n-1}\|_0^2)\\
&+\Delta t(\|\nabla\varrho^n\|_0^2-\|\nabla\varrho^{n-1}\|_0^2)\\
&=2(\rho^n-\rho^{n-1},\varrho^n-\varrho^{n-1})\\
&+\Delta t(\varrho^n+\varrho^{n-1},\varrho^n-\varrho^{n-1})\\
&=\Delta t\left(f'(\eta^n)\rho^n+f'(\eta^{n-1})\rho^{n-1},\varrho^n-\varrho^{n-1}\right)\\
&+\frac{\Delta t^5}{12}\left(w'''(\xi_n),\varrho^n-\varrho^{n-1}\right)\\
&-\frac{2\Delta t^{1-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n}a_{n-k}(\rho^k-\rho^{k-1},\varrho^n-\varrho^{n-1})\\
&+\Delta t\left(\varepsilon_0(\boldsymbol{x},t),\varrho^n-\varrho^{n-1}\right)\\
&-\frac{\Delta t^3}{8}\left(\Delta\varphi''(\varsigma_n),\varrho^n-\varrho^{n-1}\right)\\
&+\frac{\Delta t^3}{8}\left(\varphi''(\varsigma_n),\varrho^n-\varrho^{n-1}\right)\\
&-\Delta t^3\left(R(\boldsymbol{x},t),\varrho^n-\varrho^{n-1}\right)\\
&\leqslant c\Delta t^5-\Delta t(\|\varrho^n\|_0^2-\|\varrho^{n-1}\|_0^2)+c\Delta t^{5-2\alpha}\\
&+\|\nabla(\rho^n-\rho^{n-1})\|_0^2+\|\rho^n-\rho^{n-1}\|_0^2\\
&+c\Delta t^{2-2\alpha}\sum_{k=1}^{n-1}a_{n-k}[\|\nabla(\rho^k-\rho^{k-1})\|_0^2\\
&+\|\rho^k-\rho^{k-1}\|_0^2],\ \ 1\leqslant n\leqslant N.
\end{aligned}\tag{46}$$

Simplifying (46) yiels

$$\begin{aligned}
&\|\nabla(\rho^n-\rho^{n-1})\|_0^2+\|\rho^n-\rho^{n-1}\|_0^2\\
&+\Delta t(\|\nabla\varrho^n\|_0^2-\|\nabla\varrho^{n-1}\|_0^2)\\
&\leqslant c\Delta t^{2-2\alpha}\sum_{k=1}^{n-1}a_{n-k}(\|\nabla(\rho^k-\rho^{k-1})\|_0^2\\
&+\|\rho^k-\rho^{k-1}\|_0^2)+c\Delta t^{5-2\alpha},\ 1\leqslant n\leqslant N.
\end{aligned}\tag{47}$$

Summating (47) from 1 until n ($n\leqslant N$) and noting that $\rho^0=0$, we procure

$$\begin{aligned}
&\sum_{i=1}^{n}(\|\nabla(\rho^n-\rho^{n-1})\|_0^2+\|\rho^n-\rho^{n-1}\|_0^2)\\
&+\Delta t\|\nabla\varrho^n\|_0^2\\
&\leqslant c\Delta t^{2-2\alpha}\sum_{i=1}^{n}\sum_{k=1}^{i-1}a_{n-k}(\|\nabla(\rho^k-\rho^{k-1})\|_0^2\\
&+\|\rho^k-\rho^{k-1}\|_0^2)+c\Delta t^{5-2\alpha},\ 1\leqslant n\leqslant N.
\end{aligned}\tag{48}$$

Applying Lemma 1 to (48) procures

$$\begin{aligned}
&\sum_{i=1}^{n}(\|\nabla(\rho^n-\rho^{n-1})\|_0^2+\|\rho^n-\rho^{n-1}\|_0^2)\\
&+\Delta t\|\nabla\varrho^n\|_0^2\\
&\leqslant c\Delta t^{5-2\alpha}\exp\left(c\Delta t^{2-2\alpha}\sum_{i=1}^{n}\sum_{k=1}^{i-1}a_{n-k}\right)\\
&\leqslant c\Delta t^{5-2\alpha},\ \ 1\leqslant n\leqslant N.
\end{aligned}\tag{49}$$

By (49) and (42), we get (12). Theorem 1 is proved.□

Remark 2 *Theorem 1 shows that the error estimations of the TSDMCN solutions can achieve the optimal order $O(\Delta t^{2-\alpha})$.*

3 The TGCNMFE Method

To build the TGCNMFE method needs to further resort to the two-grid MFE method to discretize the spacial variables of Problem 3. To do so, let $\Im_H$ be a quasi-uniform coarse mesh partition on $\bar{\Omega}$ and $H=\sup\limits_{E\in\Im_H}\{\sup\limits_{\boldsymbol{x},\boldsymbol{y}\in E}\|\boldsymbol{x}-\boldsymbol{y}\|\}$. For $\forall E\in\Im_H$ and integer $l\geqslant 1$, if $\mathbb{P}_l(E)$ indicates the polynomial space on E with degree $\leqslant l$, then the FE subspace on the coarse meshes $\Im_H$ can be defined in the following

$$\mathbb{W}_H=\left\{w_H\in\mathbb{W}\cap C(\bar{\Omega}):w_H|_E\in\mathbb{P}_l(E),\ \forall E\in\Im_H\right\}.$$

Similarly, let $\Im_h$ be a quasi-uniform fine mesh partition on $\bar{\Omega}$ and $h=\sup\limits_{e\in\Im_h}\{\sup\limits_{\boldsymbol{x},\boldsymbol{y}\in e}\|\boldsymbol{x}-\boldsymbol{y}\|\}$ ($h\ll H$). Then the FE subspace on the fine meshes $\Im_h$ can be defined in the following

$$\mathbb{W}_h=\left\{w_h\in\mathbb{W}\cap C(\bar{\Omega}):w_h|_e\in\mathbb{P}_l(e),\ \forall e\in\Im_h\right\}.$$

Supposed that $P_\delta:\mathbb{W}\to\mathbb{W}_\delta$ ($\delta=h,H$) is two H^1-projections that for any $q\in\mathbb{W}$, there are two sole $P_\delta q\in\mathbb{W}_\delta$ satisfying the below equality

$$(\nabla(q-P_\delta q),\nabla q_\delta)=0,\ \forall q_\delta\in\mathbb{W}_\delta,\ \delta=h,\ H, \tag{50}$$

and the following error estimations

$$\begin{aligned}
&|q-P_\delta q|_r\leqslant c\delta^{l+1-r},\ \ \forall q\in H^{l+1}(\Omega)\cap\mathbb{W},\\
&\delta=h,\ H,\ \ r=-1,0,1.
\end{aligned}\tag{51}$$

Thus, a fire-new TGCNMFE method may be built in the following.

Problem 4 Step 1. *On the coarse mesh $\Im_H$, calculate $(w_H^n,\varphi_H^n)\in\mathbb{W}_H\times\mathbb{W}_H$ $(1\leqslant n\leqslant N)$ from the below* *nonlinear system:*

$$\begin{cases} (w_H^n - w_H^{n-1}, \upsilon_H) + \dfrac{\Delta t}{2}(\nabla(\varphi_H^n + \varphi_H^{n-1}), \nabla \upsilon_H) \\ = \dfrac{\Delta t}{2}(f(w_H^n) + f(w_H^{n-1}), \upsilon_H) \\ -\dfrac{\Delta t^{1-\alpha}}{\Gamma(1-\alpha)} \displaystyle\sum_{k=1}^{n} a_{n-k}(w_H^k - w_H^{k-1}, \upsilon_H), \\ \forall \upsilon_H \in \mathbb{W}_H, 1 \leqslant n \leqslant N; \\ (\nabla w_H^n, \nabla \vartheta_H) + (w_H^n, \vartheta_H) = (\varphi_H^n, \vartheta_H), \\ \quad \forall \vartheta_H \in \mathbb{W}_H, \quad 0 \leqslant n \leqslant N, \\ w_H^0 = P_H w_0(\boldsymbol{x}), \quad \varphi_H^0 = P_H \varphi^0, \boldsymbol{x} \in \Omega. \end{cases} \tag{52}$$

Step 2. *On the fine grid $\Im_h$, seek $(w_h^n, \varphi_h^n) \in \mathbb{W}_h \times \mathbb{W}_h$ $(1 \leqslant n \leqslant N)$ from the linear system:*

$$\begin{cases} (w_h^n - w_h^{n-1}, \upsilon_h) + \dfrac{\Delta t}{2}(\nabla(\varphi_h^n + \varphi_h^{n-1}), \nabla \upsilon_h) = \\ \dfrac{\Delta t}{2}(f(w_H^n) + f'(w_H^n)(w_h^n - w_H^n) + f(w_h^{n-1}), \upsilon_h) \\ -\dfrac{\Delta t^{1-\alpha}}{\Gamma(1-\alpha)} \displaystyle\sum_{k=1}^{n} a_{n-k}(w_h^k - w_h^{k-1}, \upsilon_h), \\ \forall \upsilon_h \in \mathbb{W}_h, \quad 1 \leqslant n \leqslant N; \\ (\nabla w_h^n, \nabla \vartheta_h) + (\nabla w_h^n, \nabla \vartheta_h) = (\varphi_h^n, \vartheta_h), \\ \quad \forall \vartheta_h \in \mathbb{W}_h, \quad 0 \leqslant n \leqslant N, \\ w_h^0 = P_h w_0(\boldsymbol{x}), \quad \varphi_h^0 = P_h \varphi^0, \boldsymbol{x} \in \Omega. \end{cases} \tag{53}$$

For Problem 4, we have the following result.

Theorem 2 *On the coarse grid $\Im_H$ and the fine grid $\Im_h$, from Problem 4 we can separately seek two sole sets of solutions $\{(w_H^n, \varphi_H^n)\}_{n=1}^N \subset \mathbb{W}_H \times \mathbb{W}_H$ $\{(w_h^n, \varphi_h^n)\}_{n=1}^N \subset \mathbb{W}_h \times \mathbb{W}_h$ to meet the below unconditional boundedness (unconditional stability):*

$$\begin{aligned} &\|w_H^n\|_1 + \|w_h^n\|_1 + \|\varphi_H^n\|_1 + \|\varphi_h^n\|_1 \\ &\leqslant c\|w_0\|_2, \ \ 1 \leqslant n \leqslant N, \end{aligned} \tag{54}$$

and the following error estimations

$$\begin{aligned} &\|w(t_n) - w_H^n\|_0 + \|\varphi(t_n) - \varphi_H^n\|_0 \\ &+H\|\nabla(w(t_n) - w_H^n)\|_0 + H\|\nabla(\varphi(t_n) - \varphi_H^n)\|_0 \\ &\leqslant c(\Delta t^{2-\alpha} + H^{l+1}), \ 1 \leqslant n \leqslant N, \end{aligned} \tag{55}$$

$$\begin{aligned} &\|w(t_n) - w_h^n\|_0 + \|\varphi(t_n) - \varphi_h^n\|_0 \\ &+h\|\nabla(w(t_n) - w_h^n)\|_0 + h\|\nabla(\varphi(t_n) - \varphi_h^n)\|_0 \\ &\leqslant c(\Delta t^{2-\alpha} + h^{l+1} + H^{l+3}), \ 1 \leqslant n \leqslant N. \end{aligned} \tag{56}$$

where c used subsequently is also a generical positive constant independent of H, h, and Δt, and $\Delta t = O(h) = O(H^2)$.

Proof. The demonstration for Theorem 2 is divided into the next two parts.

(1) *Analyze the existence and unconditional stability for the TGCNMFE solutions of Problem 4.*

(*i*) *The existence and unconditional stability for the TGCNMFE solutions on the coarse mesh $\Im_H$.*

Noting that (52) holds the same construction as (8)–(10), by using the same technique as proving the existence and stability of the TSDMCN solutions in Theorem 2, it can be proved that the system of equations (52) has a unique series of solutions $\{(w_H^n, \varphi_H^n)\}_{n=1}^N \subset \mathbb{W}_H \times \mathbb{W}_H$ meeting

$$\|\nabla w_H^n\|_0 + \|\nabla \varphi_H^n\|_0 \leqslant c\|w_0\|_1, \ \ 1 \leqslant n \leqslant N. \tag{57}$$

(*ii*) *The existence and unconditional stability for the TGCNMFE solutions on the fine mesh $\Im_h$.*

Let

$$\begin{aligned} &A((w,\varphi),(w,\varphi)) = (\nabla w, \nabla w) + (w,w) - (\varphi, w) \\ &+(w,\varphi) + \frac{\Delta t}{2}(\nabla\varphi, \nabla\varphi) - \frac{\Delta t}{2}(f'(w_H^n)w, \varphi) \\ &\quad + \frac{\Delta t^{1-\alpha}}{\Gamma(1-\alpha)}(w, \upsilon), \\ &F(\upsilon, \vartheta) = \left(w_h^{n-1}, \upsilon\right) - \frac{\Delta t}{2}(\nabla\varphi_h^{n-1}, \nabla\upsilon) \\ &\quad - \frac{\Delta t^{1-\alpha}}{\Gamma(1-\alpha)} \sum_{k=1}^{n-1} a_{n-k}(w_h^k, \upsilon_h) \\ &\quad + \frac{\Delta t}{2}(f(w_H^n) - f'(w_H^n)w_H^n + f(w_h^{n-1}), \upsilon). \end{aligned}$$

Thus, the linear system (53) may be rewritten into the following form.

Seek $(w_h^n, \varphi_h^n) \in \mathbb{W}_h \times \mathbb{W}_h$ $(1 \leqslant n \leqslant N)$ from the below linear system:

$$\begin{cases} A((w_h^n, \varphi_h^n), (\vartheta_h, \upsilon_h)) = F(\vartheta_h, \upsilon_h), \\ \quad \forall(\vartheta_h, \upsilon_h) \in \mathbb{W}_h \times \mathbb{W}_h, \quad 0 \leqslant n \leqslant N, \\ w_h^0 = P_h w_0(\boldsymbol{x}), \ \varphi_h^0 = P_h \varphi^0(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega. \end{cases} \tag{58}$$

Noting that there is a constant $\theta_0 > 0$ such that $\|\vartheta\|_0 \leqslant \|\vartheta\|_1 \leqslant \theta_0\|\nabla\vartheta\|_0$ $(\forall \vartheta \in \mathbb{W} = H_0^1(\Omega))$, we can assert that when Δt is small enough, there is a positive constant

$$\alpha_0 = \max\left\{1, \frac{\Delta t}{2} - \frac{\theta_0^2 \Delta t^2 \max\limits_{\nu \in \mathbb{W}_H} |f'(\nu)|}{4} - \frac{\theta_0^2 \Delta t^{2-2\alpha}}{\Gamma^2(1-\alpha)}\right\},$$

meeting

$$A((w,\varphi),(w,\varphi)) = (\nabla w, \nabla w) + (w,w) - (\varphi, w) + (w,\varphi) + \frac{\Delta t}{2}(\nabla\varphi,\nabla\varphi) - \frac{\Delta t}{2}(f'(w_H^n)w,\varphi) + \frac{\Delta t^{1-\alpha}}{\Gamma(1-\alpha)}(w,\varphi) \geqslant \alpha_0\|(w,\varphi)\|_1^2,\ \forall (w,\varphi)\in \mathbb{W}_h\times\mathbb{W}_h, \quad (59)$$

where $\|(w,\varphi)\|_1 = (\|\nabla w\|_1^2 + \|\nabla\varphi\|_1^2)^{1/2}$ is the norm in $\mathbb{W}\times\mathbb{W}$. This means that the bilinear functional $A((w,\varphi),(\vartheta,\upsilon))$ is positive definite in $\mathbb{W}_h\times\mathbb{W}_h$. It is obvious that the bilinear functional $A((w,\varphi),(\vartheta,\upsilon))$ is bounded in $\mathbb{W}_h\times\mathbb{W}_h$ and the linear functional $F(\upsilon,\vartheta)$ is bounded in $\mathbb{W}_h\times\mathbb{W}_h$ for given w_H^n, w_h^{n-1}, and φ_h^{n-1}. Hence, according to the Lax-Milgram Theorem in [33, Theorem 1.15], we assert that Step 2 in Problem 4 exists a sole solution set $\{(w_h^n,\varphi_h^n)\}_{n=1}^N \subset \mathbb{W}_h\times\mathbb{W}_h$ to meet

$$\|\nabla w_h^n\|_0 + \|\nabla\varphi_h^n\|_0 \leqslant c\|w_0\|_1,\quad 1\leqslant n\leqslant N. \quad (60)$$

This signifies that, on the fine grid $\Im_h$, the series of solutions $\{(w_h^n,\varphi_h^n)\}_{n=1}^N \subset \mathbb{W}_h\times\mathbb{W}_h$ for Problem 4 is unconditionally bounded, in other words, it is unconditionally stable.

(2) *Reckon the errors for the TGCNMFE solutions.*

(*a*) *Estimate the errors for the TGCNMFE solutions on the coarse mesh* $\Im_H$.

By subtracting (52) from (8)–(10) taking $\upsilon=\upsilon_H$ and $\vartheta=\vartheta_H$, and setting $\rho_H^n = w^n - P_H w^n$, $E_H^n = w^n - w_H^n$, $\varrho_H^n = P_H w^n - w_H^n$, $\tilde{E}_H^n = \varphi^n - \varphi_H^n$, $\tilde{\rho}_H^n = \varphi^n - P_H\varphi^n$, and $\tilde{\varrho}_H^n = P_H\varphi^n - \varphi_H^n$, and by the LDMT, we obtain

$$\frac{1}{\Delta t}\left(E_H^n - E_H^{n-1},\upsilon_H\right) + \frac{1}{2}\left(\nabla(\tilde{E}_H^n + \tilde{E}_H^{n-1}),\nabla\upsilon_H\right) = \frac{1}{2}\left(f(\chi_n)E_H^n + f'(\chi_{n-1})E_H^{n-1},\upsilon_H\right) - \frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^n a_{n-k}(E_H^k - E_H^{k-1},\upsilon_H),$$
$$\forall \upsilon_H\in\mathbb{W}_H,\quad 1\leqslant n\leqslant N, \quad (61)$$

$$(\nabla E_H^n,\nabla\vartheta_H) + (E_H^n,\vartheta_H) = (\tilde{E}_H^n,\vartheta_H),\quad \forall\vartheta_H\in\mathbb{W}_H,\quad 1\leqslant n\leqslant N, \quad (62)$$

$$E_H^0 = w_0 - P_H w_0,\ \tilde{E}_H^0 = \varphi_0 - P_H\varphi_0,\quad \text{in } \Omega. \quad (63)$$

where χ_i $(i=n,n-1)$ lies between w^i and w_H^i.

By (61), (50), (62), the second equation of (53), Taylor's formula, and the Hölder and Cauchy inequalities, as $\Delta t = O(H^{1+1/l})$, from (51) we get

$$\begin{aligned}
&\frac{1}{\Delta t}\|\nabla(E_H^n - E_H^{n-1})\|_0^2 + \frac{1}{\Delta t}\|E_H^n - E_H^{n-1}\|_0^2 + \frac{1}{2}\left(\|\nabla\tilde{E}_H^n\|_0^2 - \|\nabla\tilde{E}_H^{n-1}\|_0^2\right)\\
&= \frac{1}{\Delta t}\left(\nabla(E_H^n - E_H^{n-1}),\nabla(\rho_H^n - \rho_H^{n-1})\right) + \frac{1}{\Delta t}\left(\nabla(E_H^n - E_H^{n-1}),\nabla(\varrho_H^n - \varrho_H^{n-1})\right)\\
&+\frac{1}{\Delta t}\left(E_H^n - E_H^{n-1},\rho_H^n - \rho_H^{n-1}\right) + \frac{1}{\Delta t}\left(E_H^n - E_H^{n-1}),\varrho_H^n - \varrho_H^{n-1}\right)\\
&+\frac{1}{2}\left(\nabla(\tilde{E}_H^n + \tilde{E}_H^{n-1}),\nabla(\tilde{\rho}_H^n - \tilde{\rho}_H^{n-1})\right) + \frac{1}{2}\left(\nabla(\tilde{E}_H^n + \tilde{E}_H^{n-1}),\nabla(\tilde{\varrho}_H^n - \tilde{\varrho}_H^{n-1})\right)\\
&= \frac{1}{\Delta t}\left(\nabla(\rho_H^n - \rho_H^{n-1}),\nabla(\rho_H^n - \rho_H^{n-1})\right) + \frac{1}{\Delta t}\left(E_H^n - E_H^{n-1},\rho_H^n - \rho_H^{n-1}\right)\\
&+\frac{1}{\Delta t}\left(\tilde{E}_H^n - \tilde{E}_H^{n-1},\varrho_H^n - \varrho_H^{n-1}\right) + \frac{1}{2}\left(\nabla(\tilde{\rho}_H^n + \tilde{\rho}_H^{n-1}),\nabla(\tilde{\rho}_H^n - \tilde{\rho}_H^{n-1})\right)\\
&+\frac{1}{2}\left(\nabla(\tilde{E}_H^n + \tilde{E}_H^{n-1}),\nabla(\tilde{\varrho}_H^n - \tilde{\varrho}_H^{n-1})\right)\\
&= \frac{1}{\Delta t}\left(\nabla(\rho_H^n - \rho_H^{n-1}),\nabla(\rho_H^n - \rho_H^{n-1})\right) + \frac{1}{\Delta t}\left(E_H^n - E_H^{n-1},\rho_H^n - \rho_H^{n-1}\right)\\
&+\frac{1}{2}\left(\nabla(\tilde{\rho}_H^n + \tilde{\rho}_H^{n-1}),\nabla(\tilde{\rho}_H^n - \tilde{\rho}_H^{n-1})\right) + \frac{1}{\Delta t}\left(\tilde{\rho}_H^n - \tilde{\rho}_H^{n-1},E_H^n - E_H^{n-1}\right)\\
&-\frac{1}{\Delta t}\left(\tilde{E}_H^n - \tilde{E}_H^{n-1},\rho_H^n - \rho_H^{n-1}\right) + \frac{1}{\Delta t}\left(\tilde{\varrho}_H^n - \tilde{\varrho}_H^{n-1},E_H^n - E_H^{n-1}\right)\\
&+\frac{1}{2}\left(\nabla(\tilde{E}_H^n + \tilde{E}_H^{n-1}),\nabla(\tilde{\varrho}_H^n - \tilde{\varrho}_H^{n-1})\right)\\
&= \frac{1}{\Delta t}\left(\nabla(\rho_H^n - \rho_H^{n-1}),\nabla(\rho_H^n - \rho_H^{n-1})\right) + \frac{1}{\Delta t}\left(E_H^n - E_H^{n-1},\rho_H^n - \rho_H^{n-1}\right)\\
&+\frac{1}{2}\left(\nabla(\tilde{\rho}_H^n + \tilde{\rho}_H^{n-1}),\nabla(\tilde{\rho}_H^n - \tilde{\rho}_H^{n-1})\right) + \frac{1}{\Delta t}\left(\tilde{\rho}_H^n - \tilde{\rho}_H^{n-1},E_H^n - E_H^{n-1}\right)\\
&-\frac{1}{\Delta t}\left(\tilde{E}_H^n - \tilde{E}_H^{n-1},\rho_H^n - \rho_H^{n-1}\right) + \frac{1}{2}\left(f(\chi_n)E^n + f'(\chi_{n-1})E^{n-1},\tilde{\varrho}_H^n - \tilde{\varrho}_H^{n-1}\right)\\
&-\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^n a_{n-k}(E_H^k - E_H^{k-1},\tilde{\varrho}_H^n - \tilde{\varrho}_H^{n-1})\\
&\leqslant c\Delta t H^{2l} + \frac{1}{2\Delta t}\|E_H^n - E_H^{n-1}\|_0^2
\end{aligned}$$

$$+\frac{1}{2\Delta t}\|\nabla(E_H^n-E_H^{n-1})\|_0^2$$
$$+\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n-1}a_{n-k}(\|\nabla(E_H^n-E_H^{n-1})\|_0^2$$
$$+\|E_H^n-E_H^{n-1}\|_0^2),\quad 1\leqslant n\leqslant N. \tag{64}$$

It follows that

$$\|\nabla(E_H^n-E_H^{n-1})\|_0^2+\|E_H^n-E_H^{n-1}\|_0^2$$
$$+\Delta t\left(\|\nabla\tilde{E}_H^n\|_0^2-\|\nabla\tilde{E}_H^{n-1}\|_0^2\right)\leqslant c\Delta t^2H^{2l}$$
$$+\frac{\Delta t^{1-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n-1}a_{n-k}(\|\nabla(E_H^k-E_H^{k-1})\|_0^2$$
$$+\|E_H^k-E_H^{k-1}\|_0^2),\quad 1\leqslant n\leqslant N. \tag{65}$$

Summating (65) from 1 until n $(n\leqslant N)$ yields

$$\sum_{i=1}^{n}(\|\nabla(E_H^i-E_H^{i-1})\|_0^2+\|E_H^i-E_H^{i-1}\|_0^2)$$
$$+\Delta t\|\nabla\tilde{E}_H^n\|_0^2\leqslant cn\Delta t^2H^{2l}+c\Delta tH^{2l}$$
$$+\frac{\Delta t^{2-\alpha}}{\Gamma(1-\alpha)}\sum_{i=1}^{n}\sum_{k=1}^{i-1}a_{n-k}(\|\nabla(E_H^k-E_H^{k-1})\|_0^2$$
$$+\|E_H^k-E_H^{k-1}\|_0^2),\quad 1\leqslant n\leqslant N. \tag{66}$$

Applying Lemma 1 to (66) yields

$$\sum_{i=1}^{n}(\|\nabla(E_H^i-E_H^{i-1})\|_0^2+\|E_H^i-E_H^{i-1}\|_0^2)$$
$$+\Delta t\|\nabla\tilde{E}_H^n\|_0^2$$
$$\leqslant c\Delta tH^{2l}\exp\left(\frac{\Delta t^{2-\alpha}}{\Gamma(1-\alpha)}\sum_{i=1}^{n}\sum_{k=1}^{i-1}a_{n-k}\right)$$
$$\leqslant c\Delta tH^{2l},\quad 1\leqslant n\leqslant N. \tag{67}$$

Thereupon

$$\|\nabla\tilde{E}_H^n\|_0\leqslant cH^l,\quad 1\leqslant n\leqslant N. \tag{68}$$

By (62), (50), (51), and (78), we procure

$$\|\nabla E_H^n\|_0^2+\|E_H^n\|_0^2=\|\rho_H^n\|_0^2+(\nabla E_H^n,\nabla\varrho_H^n)$$
$$+(E_H^n,\rho_H^n)+(E_H^n,\varrho_H^n)$$
$$=\|\rho_H^n\|_0^2+(E_H^n,\rho_H^n)+(\tilde{E}_H^n,\varrho_H^n)$$
$$\leqslant cH^{2l}+\frac{1}{2}\|E_H^n\|_0^2+c\|\tilde{E}_H^n\|_0^2$$
$$\leqslant cH^{2l}+\frac{1}{2}\|E_H^n\|_0^2, 1\leqslant n\leqslant N. \tag{69}$$

It follows that

$$\|\nabla E_H^n\|_0\leqslant cH^l,\quad 1\leqslant n\leqslant N. \tag{70}$$

With the Nitsche technique in [33, Theorem 1.38], (68), and (70), we can get the blow error estimations

$$\|w^n-w_H^n\|_0+H\|\nabla(w^n-w_H^n)\|_0$$
$$\leqslant cH^{l+1},\quad 1\leqslant n\leqslant N. \tag{71}$$

Combining (71) with Theorem 1 yields (55).

(b) Reckon the errors for the TGCNMFE solutions on the fine mesh $\Im_h$.

By subtracting (53) from (8)–(10), taking $\upsilon=\upsilon_h$ and $\vartheta=\vartheta_h$, and setting $\rho_h^n=w^n-P_hw^n$, $E_h^n=w^n-w_h^n$, $\varrho_h^n=P_hw^n-w_h^n$, $\tilde{E}_h^n=\varphi^n-\varphi_h^n$, $\tilde{\rho}_h^n=\varphi^n-P_h\varphi^n$, and $\tilde{\varrho}_h^n=P_h\varphi^n-\varphi_h^n$, and by the LDMT, we obtain

$$\frac{1}{\Delta t}\left(E_h^n-E_h^{n-1},\upsilon_h\right)+\frac{1}{2}\left(\nabla(\tilde{E}_h^n+\tilde{E}_h^{n-1}),\nabla\upsilon_h\right)$$
$$=\frac{1}{2}(f'(\chi_n)E_H^n+f'(\zeta_{n-1})E_h^{n-1},\upsilon_h)$$
$$-\frac{1}{2}(f'(w_H^n)(w_h^n-w_H^n),\upsilon_h)$$
$$-\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n-1}a_{n-k}(E_h^k-E_h^{k-1},\upsilon_h),$$
$$\forall\upsilon_h\in\mathbb{W}_h,\ 1\leqslant n\leqslant N, \tag{72}$$
$$(\nabla E_h^n,\nabla\vartheta_h)+(E_h^n,\vartheta_h)$$
$$=(\tilde{E}_h^n,\vartheta_h),\quad\forall\vartheta_H\in\mathbb{W}_h,\ 1\leqslant n\leqslant N, \tag{73}$$
$$E_h^0=w_0-P_hw_0,\ \tilde{E}_h^0=\varphi_0-P_h\varphi_0,\ \text{ in }\Omega, \tag{74}$$

where ζ_{n-1} lies between w^{n-1} with w_h^{n-1}.

By (72), (73), (50), the Hölder and Cauchy inequalities, Taylor's formula, and (55) or (71), when $\Delta t=O(h)=O(H^{1+1/l})$, we get

$$\frac{1}{\Delta t}[\|\nabla(E_h^n-E_h^{n-1})\|_0^2+\|\nabla(E_h^n-E_h^{n-1})\|_0^2]$$
$$+\frac{1}{2}\left(\|\nabla\tilde{E}_h^n\|_0^2-\|\nabla\tilde{E}_h^{n-1}\|_0^2\right)$$
$$=\frac{1}{\Delta t}\left(\nabla(\rho_h^n-\rho_h^{n-1}),\nabla(\rho_h^n-\rho_h^{n-1})\right)$$
$$+\frac{1}{\Delta t}\left(E_h^n-E_h^{n-1},\rho_h^n-\rho_h^{n-1}\right)$$
$$+\frac{1}{2}\left(\nabla(\tilde{\rho}_h^n+\tilde{\rho}_h^{n-1}),\nabla(\tilde{\rho}_h^n-\tilde{\rho}_h^{n-1})\right)$$
$$+\frac{1}{\Delta t}\left(\tilde{\rho}_h^n-\tilde{\rho}_h^{n-1},E_h^n-E_h^{n-1}\right)$$
$$-\frac{1}{\Delta t}\left(\tilde{E}_h^n-\tilde{E}_h^{n-1},\rho_h^n-\rho_h^{n-1}\right)$$
$$+\frac{1}{2}\left(f'(\chi_n)E_H^n+f'(\zeta_{n-1})E_h^{n-1},\tilde{E}_h^n-\tilde{E}_h^{n-1}\right)$$
$$-\frac{1}{2}\left(f'(w_H^n)(E_H^n-E_h^n),\tilde{E}_h^n-\tilde{E}_h^{n-1}\right)$$
$$-\frac{1}{2}\left(f'(\chi_n)E_H^n+f'(\zeta_{n-1})E_h^{n-1},\tilde{\rho}_h^n-\tilde{\rho}_h^{n-1}\right)$$

$$+\frac{1}{2}\left(f'(w_H^n)(E_H^n - E_h^n), \tilde{\rho}_h^n - \tilde{\rho}_h^{n-1}\right)$$
$$-\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n} a_{n-k}(E_h^k - E_h^{k-1}, \tilde{E}_h^n - \tilde{E}_h^{n-1})$$
$$-\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n-1} a_{n-k}(E_h^k - E_h^{k-1}, \tilde{\varrho}_h^n - \tilde{\varrho}_h^{n-1})$$
$$\leqslant \frac{1}{2\Delta t}[\|\nabla(E_h^n - E_h^{n-1})\|_0^2 + \|E_h^n - E_h^{n-1}\|_0^2]$$
$$+c\Delta t h^{2l} + \frac{\Delta t^{1-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n} a_{n-k}[\|\nabla(E_h^k - E_h^{k-1})\|_0^2$$
$$+\|E_h^k - E_h^{k-1}\|_0^2, \ 1 \leqslant n \leqslant N. \quad (75)$$

Thus, from (75) we obtain

$$\|\nabla(E_h^n - E_h^{n-1})\|_0^2 + \|\nabla(E_h^n - E_h^{n-1})\|_0^2$$
$$+\Delta t\left(\|\nabla\tilde{E}_h^n\|_0^2 - \|\nabla\tilde{E}_h^{n-1}\|_0^2\right)$$
$$\leqslant c\Delta t^2 h^{2l} + \frac{\Delta t^{2-\alpha}}{\Gamma(1-\alpha)}\sum_{k=1}^{n} a_{n-k}[\|\nabla(E_h^k - E_h^{k-1})\|_0^2$$
$$+\|E_h^k - E_h^{k-1}\|_0^2, \quad 1 \leqslant n \leqslant N. \quad (76)$$

Summating (76) from 1 until n $(n \leqslant N)$ yields

$$\sum_{i=1}^{n}(\|\nabla(E_h^i - E_h^{i-1})\|_0^2 + \|E_h^i - E_h^{i-1}\|_0^2)$$
$$+\Delta t\|\nabla\tilde{E}_h^n\|_0^2 \leqslant cn\Delta t^2 h^{2l} + c\Delta t h^{2l}$$
$$+\frac{\Delta t^{2-\alpha}}{\Gamma(1-\alpha)}\sum_{i=1}^{n}\sum_{k=1}^{i-1} a_{n-k}(\|\nabla(E_h^k - E_h^{k-1})\|_0^2$$
$$+\|E_h^k - E_h^{k-1}\|_0^2), \quad 1 \leqslant n \leqslant N. \quad (77)$$

Applying Lemma 1 to (77) yields

$$\sum_{i=1}^{n}(\|\nabla(E_h^i - E_h^{i-1})\|_0^2 + \|E_h^i - E_h^{i-1}\|_0^2)$$
$$+\Delta t\|\nabla\tilde{E}_h^n\|_0^2$$
$$\leqslant c\Delta t h^{2l}\exp\left(\frac{\Delta t^{2-\alpha}}{\Gamma(1-\alpha)}\sum_{i=1}^{n}\sum_{k=1}^{i-1} a_{n-k}\right)$$
$$\leqslant c\Delta t h^{2l}, \quad 1 \leqslant n \leqslant N. \quad (78)$$

Thereupon, we obtain

$$\|\nabla\tilde{E}_h^n\|_0 \leqslant ch^l, \quad 1 \leqslant n \leqslant N. \quad (79)$$

By (73), (50), (51), and (79), we procure

$$\|\nabla E_h^n\|_0^2 + \|E_h^n\|_0^2 = \|\rho_h^n\|_0^2 + (\nabla E_h^n, \nabla\varrho_h^n)$$
$$+(E_h^n, \rho_h^n) + (E_h^n, \varrho_h^n)$$
$$= \|\rho_h^n\|_0^2 + (E_h^n, \rho_h^n) + (\tilde{E}_h^n, \varrho_h^n)$$
$$\leqslant ch^{2l} + \frac{1}{2}\|E_h^n\|_0^2 + c\|\tilde{E}_h^n\|_0^2$$
$$\leqslant ch^{2l} + \frac{1}{2}\|E_h^n\|_0^2, 1 \leqslant n \leqslant N. \quad (80)$$

It follows that

$$\|\nabla E_h^n\|_0 \leqslant ch^l, \quad 1 \leqslant n \leqslant N. \quad (81)$$

Through the Nitsche method in [33, Theorem 1.38 or Remark 3.1]), (79), and (81), we can obtain the below error estimations

$$\|w^n - w_h^n\|_0 + \|\varphi^n - \varphi_h^n\|_0$$
$$+h(\|\nabla(w^n - w_h^n)\|_0 + \|\nabla(\varphi^n - \varphi_h^n)\|_0)$$
$$= \|E_h^n\|_0\|\tilde{E}_h^n\|_0 + h(\|\nabla E_h^n\|_0 + \|\nabla\tilde{E}_h^n\|_0)$$
$$\leqslant ch^{l+1}, \ 1 \leqslant n \leqslant N. \quad (82)$$

Thus, (56) is obtained by combining Theorem 1 with (82). This finishes the demonstration of Theorem 2. □

Remark 3 *Theorem 2 explains that the theory errors of the TGCNMFE solutions achieve optimal order and the TGCNMFE solutions are unconditionally stable. In the next section, we will conduct numerical experiments to verify that the theory errors of the TGCNMFE solution is consistent with the computing errors.*

4 Some Numerical Experiments

In this section, we give a set of numerical tests to verify the rightness for the procured theory results and to reveal the advantage for the TGCNMFE method.

Set that $\bar{\Omega} = [0,\ 1]\times[0,\ 1]$ and the initial value functions $w_0(\boldsymbol{x}) = \sin(2\pi x_1)\sin(2\pi x_2)$ in GNTFFORDE (i.e., Problem 1).

The fine mesh partition $\Im_h$ consists of all squares with equal side length $1/1000$ and the all sides parallel to the coordinate axis. When $l = 1$, in order to satisfy the condition of optimal error estimates $h = O(H^{1+1/l}) = O(H^2)$, the coarse mesh partition $\Im_H$ was taken as the squares with equal side length $1/\sqrt{1000}$ and the all sides parallel to the coordinate axis. When $\alpha = 0.2$, 0.4, 0.6, and 0.8, $l = 1$, and $\Delta t = h^{1+\alpha/2}$, according to Theorems 2, the L^2 morn error estimations for the TGCNMFE solutions to the GNTFFORDE can achieve $O(10^{-6})$, theoretically.

First, when $\alpha = 0.2, 0.4, 0.6, 0.8$, we calculated four series of TGCNMFE solutions $\{(w_{\alpha h}^n, \varphi_{\alpha h}^n)\}$ at $n = 1000$ (i.e., $t = 1000\Delta t$) and $n = 2000$ (i.e., $t = 2000\Delta t$) by the TGCNMFE method (i.e., Problem 4), and recorded

Table 1. The errors of the MGCNMFE and TGCNMFE solutions and CPU running-time at $t = 1000\Delta t$.

α	MGCNMFE solutions Errors	TGCNMFE solutions Errors	MGCNMFE method CPU Running-time	TGCNMFE method CPU Running-time
0.2	2.2316×10^{-6}	1.0273×10^{-6}	215.332 s	114.028 s
0.4	2.4187×10^{-6}	1.1438×10^{-6}	216.662 s	113.312 s
0.6	2.8665×10^{-6}	1.2662×10^{-6}	217.153 s	113.634 s
0.8	2.9782×10^{-6}	1.3861×10^{-6}	216.709 s	112.451 s

Table 2. The errors of the MGCNMFE and TGCNMMFE solutions and CPU running-time at $t = 2000\Delta t$.

α	MGCNMFE solutions Errors	TGCNMFE solutions Errors	MGCNMFE method CPU Running-time	TGCNMFE method CPU Running-time
0.2	3.9781×10^{-6}	2.1563×10^{-6}	427.562 s	214.731 s
0.4	3.6436×10^{-6}	2.2253×10^{-6}	426.826 s	213.813 s
0.6	3.8841×10^{-6}	2.3265×10^{-6}	425.716 s	212.832 s
0.8	3.9862×10^{-6}	2.4453×10^{-6}	425.261 s	212.764 s

the CPU running-time and errors, which are estimated by $\|w^n_{\alpha h} - w^{n-1}_{\alpha h}\|_0 + \|\varphi^n_{\alpha h} - \varphi^{n-1}_{\alpha h}\|_0$, listed in the third and fifth columns in Tables 1 and 2.

Next, in order to exhibit that the TGCNMFE method is superior to the monolayer-grid CNMFE (MGCNMFE) method (i.e., the Step 1 of Problem 4 when $H = \sqrt{2}/1000$), we used the MGCNMFE method to calculate four series of MGCNMFE solutions $\{(w^n_{\alpha H}, \varphi^n_{\alpha H})\}$ at $n = 1000$ (i.e., $t = 1000\Delta t$) and $n = 2000$ (i.e., $t = 2000\Delta t$) when $\alpha = 0.2$, 0.4, 0.6, 0.8, and recorded the CPU runtime and errors, which are estimated by $\|w^n_{\alpha h} - w^{n-1}_{\alpha h}\|_0 + \|\varphi^n_{\alpha h} - \varphi^{n-1}_{\alpha h}\|_0$, listed in the second and fourth columns in Tables 1 and 2.

The data of Tables 1 and 2 manifest that when $n = 1000$ (i.e., $t = 1000\Delta t$) and $n = 2000$ (i.e., $t = 2000\Delta t$), and $\alpha = 0.2$, 0.4, 0.6, 0.8, the numerical computing errors of the MGCNMFE and TGCNMFE solutions are coincided with the theory errors $O(10^{-6})$, but the CPU running-time of TGCNMFE method is nearly half that of MGCNMFE method. Therefore, the TGCNMFE method is markedly superior over the MGCNMFE method and the TGCNMFE method is feasible and effective to solve GNTFFORDE.

5 Conclusions and Prospect

Above, we have proposed a new TSDMCN method and a new TGCNMFE method for the GNTFFORDE, and have strictly analyzed the existence, stability, and errors for the TSDMCN and TGCNMFE solutions, theoretically. We have also provided the numerical experiments to confirm the correctness of theory results and shown the superiorities for the TGCNMFE method. The TSDMCN and TGCNMFE methods for the GNTFFORDE are firstly proposed in this paper. Hence, they completely differ from the existed time semi-discrete scheme and the MFE method, including that in [1]. Of course, the new TGCNMFE method is also distinct from the existing FE methods with only first-order time precision and conditional convergence in [23–26]. Therefore, the TSDMCN and TGCNMFE methods herein are original and fire-new.

Although the TGCNMFE method here can greatly simplify computation, save CPU-time, and improve calculation efficiency, when it is applied to settling the GNTFFORDE of practical engineering problem, it usually contains many (often more than tens of millions) unknowns and needs to take a long time to calculate the result on a computer. Thus, after the computer has been running for a long time, due to the accumulation of calculation errors, the obtained TGCNMFE solution may deviate from the correct solution, and even floating-point overflow may occur, resulting in incorrect calculation results. Therefore, in future research, we will adopt appropriate orthogonal decomposition (POD) methods to reduce the unknowns in the TGCNMFE method, and create some new POD dimension reduction methods for the GNTFFORDE and other nonlinear and unsteady fractional PDEs.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

Ethical Approval and Consent to Participate

Not applicable.

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Yuejie Li received the PhD. degree in pattern recognition and intelligent system from North China Electric Power University, BEIJING 102206, CHINA, in 2023. Her research interests include the finite element methods for the various PDEs as well as their reduced-dimension methods based on the proper orthogonal decomposition, and computer applications. She served as an Associate Editor for the Journal of Numerical Simulations in Physics and Mathematics in the ICCK. (Email: lyj@oit.edu.cn)

Zhendong Luo received the PhD. degree in computational mathematics from University of Science and Technology of China (USTC), HDFEI 230026, CHINA, in 1997. His research interests include almost all numerical calculation methods such as the finite element method, the finite difference scheme, the finite volume method, the space-time finite method, the natural boundary element method, the spectral method, and the spectral element method, as well as their reduced-dimension methods based on the proper orthogonal decomposition, and their applications. He served as an Editor-in-Chief for the Journal of Numerical Simulations in Physics and Mathematics in the ICCK. (Email: zhdluo@ncepu.edu.cn)