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RESEARCH ARTICLE

A New Two-grid Crank-Nicolson Mixed Finite Element Method for Convective FitzHugh-Nagumo Equation

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Abstract

In this article, we mainly develop a new two-grid Crank-Nicolson (CN) mixed finite element (FE) (TGCNMFE) method for the convective FitzHugh-Nagumo equation. For the purpose, a new time semi-discrete CN mixed (TSDCNM) scheme is created, and the existence, steadiness, and estimates of errors for the TSDCNM solutions are attested. Thereafter, a new TGCNMFE method is developed, and the existence, steadiness, and estimates of errors for the TGCNMFE solutions are discussed. Lastly, the rightness of the procured theoretical results and the effectiveness of the TGCNMFE method are attested through several numerical experiments.

Keywords: two-grid Crank-Nicolson finite element method, the convective FitzHugh-Nagumo equation, time semi-discrete Crank-Nicolson scheme, existence and steadiness, error estimate.

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1 Introduction

Let $\Omega \subset R^2$ be a two-dimensional bounded domain with the boundary $\partial\Omega$. For a known final time t_e, we research the following the convective FitzHugh-Nagumo equation.

Problem 1 *Seek* $(w, \varpi) \in [C^1(0, t_e; C^2(\bar{\Omega}))]^2$ *from the following system of equations:*

$$\begin{cases} w_t(\boldsymbol{x},t) - \Delta w(\boldsymbol{x},t) + \varpi(\boldsymbol{x},t) \\ = f(w(\boldsymbol{x},t)) + g(\boldsymbol{x},t), \ (\boldsymbol{x},t) \in \Omega \times (0,t_e), \\ \varpi_t(\boldsymbol{x},t) + \varpi(\boldsymbol{x},t) \\ = w(\boldsymbol{x},t), \ (\boldsymbol{x},t) \in \Omega \times (0,t_e), \\ w(\boldsymbol{x},t) = \varpi(\boldsymbol{x},t) = 0, \ (\boldsymbol{x},t) \in \partial\Omega \times (0,t_e), \\ w(\boldsymbol{x},0) = w_0(\boldsymbol{x}), \ \varpi(\boldsymbol{x},0) = \varpi_0(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega, \end{cases} \tag{1}$$

in which $C^1(0, t_e; C^2(\bar{\Omega}))$ *is the space formed by functions with continuous 2nd-order derivatives about spatial variables on* $\bar{\Omega}$ *and continuous 1rst-order derivative about time on* $[0,\ t_e]$, $w_t(\boldsymbol{x},t) = \partial w(\boldsymbol{x},t)/\partial t$, $\boldsymbol{x} = (x_1, x_2)$, Δ *indicates the Laplacian operator,* $f(w) = w(1-w)(w-a)$ *and* a *is a known constant, the source term* $g(\boldsymbol{x},t)$ *and the initial functions* $w_0(\boldsymbol{x})$ *and* $\varpi_0(\boldsymbol{x})$ *are sufficiently smooth and known.*

The convective FitzHugh-Nagumo equation is a very important mathematical model to depict the neuronal

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excitation system and plays a significant role in depicting the neuronal nonlinear behavior (see [1–3]). It has a very high efficiency to simulate the electrical activity of neurons. Specially, it has a significant feature that can be stimulated through controlling the range of parameters. Hence, it is also a mathematical model to study the excitable systems and helical waves [1]. Thereby, it is of meanings to study the convective FitzHugh-Nagumo equation.

Unfortunately, owing to the complexity of the convective FitzHugh-Nagumo equation with a strong nonlinear term $f(w) = w(1-w)(w-a)$, it usually cannot be solved analytically. It is a best choice to compute its approximate solutions through numerical approaches. Though some finite-difference (FD) schemes for the convective FitzHugh-Nagumo equation were proposed in [4], their accuracy is very poor, for instance, they have only 1rst-order time accuracy.

Massive numerical tests indicate that two-grid FE technique is one of the most effective algorithms to calculate nonlinear PDEs including the convective FitzHugh-Nagumo equation. It is composed of a nonlinear system including a few equations on coarser meshes and a linear system on adequately fine meshes. Thus, its computation process can be greatly simplified and its computation efficiency is improved. It was initially used to calculate the numeric solutions for some quasi-linear second-order elliptic problems (see [5]). Lately, in [6–8], it was accustomed to calculate some slightly intricate nonlinear PDEs.

As far as we know, at the moment there has been no report that the convective FitzHugh-Nagumo equation is solved by TGCNMFE method. Though a two-grid FE algorithm for the convective FitzHugh-Nagumo equation was proposed in [3], it is not the TGCNMFE method. Therefore, in this article, we develop a new TGCNMFE method with time second-order accuracy and unconditional stability for the convective FitzHugh-Nagumo equation, which is completely distant from the methods with only first-order time accuracy in [6–8] and the method in [3].

The arrangement of the remaining content of this article is as follows. In Section 2, we develop a new TSDCNM scheme for the convective FitzHugh-Nagumo equation and analyze the existence, stability, and estimates of errors of the TSDCNM solutions. In Section 3, we create a new TGCNMFE method for the convective FitzHugh-Nagumo equation and discuss the existence, unconditional stability, and estimates of errors for the TGCNMFE solutions. In Section 4, we employ several numeric tests to attest the rightness of the procured theory results and the effectiveness of the TGCNMFE method. Finally, we give the dominating conclusions of this article and the future research prospects in Section 5.

2 A New TSDCNM Scheme

The Sobolev spaces as well as their norms presented in the following text are classic (see [9–11]). Let $\mathbb{W} = H_0^1(\Omega)$ and $\widetilde{\mathbb{W}} = L^2(\Omega)$. In order to facilitate theory analysis and without loss of generality, we assume that $g(\boldsymbol{x},t)=0$. Thereupon, with the Green formula, we may build the below weak form of Problem 1.

Problem 2 *$\forall t \in (0,\ t_e)$, seek $(w,\varpi) \in \mathbb{W}\times\widetilde{\mathbb{W}}$ from the following system of equations:*

$$\begin{cases} (v, w_t) + (\nabla v, \nabla w) + (v, \varpi) \\ \quad = (v, f(w)),\ \forall v \in \mathbb{W}, \\ (\vartheta, \varpi_t) + (\vartheta, \varpi) = (\vartheta, w),\ \forall \vartheta \in \widetilde{\mathbb{W}}, \\ w(\boldsymbol{x},0) = w_0(\boldsymbol{x}),\ \varpi(\boldsymbol{x},0) = \varpi_0(\boldsymbol{x}),\ \boldsymbol{x}\in\Omega, \end{cases} \tag{2}$$

wherein $(v,w) = \int_\Omega v\cdot w\mathrm{d}\boldsymbol{x}$.

By using the proof method in [9, Theorem 3.5.1] or the same proof method as the next Theorem 1, we may deduce the existence and stableness for generalized solutions to Problem 2.

In order to develop the TGCNMFE method, we first need to create a fire-new TSDCNM scheme. Thereby, we assume that $N>0$ is a natural number, $\Delta t = t_e/N$ denotes the time-step, and w^n and ϖ^n indicate the approximations to $w(\boldsymbol{x},t)$ and $\varpi(\boldsymbol{x},t)$ at moment $t_n = n\Delta t\ (0 \leqslant n \leqslant N)$ separately.

Using a time implicit FD scheme with only time 1rst-order accuracy to discretize the 1rst and 2nd equations in Problem 2 procures

$$\begin{cases} \dfrac{1}{\Delta t}\left(w^n - w^{n-1}, v\right) + (\nabla v, \nabla w^n) + (v, \varpi^n) \\ = (f(w^n), v), \quad \forall v \in \mathbb{W}. \\ \dfrac{1}{\Delta t}\left(\varpi^n - \varpi^{n-1}, \vartheta\right) + (\vartheta, \varpi^n) \\ = (w^n, \vartheta), \quad \forall \vartheta \in \widetilde{\mathbb{W}}. \end{cases} \tag{3}$$

Using a time explicit FD scheme with only time 1rst-order accuracy to discretize the 1rst and 2nd equations in Problem 2 procures

$$\begin{cases} \dfrac{1}{\Delta t}\left(w^n - w^{n-1}, \upsilon\right) + (\nabla \upsilon, \nabla w^{n-1}) \\ +(\upsilon, \varpi^{n-1}) = (f(w^{n-1}), \upsilon), \quad \forall \upsilon \in \mathbb{W}. \\ \dfrac{1}{\Delta t}\left(\varpi^n - \varpi^{n-1}, \vartheta\right) + (\vartheta, \varpi^{n-1}) \\ = (w^{n-1}, \vartheta), \quad \forall \vartheta \in \widetilde{\mathbb{W}}. \end{cases} \tag{4}$$

By adding (3) to (4), we procure a fire-new TSDCNM scheme with time 2nd-order accuracy, which is completely distent from the existing time semi-discrete schemes such as those in [3, 6–8], since $\frac{1}{2}(f(w^n) + f(w^{n-1}) \neq \left(\frac{1}{2}(f(w^n) + f(w^{n-1})\right)$.

Problem 3 *Calculate* $\{(w^n, \varpi^n)\} \in \mathbb{W} \times \widetilde{\mathbb{W}}\ (1 \leqslant n \leqslant N)$ *by the following system of equations*

$$\frac{1}{\Delta t}(\upsilon, w^n - w^{n-1}) + \frac{1}{2}(\nabla \upsilon, \nabla(w^n + w^{n-1})) + \frac{1}{2}(\upsilon, \varpi^n + \varpi^{n-1}) = \frac{1}{2}(\upsilon, f(w^n) + f(w^{n-1})), \quad \forall \upsilon \in \mathbb{W},\ 1 \leqslant n \leqslant N, \tag{5}$$

$$\frac{1}{\Delta t}\left(\vartheta, \varpi^n - \varpi^{n-1}\right) + \frac{1}{2}(\vartheta, \varpi^n + \varpi^{n-1}) = \frac{1}{2}(\vartheta, w^n + w^{n-1}), \quad \forall \vartheta \in \widetilde{\mathbb{W}},\ 1 \leqslant n \leqslant N, \tag{6}$$

$$w^0 = w_0(\boldsymbol{x}), \quad \varpi^0 = \varpi_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{7}$$

The following discrete Gronwall lemma (see [12, Lemma 3.1]) is repeatedly adopted in after theory proofs.

Lemma 1 *Let* $\{a_n\}$, $\{b_n\}$, *and* $\{\delta_n\}$ *be three nonnegative real sequences, and meet* $a_n \leqslant b_n + \sum_{j=1}^{n-1} \delta_j a_j\ (n \geqslant 1)$, *then they also meet* $a_n \leqslant b_n \exp\left(\sum_{j=1}^{n-1} \delta_j\right)\ (n \geqslant 1)$.

For Problem 3, we obtain the following results.

Theorem 1 *A solution set* $\{(w^n, \varpi^n)\}_{n=1}^N \subset \mathbb{W} \times \widetilde{\mathbb{W}}$ *can be uniquely calculated from Problem 3 to satisfy the next boundedness (i.e., stableness):*

$$\|w^n\|_1 + \|\varpi^n\|_0 \leqslant c(\|w_0\|_1 + \|\varpi\|_0), \ 1 \leqslant n \leqslant N. \tag{8}$$

where and subsequent $c > 0$ *is a constant independent of* Δt. *In addition, if* $w_0(\boldsymbol{x})$ *and* $\varpi_0(\boldsymbol{x})$ *is smooth enough, then the errors of the TSDCNM solution set* $\{(w^n, \varpi^n)\}_{n=1}^N$ *are reckoned via the next inequalities*

$$\|\nabla(w(t_n) - w^n)\|_0 + \|\varpi(t_n) - \varpi^n\|_0 \leqslant c\Delta t^2, \ 1 \leqslant n \leqslant N, \tag{9}$$

wherein $w(t_n) = w(\boldsymbol{x}, t_n)$ *and* $\varpi(t_n) = \varpi(\boldsymbol{x}, t_n)$.

Proof. The demonstration of Theorem 1 is executed by the next three steps.

Step 1. *Discuss the existence and uniqueness of the TSDCNM solutions.*

Taking $\vartheta = \varpi^n + \varpi^{n-1}$ in (6), by Hölder and Cauchy inequalities, we obtain

$$\begin{aligned} &\|\varpi^n\|_0^2 - \|\varpi^{n-1}\|_0^2 + \frac{\Delta t}{2}\|\varpi^n + \varpi^{n-1}\|_0^2 \\ &= \frac{\Delta t}{2}(w^n + w^{n-1}, \varpi^n + \varpi^{n-1}) \\ &\leqslant \frac{\Delta t}{4}\|w^n + w^{n-1}\|_0^2 + \frac{\Delta t}{4}\|\varpi^n + \varpi^{n-1}\|_0^2. \end{aligned} \tag{10}$$

Thereupon, we get

$$\|\varpi^n\|_0^2 - \|\varpi^{n-1}\|_0^2 \leqslant \frac{\Delta t}{2}(\|w^n\|_0^2 + \|w^{n-1}\|_0^2). \tag{11}$$

Summing (11) from 1 until n procures

$$\|\varpi^n\|_0^2 \leqslant \|\varpi^0\|_0^2 + \Delta t \sum_{i=0}^{n} \|w^i\|_0^2, \ 1 \leqslant n \leqslant N. \tag{12}$$

Taking $\vartheta = w^n - w^{n-1}$ in (6), by Hölder and Cauchy inequalities, we procure

$$\begin{aligned} &\frac{1}{2}(\varpi^n + \varpi^{n-1}, w^n - w^{n-1}) \\ &= \frac{1}{2}(w^n + w^{n-1}, w^n - w^{n-1}) \\ &\quad - \frac{1}{\Delta t}(\varpi^n - \varpi^{n-1}, w^n - w^{n-1}) \\ &= -\frac{1}{\Delta t}(\varpi^n - \varpi^{n-1}, w^n - w^{n-1}) \\ &\quad + \frac{1}{2}(\|w^n\|_0^2 - \|w^{n-1}\|_0^2), 1 \leqslant n \leqslant N. \end{aligned} \tag{13}$$

Taking $\vartheta = \varpi^n - \varpi^{n-1}$ in (6) and using the Hölder and Cauchy inequalities, we procure

$$\begin{aligned} &\frac{1}{\Delta t}\|\varpi^n - \varpi^{n-1}\|_0^2 + \frac{1}{2}(\|\varpi^n\|_0^2 - \|\varpi^{n-1}\|_0^2) \\ &= (\varpi^n - \varpi^{n-1}, w^n - w^{n-1}) \\ &\leqslant \frac{1}{2\Delta t}\|\varpi^n - \varpi^{n-1}\|_0^2 \\ &\quad + \frac{\Delta t}{2}\|w^n - w^{n-1}\|_0^2, \quad 1 \leqslant n \leqslant N. \end{aligned} \tag{14}$$

Thereupon, we procure

$$\begin{aligned} &\frac{1}{\Delta t}\|\varpi^n - \varpi^{n-1}\|_0^2 + \|\varpi^n\|_0^2 - \|\varpi^{n-1}\|_0^2 \\ &\leqslant \Delta t\|w^n - w^{n-1}\|_0^2, \quad 1 \leqslant n \leqslant N. \end{aligned} \tag{15}$$

Summing (15) from 1 until n procures

$$\frac{1}{\Delta t}\sum_{i=1}^{n}\|\varpi^i-\varpi^{i-1}\|_0^2+\|\varpi^n\|_0^2\leqslant\|\varpi^0\|_0^2 + \Delta t\sum_{i=1}^{n}\|w^i-w^{i-1}\|_0^2,\ 1\leqslant n\leqslant N. \quad (16)$$

Taking $\upsilon=w^n-w^{n-1}$ in (5), and using the Hölder and Cauchy inequalities, (13), and Lagrange's Mean Value Theorem of Differential Calculus (LMVTDC), we procure

$$\begin{aligned}&\frac{1}{\Delta t}\|w^n-w^{n-1}\|_0^2+\frac{1}{2}(\|\nabla w^n\|_0^2-\|\nabla w^{n-1}\|_0^2)\\&+\frac{1}{2}(\|w^n\|_0^2-\|w^{n-1}\|_0^2)\\&=\frac{1}{\Delta t}(\varpi^n-\varpi^{n-1},w^n-w^{n-1})\\&+\frac{1}{2}(f(w^n)+f(w^{n-1}),w^n-w^{n-1})\\&\leqslant\frac{1}{2\Delta t}(\|w^n-w^{n-1}\|_0^2+\|\varpi^n-\varpi^{n-1}\|_0^2)\\&+\frac{\Delta t}{4}\|f'\|_{0,\infty}(\|w^n\|_0^2+\|w^{n-1}\|_0^2). \end{aligned}\quad (17)$$

Simplifying (17) yields

$$\begin{aligned}&\frac{1}{\Delta t}\|w^n-w^{n-1}\|_0^2+\|w^n\|_1^2-\|w^{n-1}\|_1^2\\&\leqslant\frac{\Delta t}{2}\|f'\|_{0,\infty}(\|w^n\|_0^2+\|w^{n-1}\|_0^2)\\&\quad+\frac{1}{\Delta t}\|\varpi^n-\varpi^{n-1}\|_0^2,1\leqslant n\leqslant N. \end{aligned}\quad (18)$$

Summating (18) from 1 until n ($n\leqslant N$), while $\triangle t$ is sufficiently small to meet $c\Delta t\leqslant 1/2$, using (16), we obtain

$$\begin{aligned}&\frac{1}{\Delta t}\sum_{i=1}^{n}\|w^i-w^{i-1}\|_0^2+\|w^n\|_1^2\\&\leqslant\|w^0\|_1^2+\Delta t\|f'\|_{0,\infty}\sum_{i=0}^{n-1}\|w^i\|_1^2\\&\quad+\frac{1}{\Delta t}\sum_{i=1}^{n-1}\|\varpi^i-\varpi^{i-1}\|_0^2\\&\leqslant\|w^0\|_1^2+\|\varpi^0\|_0^2+\Delta t\|f'\|_{0,\infty}\sum_{i=0}^{n-1}\|w^i\|_1^2\\&\quad+\sum_{i=0}^{n-1}\|w^i-w^{i-1}\|_0^2,\ \ 1\leqslant n\leqslant N. \end{aligned}\quad (19)$$

Applying Lemma 1 to (19), we procure

$$\begin{aligned}&\frac{1}{\Delta t}\sum_{i=1}^{n}\|w^i-w^{i-1}\|_0^2+\|w^n\|_1^2\\&\leqslant(\|w^0\|_1^2+\|\varpi^0\|_0^2)\exp\left(n\Delta t(1+\Delta t\|f'\|_{0,\infty})\right)\\&\leqslant C_0(\|w^0\|_1^2+\|\varpi^0\|_0^2),1\leqslant n\leqslant N, \end{aligned}\quad (20)$$

where $C_0=\exp\left(n\Delta t(1+\Delta t\|f'\|_{0,\infty})\right)$. Thereupon, by (12) and (20), we procure

$$\|w^n\|_1+\|\varpi^n\|_0\leqslant\tilde{C}_0(\|w^0\|_1+\|\varpi^0\|_0),1\leqslant n\leqslant N, \quad (21)$$

where $\tilde{C}_0=\max\{\sqrt{C_0},\sqrt{1+TC_0}\}$, which is denoted the general c. Thus, if $w^0=\varpi^0=0$, then from (21) we can assert that $w^n=\varpi^n=0$ ($1\leqslant n\leqslant N$). This implies that Problem 3 has at the least a solution set $\{(w^n,\varpi^n)\}_{n=1}^N$.

If Problem 3 has another solution set $\{(\tilde{w}^n,\tilde{\varpi}^n)\}_{n=1}^N$, it should meet the below system of equations:

$$\begin{cases}\frac{1}{\Delta t}\left(\tilde{w}^n-\tilde{w}^{n-1},\upsilon\right)+\frac{1}{2}(\tilde{\varpi}^n+\tilde{\varpi}^{n-1},\upsilon)\\+\frac{1}{2}\left(\nabla(\tilde{w}^n+\tilde{w}^{n-1}),\nabla\upsilon\right)\\=\frac{1}{2}\left(f(\tilde{w}^n)+f(\tilde{w}^{n-1}),\upsilon\right),\forall\upsilon\in\mathbb{W};\\\frac{1}{\Delta t}\left(\vartheta,\tilde{\varpi}^n-\tilde{\varpi}^{n-1}\right)+\frac{1}{2}(\vartheta,\tilde{\varpi}^n+\tilde{\varpi}^{n-1})\\=\frac{1}{2}(\vartheta,\tilde{w}^n+\tilde{w}^{n-1}),\quad\forall\vartheta\in\widetilde{\mathbb{W}},\ 1\leqslant n\leqslant N,\\\tilde{w}^0=w_0(\boldsymbol{x}),\ \ \tilde{\varpi}^0=\varpi_0(\boldsymbol{x}),\quad\boldsymbol{x}\in\Omega.\end{cases}\quad (22)$$

If we set $R^n=w^n-\tilde{w}^n$ and $r^n=\varpi^n-\tilde{\varpi}^n$ and subtract (22) from (5)–(5), then by the LMVTDC, we procure

$$\begin{cases}\frac{1}{\Delta t}\left(R^n-R^{n-1},\upsilon\right)+\frac{1}{2}(r^n+r^{n-1},\upsilon)\\\quad+\frac{1}{2}\left(\nabla(R^n+R^{n-1}),\nabla\upsilon\right)\\=\frac{1}{2}\left(R^nf'(\xi_n)+R^{n-1}f'(\xi_{n-1}),\upsilon\right),\\\forall\upsilon\in\mathbb{W},\ \ 1\leqslant n\leqslant N,\\\frac{1}{\Delta t}\left(\vartheta,r^n-r^{n-1}\right)+\frac{1}{2}(\vartheta,r^n+r^{n-1})\\=\frac{1}{2}(\vartheta,R^n+R^{n-1}),\quad\forall\vartheta\in\widetilde{\mathbb{W}},\ 1\leqslant n\leqslant N,\\R^0=0,\ \ r^0=0,\quad\boldsymbol{x}\in\Omega,\end{cases}\quad (23)$$

in which ξ_i are located between w^i and $\tilde{w}^i$ ($i=n,n-1$).

Since (23) has the same form as (5)–(7), using the completely same technique as proving (21), we can get

$$\|R^n\|_1+\|r^n\|_0\leqslant\tilde{C}_0(\|R^0\|_1+\|r^0\|_0),1\leqslant n\leqslant N. \quad (24)$$

Noting that $r^0=R^0=0$, from (24), we obtain $r^n=R^n=0$, i.e, $w^n=\tilde{w}^n$ and $\varpi^n=\tilde{\varpi}^n$ ($1\leqslant n\leqslant$

N). Therefore, Problem 3 has a unique solution set $\{(w^n,\varpi^n)\}_{n=1}^N$.

Step 2. *Discuss the stability of solutions* $\{(w^n,\varpi^n)\}_{n=1}^N$.

Step 1 has proved that Problem 3 has a unique TSDCNM solution series $\{(w^n,\varpi^n)\}_{n=1}^N$. By (21), we assert that the solution series $\{w^n,\varpi^n\}_{n=1}^N$ to Problem 3 is bounded, i.e., stabilized.

Step 3. *Evaluate the errors of the TSDCNM solution series* $\{(w^n,\varpi^n)\}_{n=1}^N$.

With the Taylor expansion, we procure

$$u'(t_{n-\frac{1}{2}})=\frac{u(t_n)-u(t_{n-1})}{\Delta t}-\frac{\Delta t^2}{24}u'''(\xi_n), \quad (25)$$
$$u(t_{n-\frac{1}{2}})=\frac{u(t_n)+u(t_{n-1})}{2}-\frac{\Delta t^2}{16}u''(\varsigma_n), \quad (26)$$
$$f(u(t_{n-\frac{1}{2}}))=\frac{1}{2}(f(u(t_n))+f(u(t_{n-1}))) -\Delta t^2R(\boldsymbol{x},t), \quad (27)$$

in which $u=w$ or ϖ, $t_{n-1}\leqslant\xi_n\leqslant t_{n+1}$, $t_{n-1}\leqslant\varsigma_n\leqslant t_{n-\frac{1}{2}}$, and $R(\boldsymbol{x},t)$ is a bounded remainder.

Thereupon, subtracting (5)–(7) from (2), taking $t=t_{n-\frac{1}{2}}$, using the LMVTDC, and setting $\varepsilon^n=w(t_n)-w^n$ and $\epsilon^n=\varpi(t_n)-\varpi^n$, we obtain the following system of error equations.

$$\begin{aligned}&\frac{1}{\Delta t}\left(\varepsilon^n-\varepsilon^{n-1},\upsilon\right)+\frac{1}{2}\left(\nabla(\varepsilon^n+\varepsilon^{n-1}),\nabla\upsilon\right)\\&=\frac{1}{2}\left(f'(\eta^n)\varepsilon^n+f'(\eta^{n-1})\varepsilon^{n-1},\upsilon\right)\\&-\frac{1}{2}(\epsilon^n+\epsilon^{n-1},\upsilon)+\frac{\Delta t^2}{24}\left(w'''(\xi_n),\upsilon\right)\\&+\frac{\Delta t^2}{16}\left(\varpi''(\varsigma_n),\upsilon\right)+\frac{\Delta t^2}{16}\left(\nabla w''(\varsigma_n),\nabla\upsilon\right)\\&-\Delta t^2\left(R(\boldsymbol{x},t),\upsilon\right),1\leqslant n\leqslant N,\forall\upsilon\in\mathbb{W}, \quad (28)\\&\frac{1}{\Delta t}\left(\epsilon^n-\epsilon^{n-1},\vartheta\right)+\frac{1}{2}\left(\epsilon^n+\epsilon^{n-1},\vartheta\right)\\&=\frac{1}{2}(\varepsilon^n+\varepsilon^{n-1},\vartheta)+\frac{\Delta t^2}{24}\left(\varpi'''(\xi_n),\vartheta\right)\\&+\frac{\Delta t^2}{16}\left(\varpi''(\varsigma_n),\vartheta\right)-\frac{\Delta t^2}{16}\left(w''(\varsigma_n),\vartheta\right),\\&\quad 1\leqslant n\leqslant N,\quad\forall\vartheta\in\widetilde{\mathbb{W}}, \quad (29)\\&\varepsilon^0=\epsilon^0=0, \quad (30)\end{aligned}$$

where η_i $(i=n\eta,n-1)$ are located between $w(t_n)$ and w^n. Taking $\vartheta=\epsilon^n+\epsilon^{n-1}$ in (29), by Hölder and Cauchy inequalities, we obtain

$$\begin{aligned}&\|\epsilon^n\|_0^2-\|\epsilon^{n-1}\|_0^2+\frac{\Delta t}{2}\|\epsilon^n+\epsilon^{n-1}\|_0^2\\&=\frac{\Delta t}{2}(\varepsilon^n+\varepsilon^{n-1},\epsilon^n+\epsilon^{n-1})\\&+\frac{\Delta t^3}{24}\left(\varpi'''(\xi_n),\epsilon^n+\epsilon^{n-1}\right)\\&+\frac{\Delta t^3}{16}\left(\varpi''(\varsigma_n),\epsilon^n+\epsilon^{n-1}\right)\\&-\frac{\Delta t^3}{16}\left(w''(\varsigma_n),\epsilon^n+\epsilon^{n-1}\right)\\&\leqslant\frac{\Delta t}{4}\|\varepsilon^n+\varepsilon^{n-1}\|_0^2+\frac{\Delta t}{4}\|\epsilon^n+\epsilon^{n-1}\|_0^2\\&+c\Delta t^5,\quad 1\leqslant n\leqslant N. \quad (31)\end{aligned}$$

Thereupon, we get

$$\begin{aligned}&\|\epsilon^n\|_0^2-\|\epsilon^{n-1}\|_0^2\leqslant\frac{\Delta t}{2}(\|\varepsilon^n\|_0^2+\|\varepsilon^{n-1}\|_0^2)\\&+c\Delta t^5,\quad 1\leqslant n\leqslant N. \quad (32)\end{aligned}$$

Summing (32) from 1 until n $(n\leqslant N)$ and noting that $\epsilon^0=0$ procure

$$\|\epsilon^n\|_0^2\leqslant c\Delta t^4+\Delta t\sum_{i=0}^{n}\|w^i\|_0^2,\ 1\leqslant n\leqslant N. \quad (33)$$

Taking $\vartheta=\varepsilon^n-\varepsilon^{n-1}$ in (29), by the Hölder and Cauchy inequalities, we attain

$$\begin{aligned}&\frac{1}{2}(\epsilon^n+\epsilon^{n-1},\varepsilon^n-\varepsilon^{n-1})\\&=\frac{1}{2}(\varepsilon^n+\varepsilon^{n-1},\varepsilon^n-\varepsilon^{n-1})\\&-\frac{1}{\Delta t}(\epsilon^n-\epsilon^{n-1},\varepsilon^n-\varepsilon^{n-1})\\&-\frac{\Delta t^2}{24}\left(\varpi'''(\xi_n),\varepsilon^n-\varepsilon^{n-1}\right)\\&-\frac{\Delta t^2}{16}\left(\varpi''(\varsigma_n),\varepsilon^n-\varepsilon^{n-1}\right)\\&+\frac{\Delta t^2}{16}\left(w''(\varsigma_n),\varepsilon^n-\varepsilon^{n-1}\right)\\&=-\frac{1}{\Delta t}(\epsilon^n-\epsilon^{n-1},w^n-w^{n-1})\\&+\frac{1}{2}(\|\varepsilon^n\|_0^2-\|\varepsilon^{n-1}\|_0^2)\\&-\frac{\Delta t^2}{24}\left(\varpi'''(\xi_n),\varepsilon^n-\varepsilon^{n-1}\right)\\&-\frac{\Delta t^2}{16}\left(\varpi''(\varsigma_n),\varepsilon^n-\varepsilon^{n-1}\right)\\&+\frac{\Delta t^2}{16}\left(w''(\varsigma_n),\varepsilon^n-\varepsilon^{n-1}\right),1\leqslant n\leqslant N. \quad (34)\end{aligned}$$

Taking $\vartheta=\epsilon^n-\epsilon^{n-1}$ in (29) and using the Hölder and Cauchy inequalities, we attain

$$\frac{1}{\Delta t}\|\epsilon^n-\epsilon^{n-1}\|_0^2+\frac{1}{2}(\|\epsilon^n\|_0^2-\|\epsilon^{n-1}\|_0^2)$$

$$
\begin{aligned}
&= (\epsilon^n - \epsilon^{n-1}, \varepsilon^n - \varepsilon^{n-1}) \\
&\quad + \varepsilon^{n-1}, \vartheta) + \frac{\Delta t^2}{24}\left(\varpi'''(\xi_n), \epsilon^n - \epsilon^{n-1}\right) \\
&\quad + \frac{\Delta t^2}{16}\left(\varpi''(\varsigma_n), \epsilon^n - \epsilon^{n-1}\right) \\
&\quad - \frac{\Delta t^2}{16}\left(w''(\varsigma_n), \epsilon^n - \epsilon^{n-1}\right) \\
&\leqslant \frac{1}{2\Delta t}\|\epsilon^n - \epsilon^{n-1}\|_0^2 + \frac{\Delta t}{2}\|\varepsilon^n - \varepsilon^{n-1}\|_0^2 \\
&\quad + c\Delta t^5, \quad 1 \leqslant n \leqslant N. \qquad (35)
\end{aligned}
$$

Thereupon, we procure

$$
\begin{aligned}
&\frac{1}{\Delta t}\|\epsilon^n - \epsilon^{n-1}\|_0^2 + \|\epsilon^n\|_0^2 - \|\epsilon^{n-1}\|_0^2 \\
&\leqslant \Delta t\|\varepsilon^n - \varepsilon^{n-1}\|_0^2 + c\Delta t^5, \quad 1 \leqslant n \leqslant N. \qquad (36)
\end{aligned}
$$

Summing (36) from 1 until n ($n \leqslant N$) and noting that $\epsilon^0 = 0$ procure

$$
\begin{aligned}
&\frac{1}{\Delta t}\sum_{i=1}^{n}\|\epsilon^i - \epsilon^{i-1}\|_0^2 + \|\epsilon^n\|_0^2 \leqslant c\Delta t^5 \\
&\quad + \Delta t\sum_{i=1}^{n}\|\varepsilon^i - \varepsilon^{i-1}\|_0^2, \ 1 \leqslant n \leqslant N. \qquad (37)
\end{aligned}
$$

Taking $v = \varepsilon^n - \varepsilon^{n-1}$ in (28), and using the Hölder and Cauchy inequalities, (34), and Green's formula, we obtain

$$
\begin{aligned}
&\frac{1}{\Delta t}\|\varepsilon^n - \varepsilon^{n-1}\|_0^2 + \frac{1}{2}(\|\nabla\varepsilon^n\|_0^2 - \|\nabla\varepsilon^{n-1}\|_0^2) \\
&\quad + \frac{1}{2}(\|\varepsilon^n\|_0^2 - \|\varepsilon^{n-1}\|_0^2) \\
&= \frac{1}{\Delta t}(\epsilon^n - \epsilon^{n-1}, \varepsilon^n - \varepsilon^{n-1}) \\
&\quad + \frac{1}{2}(f'(\eta^n)\varepsilon^n + f'(\eta^{n-1})\varepsilon^{n-1}, \varepsilon^n - \varepsilon^{n-1}) \\
&\quad + \frac{\Delta t^2}{24}\left(w'''(\xi_n), \varepsilon^n - \varepsilon^{n-1}\right) \\
&\quad + \frac{\Delta t^2}{16}\left(\varpi''(\varsigma_n), \varepsilon^n - \varepsilon^{n-1}\right) \\
&\quad - \frac{\Delta t^2}{16}\left(\Delta w''(\varsigma_n), \varepsilon^n - \varepsilon^{n-1}\right) \\
&\quad - \Delta t^2\left(R(\boldsymbol{x}, t), \varepsilon^n - \varepsilon^{n-1}\right) \\
&\leqslant \frac{1}{2\Delta t}(\|\varepsilon^n - \varepsilon^{n-1}\|_0^2 + 2\|\epsilon^n - \epsilon^{n-1}\|_0^2) \\
&\quad + \frac{\Delta t}{4}\|f'\|_{0,\infty}(\|\varepsilon^n\|_0^2 + \|\varepsilon^{n-1}\|_0^2) \\
&\quad + c\Delta t^5, 1 \leqslant n \leqslant N. \qquad (38)
\end{aligned}
$$

Simplifying (38) yields

$$
\begin{aligned}
&\frac{1}{\Delta t}\|\varepsilon^n - \varepsilon^{n-1}\|_0^2 + \|\varepsilon^n\|_1^2 - \|\varepsilon^{n-1}\|_1^2 \\
&\leqslant \frac{\Delta t}{2}\|f'\|_{0,\infty}(\|\varepsilon^n\|_0^2 + \|\varepsilon^{n-1}\|_0^2) \\
&\quad + \frac{2}{\Delta t}\|\epsilon^n - \epsilon^{n-1}\|_0^2 + c\Delta t^5, 1 \leqslant n \leqslant N. \qquad (39)
\end{aligned}
$$

Summating (39) from 1 until n ($n \leqslant N$), while $\triangle t$ is sufficiently small to meet $\Delta t\|f'\|_{0,\infty} \leqslant 1/2$, using (37), and noticing that $\varepsilon^0 = \epsilon^0 = 0$, we procure

$$
\begin{aligned}
&\frac{1}{\Delta t}\sum_{i=1}^{n}\|\varepsilon^i - \varepsilon^{i-1}\|_0^2 + \|\varepsilon^n\|_1^2 \\
&\leqslant cn\Delta t^5 + \Delta t\|f'\|_{0,\infty}\sum_{i=0}^{n-1}\|\varepsilon^i\|_1^2 \\
&\quad + \frac{4}{\Delta t}\sum_{i=1}^{n-1}\|\epsilon - \epsilon^{i-1}\|_0^2 \\
&\leqslant c\Delta t^4 + \Delta t\|f'\|_{0,\infty}\sum_{i=0}^{n-1}\|\varepsilon^i\|_1^2 \\
&\quad + 4\sum_{i=0}^{n-1}\|\varepsilon^i - \varepsilon^{i-1}\|_0^2, 1 \leqslant n \leqslant N. \qquad (40)
\end{aligned}
$$

Applying Lemma 1 to (40) produces

$$
\begin{aligned}
&\frac{1}{\Delta t}\sum_{i=1}^{n}\|\varepsilon^i - \varepsilon^{i-1}\|_0^2 + \|\varepsilon^n\|_1^2 \\
&\leqslant c\Delta t^4 \exp\left(n\Delta t(4 + \Delta t\|f'\|_{0,\infty})\right) \\
&\leqslant c\Delta t^4, 1 \leqslant n \leqslant N. \qquad (41)
\end{aligned}
$$

Thereupon, by (33) and (41), we procure

$$
\|w^n\|_1 + \|\varpi^n\|_0 \leqslant c\Delta t^2, 1 \leqslant n \leqslant N, \qquad (42)
$$

which gets (9). The proof of Theorem 1 ends. □

Remark 1 *The results of Theorem 1 show that the estimates of errors for the TSDCNM solutions can reach the optimal order $O(\Delta t^2)$ and the TSDCNM solutions are unconditionally stabilized.*

3 A New TGCNMFE Method

To establish the TGCNMFE method, it is necessary to further discretize the spatial variables for the TSDCNM scheme through the two-grid FE technique. Therefore, we assume that $\mathcal{J}_H$ is a quasi-regular subdivision for the coarse mesh on $\bar{\Omega}$, $H = \sup_{K\in\mathcal{J}_H}\{\sup_{\boldsymbol{x},\boldsymbol{y}\in K}\|\boldsymbol{x}-\boldsymbol{y}\|\}$, $l \geqslant 1$ is an integer, and $\mathbb{P}_l(K)$ indicates the polynomial space on $K \in \Im_H$. Thus, the FE subspaces of $\mathbb{W}$ and $\widetilde{\mathbb{W}}$ on the coarse mesh subdivision $\mathcal{J}_H$ may be separately expressed as

$$\mathbb{W}_H = \{w_H \in \mathbb{W} : w_H|_K \in \mathbb{P}_l(K), \forall K \in \mathcal{J}_H\},$$
$$\widetilde{\mathbb{W}}_H = \left\{\tilde{w}_H \in \widetilde{\mathbb{W}} : \tilde{w}_H|_K \in \mathbb{P}_l(K), \forall K \in \mathcal{J}_H\right\}.$$

Similarly, suppose that $\mathcal{J}_h$ is a quasi-regular subdivision for the fine mesh on $\bar{\Omega}$ and $h = \sup\limits_{\kappa\in\mathcal{J}_h}\{\sup\limits_{\boldsymbol{x},\boldsymbol{y}\in\kappa}\|\boldsymbol{x}-\boldsymbol{y}\|\}$ $(h \ll H)$. Thereupon, the FE subspaces of $\mathbb{W}$ and $\widetilde{\mathbb{W}}$ on the fine mesh subdivision $\mathcal{J}_h$ may be separately expressed as

$$\mathbb{W}_h = \{w_h \in \mathbb{W} : w_h|_\kappa \in \mathbb{P}_l(\kappa),\ \forall\kappa \in \mathcal{J}_h\},$$
$$\widetilde{\mathbb{W}}_h = \left\{\tilde{w}_h \in \widetilde{\mathbb{W}} : \tilde{w}_h|_\kappa \in \mathbb{P}_l(\kappa),\ \forall\kappa \in \mathcal{J}_h\right\}.$$

Suppose that $T_\gamma : \mathbb{W} \to \mathbb{W}_\gamma$ $(\gamma = h, H)$ are two H^1-operators that $\forall w \in \mathbb{W}$, there are two unique $T_\gamma w \in \mathbb{W}_\gamma$ to satisfy the below formulas

$$(\nabla(w - T_\gamma w), \nabla w_\gamma) = 0,\ \forall w_\gamma \in \mathbb{W}_\gamma,\ \gamma = h,\ H, \tag{43}$$

and the below estimates of errors

$$|w - T_\gamma w|_r \leqslant c\gamma^{l+1-r},\ \ \forall w \in H^{l+1}(\Omega)\cap\mathbb{W},$$
$$\gamma = h,\ H,\ \ r = -1, 0, 1. \tag{44}$$

Suppose that $P_\gamma : \widetilde{\mathbb{W}} \to \widetilde{\mathbb{W}}_\gamma$ $(\gamma = h, H)$ are two L^2-operators that $\forall v \in \widetilde{\mathbb{W}}$, there are two unique $P_\gamma v \in \widetilde{\mathbb{W}}_\gamma$ to meet the below formulas

$$(v - P_\gamma v, v_\gamma) = 0,\ \forall v_\gamma \in \widetilde{\mathbb{W}}_\gamma,\ \gamma = h,\ H, \tag{45}$$

and the below estimates of errors

$$|v - P_\gamma v|_r \leqslant c\gamma^{l+1-r},\ \ \forall v \in H^{l+1}(\Omega)\cap\widetilde{\mathbb{W}},$$
$$\gamma = h,\ H,\ \ r = -1, 0, 1. \tag{46}$$

Whereupon, a brand-new TGCNMFE method may be constructed as below.

Problem 4 **Step 1**. *On the coarse grid subdivision $\mathcal{J}_H$, compute $(w_H^n, \varpi_H^n) \in \mathbb{W}_H \times \mathbb{W}_H$ $(1 \leqslant n \leqslant N)$ through the below nonlinear system:*

$$\begin{cases} \dfrac{2}{\Delta t}\left(w_H^n - w_H^{n-1}, v_H\right) + (\varpi_H^n + \varpi_H^{n-1}, v_H) \\ +(\nabla(w_H^n + w_H^{n-1}), \nabla v_H) \\ = (f(w_H^n) + f(w_H^{n-1}), v_H), \\ \quad \forall v_H \in \mathbb{W}_H, 1 \leqslant n \leqslant N; \\ \dfrac{2}{\Delta t}\left(\vartheta_H, \varpi_H^n - \varpi_H^{n-1}\right) + \left(\vartheta_H, \varpi_H^n + \varpi_H^{n-1}\right) \\ = (\vartheta_H, w_H^n + w_H^{n-1}), \forall\vartheta_H \in \widetilde{\mathbb{W}}_H, 1 \leqslant n \leqslant N, \\ w_H^0 = T_H w_0, \quad \varpi_H^0 = P_H\varpi^0,\ in\ \Omega. \end{cases} \tag{47}$$

Step 2. *On the fine grid subdivision $\mathcal{J}_h$, compute $(w_h^n, \varpi_h^n) \in \mathbb{W}_h \times \mathbb{W}_h$ $(1 \leqslant n \leqslant N)$ through the linear system:*

$$\begin{cases} \dfrac{2}{\Delta t}(w_h^n - w_h^{n-1}, v_h) + (\varpi_h^n + \varpi_h^{n-1}, v_h) \\ +(\nabla(w_h^n + w_h^{n-1}), \nabla v_h) = \\ (f(w_H^n) + f'(w_H^n)(w_h^n - w_H^n) + f(w_h^{n-1}), v_h), \\ \quad \forall v_h \in \mathbb{W}_h, 1 \leqslant n \leqslant N; \\ \dfrac{2}{\Delta t}\left(\vartheta_h, \varpi_h^n - \varpi_h^{n-1}\right) + \left(\vartheta_h, \varpi_h^n + \varpi_h^{n-1}\right) \\ = (\vartheta, w_h^n + w_h^{n-1}), \forall\vartheta_h \in \widetilde{\mathbb{W}}_h, \quad 1 \leqslant n \leqslant N, \\ w_h^0 = T_h w_0(\boldsymbol{x}), \quad \varpi_h^0 = P_h\varpi^0, \boldsymbol{x}\in\Omega. \end{cases} \tag{48}$$

By Problem 4, we procure the below results.

Theorem 2 *On the coarse grid $\mathcal{J}_H$ and the fine grid $\mathcal{J}_h$, two sole solution sets $\{(w_H^n, \varpi_H^n)\}_{n=1}^N \subset \mathbb{W}_H \times \widetilde{\mathbb{W}}_H$ and $\{(w_h^n, \varpi_h^n)\}_{n=1}^N \subset \mathbb{W}_h \times \widetilde{\mathbb{W}}_h$ can be separately calculated from Problem 4 to satisfy the below unconditional boundedness (unconditional stableness):*

$$\|w_H^n\|_1 + \|w_h^n\|_1 + \|\varpi_H^n\|_0 + \|\varpi_h^n\|_0$$
$$\leqslant c(\|w_0\|_1 + \|\varpi_0\|_0),\ \ 1 \leqslant n \leqslant N, \tag{49}$$

and the next estimates of errors

$$\|w(t_n) - w_H^n\|_0 + \|\varpi(t_n) - \varpi_H^n\|_0$$
$$+H\|\nabla(w(t_n) - w_H^n)\|_0$$
$$\leqslant c(\Delta t^2 + H^{l+1}),\ 1 \leqslant n \leqslant N, \tag{50}$$
$$\|w(t_n) - w_h^n\|_0 + \|\varpi(t_n) - \varpi_h^n\|_0$$
$$+h\|\nabla(w(t_n) - w_h^n)\|_0$$
$$\leqslant c(\Delta t^2 + h^{l+1}),\ 1 \leqslant n \leqslant N. \tag{51}$$

where c used subsequently is also a generical positive constant independent of Δt, H, and h, and $\Delta t = O(h) = O(H^{1+1/l})$.

Proof. The proof of Theorem 2 is finished by the next two steps.

Step 1. *Discuss the existence and unconditional stableness to the TGCNMFE solutions.*

(1) Study the existence and unconditional stableness of the TGCNMFE solutions on the coarse mesh $\mathcal{J}_H$.

Noting that the structure of (47) is the completely same as that of (5)–(7), by using the same proof as the existence and stableness for the TSDCNM solutions in Theorem 2, we can prove that the equation (47) exists a unique solution series $\{(w_H^n, \varpi_H^n)\}_{n=1}^N \subset \mathbb{W}_H \times \widetilde{\mathbb{W}}_H$ to meet

$$\|w_H^n\|_1 + \|\varpi_H^n\|_0 \leqslant c(\|w_0\|_1 + \|\varpi_0\|_0),\ \ 1 \leqslant n \leqslant N. \tag{52}$$

(2) *Study the existence and unconditional stableness to the TGCNMFE solutions on the fine mesh $\mathcal{J}_h$.*

Let

$$\begin{aligned}&A((w,\varpi),(\upsilon,\vartheta))=2(w,\upsilon)+\Delta t\,(\nabla w,\nabla\upsilon)\\&\quad-\Delta t(w,\vartheta)+\Delta t\,(\varpi,\vartheta)+2(\varpi,\vartheta)\\&\quad+\Delta t(\varpi,\upsilon)-\Delta t(f'(w_H^n)w,\upsilon),\\&F(\upsilon,\vartheta)=2\left(w_h^{n-1},\upsilon\right)-\Delta t(\nabla w^{n-1},\nabla\upsilon)\\&\quad-\Delta t(w_h^{n-1},\upsilon_h)\\&\quad+\Delta t(f(w_H^n)-f'(w_H^n)w_H^n+f(w_h^{n-1}),\upsilon).\end{aligned}$$

Thereupon, (48) can be rewritten as the next problem.

Problem 5 *Seek $(w_h^n,\varpi_h^n)\in\mathbb{W}_h\times\mathbb{W}_h$ $(1\leqslant n\leqslant N)$ from the below linear system:*

$$\begin{cases}A((w_h^n,\varpi_h^n),(\vartheta_h,\upsilon_h))=F(\vartheta_h,\upsilon_h),\\\quad\forall(\vartheta_h,\upsilon_h)\in\mathbb{W}_h\times\mathbb{W}_h,\quad 0\leqslant n\leqslant N,\\ w_h^0=T_hw_0(\boldsymbol{x}),\ \varpi_h^0=P_h\varpi^0(\boldsymbol{x}),\ \boldsymbol{x}\in\Omega.\end{cases}\tag{53}$$

When Δt is small enough to meet $\Delta t\|f'\|_{0,\infty}<1$, there is a constant $\alpha_0=\min\{1,\Delta t,2+\Delta t\}>0$ to meet

$$\begin{aligned}&A((w,\varpi),(w,\varpi))=2(w,w)+\Delta t\,(\nabla w,\nabla w)\\&\quad-\Delta t(w,\varpi)+\Delta t\,(\varpi,\varpi)+2(\varpi,\varpi)\\&\quad+\Delta t(\varpi,w)-\Delta t(f'(w_H^n)w,w)\\&\geqslant 2\|w\|_0^2+\Delta t\|\nabla w\|_0^2+(\Delta t+2)\|\varpi\|_0^2\\&\quad-\Delta t\|f'\|_{0,\infty}\|w\|_0^2\\&\geqslant\alpha_0\|(w,\varpi)\|_1^2,\ \forall(w,\varpi)\in\mathbb{W}_h\times\mathbb{W}_h,\end{aligned}\tag{54}$$

where $\|(w,\varpi)\|_1=(\|w\|_1^2+\|\varpi\|_0^2)^{1/2}$ is the norm in $\mathbb{W}\times\widetilde{\mathbb{W}}$. This implies that $A((w,\varpi),(\upsilon,\vartheta))$ is a positive definite bilinear function in $\mathbb{W}_h\times\widetilde{\mathbb{W}}_h$. It is obvious that $A((w,\varpi),(\upsilon,\vartheta))$ is bounded in $\mathbb{W}_h\times\widetilde{\mathbb{W}}_h$ and the linear function $F(\upsilon,\vartheta)$ is bounded in $\mathbb{W}_h\times\widetilde{\mathbb{W}}_h$ for given w_H^n, w_h^{n-1}, and ϖ_h^{n-1}. Thereby, with the Lax-Milgram Theorem in [9, Theorem 1.15], we claim that Problem 5, i.e., Step 2 in Problem 4 has a sole solution series $\{(w_h^n,\varpi_h^n)\}_{n=1}^N\subset\mathbb{W}_h\times\widetilde{\mathbb{W}}_h$ to meet

$$\|w_h^n\|_1+\|\varpi_h^n\|_0\leqslant c(\|w_0\|_1+\|\varpi_0\|_0),1\leqslant n\leqslant N.\tag{55}$$

This implies that, on the fine grid $\mathcal{J}_h$, the solution series $\{(w_h^n,\varpi_h^n)\}_{n=1}^N$ of Problem 4 is unconditionally bounded, namely unconditionally stabilized.

Step 2. *Evaluate the errors of the TGCNMFE solutions.*

(1) *Evaluate the errors of the TGCNMFE solutions on the coarse grid subdivision $\mathcal{J}_H$.*

By subtracting (47) from (5)–(7), taking $\upsilon=\upsilon_H$ and $\vartheta=\vartheta_H$, and setting $\varepsilon_H^n=w^n-T_Hw^n$, $R_H^n=w^n-w_H^n$, $\epsilon_H^n=T_Hw^n-w_H^n$, $\tilde{R}_H^n=\varpi^n-\varpi_H^n$, $\tilde{\varepsilon}_H^n=\varpi^n-T_H\varpi^n$, and $\tilde{\epsilon}_H^n=T_H\varpi^n-\varpi_H^n$, and by the LMVTDC, we obtain

$$\begin{aligned}&\frac{1}{\Delta t}\left(R_H^n-R_H^{n-1},\upsilon_H\right)+\frac{1}{2}\left(\nabla(R_H^n+R_H^{n-1}),\nabla\upsilon_H\right)\\&=\frac{1}{2}\left(f(\chi_n)R_H^n+f'(\chi_{n-1})R_H^{n-1},\upsilon_H\right)\\&-\frac{1}{2}(\tilde{R}_H^n-\tilde{R}_H^{n-1},\upsilon_H),\forall\upsilon_H\in\mathbb{W}_H,1\leqslant n\leqslant N,\end{aligned}\tag{56}$$

$$\begin{aligned}&\frac{2}{\Delta t}(\tilde{R}_H^n-\tilde{R}_H^{n-1},\vartheta_H)+(\tilde{R}_H^n+\tilde{R}_H^{n-1},\vartheta_H)\\&=(R_H^n+R_H^{n-1},\vartheta_H),\forall\vartheta_H\in\mathbb{W}_H,1\leqslant n\leqslant N,\end{aligned}\tag{57}$$

$$R_H^0=w_0-T_Hw_0,\tilde{R}_H^0=\varpi_0-P_H\varpi_0,\quad\text{in }\Omega,\tag{58}$$

where χ_i $(i=n,n-1)$ are located between w^i and w_H^i.

By (56), (43), (57), the second equation of (48), Taylor's formula, and the Hölder and Cauchy inequalities, as $\Delta t=O(H^{1+1/l})$, from (44) we get

$$\begin{aligned}&\frac{1}{\Delta t}\|R_H^n-R_H^{n-1}\|_0^2+\frac{1}{2}\left(\|\nabla R_H^n\|_0^2-\|R_H^{n-1}\|_0^2\right)\\&=\frac{1}{\Delta t}\left(R_H^n-R_H^{n-1},\varepsilon_H^n-\varepsilon_H^{n-1}\right)\\&\quad+\frac{1}{\Delta t}\left(R_H^n-R_H^{n-1}),\epsilon_H^n-\epsilon_H^{n-1}\right)\\&\quad+\frac{1}{2}\left(\nabla(R_H^n-R_H^{n-1}),\nabla(\varepsilon_H^n-\varepsilon_H^{n-1})\right)\\&\quad+\frac{1}{2}\left(\nabla(R_H^n-R_H^{n-1}),\nabla(\epsilon_H^n-\epsilon_H^{n-1})\right)\\&=\frac{1}{\Delta t}\left(R_H^n-R_H^{n-1},\varepsilon_H^n-\varepsilon_H^{n-1}\right)\\&\quad+\frac{1}{2}\left(\nabla(\varepsilon_H^n-\varepsilon_H^{n-1}),\nabla(\varepsilon_H^n-\varepsilon_H^{n-1})\right)\\&\quad-\frac{1}{2}\left(\tilde{R}_H^n+\tilde{R}_H^{n-1},\epsilon_H^n-\epsilon_H^{n-1})\right)\\&\quad+\frac{1}{2}\left(f(\chi_n)R_H^n+f'(\chi_{n-1})R_H^{n-1},\epsilon_H^n-\epsilon_H^{n-1})\right)\\&\leqslant c\Delta tH^{2l}+\frac{1}{2\Delta t}\|R_H^n-R_H^{n-1}\|_0^2+c\Delta t(\|R_H^n\|_0^2\\&\quad+\|R_H^{n-1}\|_0^2)+c\Delta t(\|\tilde{R}_H^n\|_0^2+\|\tilde{R}_H^{n-1}\|_0^2).\end{aligned}\tag{59}$$

It follows that

$$\begin{aligned}&\frac{1}{\Delta t}\|R_H^n-R_H^{n-1}\|_0^2+\|\nabla R_H^n\|_0^2-\|\nabla R_H^{n-1}\|_0^2\\&\leqslant c\Delta tH^{2l}+c\Delta t(\|R_H^n\|_0^2+\|R_H^{n-1}\|_0^2)\\&\quad+c\Delta t(\|\tilde{R}_H^n\|_0^2+\|\tilde{R}_H^{n-1}\|_0^2),\quad 1\leqslant n\leqslant N.\end{aligned}\tag{60}$$

Summating (60) from 1 until n $(n\leqslant N)$ yields

$$\frac{1}{\Delta t}\sum_{i=1}^{n}\|R_H^i-R_H^{i-1}\|_0^2+\|\nabla R_H^n\|_0^2$$

$$\leqslant cn\Delta t H^{2l} + cH^{2l} + c\Delta t \sum_{i=0}^{n} \|\nabla R_H^i\|_0^2$$
$$+c\Delta t \sum_{i=0}^{n} \|\tilde{R}_H^i\|_0^2, \quad 1 \leqslant n \leqslant N. \tag{61}$$

Applying Lemma 1 to (61) yields

$$\frac{1}{\Delta t}\sum_{i=1}^{n} \|R_H^i - R_H^{i-1}\|_0^2 + \|\nabla R_H^n\|_0^2$$
$$\leqslant \left(cH^{2l} + c\Delta t \sum_{i=0}^{n} \|\tilde{R}_H^i\|_0^2\right) \exp\left(cn\Delta t\right)$$
$$\leqslant cH^{2l} + c\Delta t \sum_{i=0}^{n} \|\tilde{R}_H^i\|_0^2, 1 \leqslant n \leqslant N. \tag{62}$$

By (57), (45), (46), Taylor's formula, and the Hölder and Cauchy inequalities, and (62), as $\Delta t = O(H^{1+1/l})$, we procure

$$\frac{1}{\Delta t}\|\tilde{R}_H^n - \tilde{R}_H^{n-1}\|_0^2 + \frac{1}{2}(\|\tilde{R}_H^n\|_0^2 - \|\tilde{R}_H^{n-1}\|_0^2)$$
$$= \frac{1}{\Delta t}(\tilde{R}_H^n - \tilde{R}_H^{n-1}, \tilde{\varepsilon}_H^n - \tilde{\varepsilon}_H^{n-1})$$
$$+ \frac{1}{\Delta t}(\tilde{R}_H^n - \tilde{R}_H^{n-1}, \tilde{\epsilon}_H^n - \tilde{\epsilon}_H^{n-1})$$
$$+ \frac{1}{2}(\tilde{R}_H^n + \tilde{R}_H^{n-1}, \tilde{\varepsilon}_H^n - \tilde{\varepsilon}_H^{n-1})$$
$$+ \frac{1}{2}(\tilde{R}_H^n + \tilde{R}_H^{n-1}, \tilde{\epsilon}_H^n - \tilde{\epsilon}_H^{n-1})$$
$$= \frac{1}{\Delta t}(\tilde{\varepsilon}_H^n - \tilde{\varepsilon}_H^{n-1}, \tilde{\varepsilon}_H^n - \tilde{\varepsilon}_H^{n-1})$$
$$+ \frac{1}{2}(\tilde{\varepsilon}_H^n + \tilde{\varepsilon}_H^{n-1}, \tilde{\varepsilon}_H^n - \tilde{\varepsilon}_H^{n-1})$$
$$+ \frac{1}{2}(R_H^n + R_H^{n-1}, \tilde{\epsilon}_H^n - \tilde{\epsilon}_H^{n-1})$$
$$\leqslant c\Delta t H^{2(l+1)} + c\Delta t(\|R_H^n\|_0^2 + \|R_H^{n-1}\|_0^2)$$
$$+ \frac{1}{2\Delta t}\|\tilde{R}_H^n - \tilde{R}_H^{n-1}\|_0^2$$
$$\leqslant c\Delta t H^{2(l+1)} + c\Delta t^2 \sum_{i=0}^{n} \|\tilde{R}_H^i\|_0^2$$
$$+ \frac{1}{2\Delta t}\|\tilde{R}_H^n - \tilde{R}_H^{n-1}\|_0^2, 1 \leqslant n \leqslant N. \tag{63}$$

When Δt is sufficiently small to meet $c\Delta t \leqslant 1/2$, by simplifying (63), we get

$$\frac{1}{\Delta t}\|\tilde{R}_H^n - \tilde{R}_H^{n-1}\|_0^2 + \|\tilde{R}_H^n\|_0^2 - \|\tilde{R}_H^{n-1}\|_0^2$$
$$\leqslant c\Delta t H^{2(l+1)}$$
$$+ c\Delta t^2 \sum_{i=0}^{n-1} \|\tilde{R}_H^i\|_0^2, \quad 1 \leqslant n \leqslant N. \tag{64}$$

Summating (64) from 1 until n ($n \leqslant N$) and using (46), we procure

$$\frac{1}{\Delta t}\sum_{i=1}^{n} \|\tilde{R}_H^i - \tilde{R}_H^{i-1}\|_0^2 + \|\tilde{R}_H^n\|_0^2$$
$$\leqslant cn\Delta t H^{2(l+1)} + cH^{2(l+1)} + cn\Delta t^2 \sum_{i=0}^{n-1} \|\tilde{R}_H^i\|_0^2$$
$$\leqslant cH^{2(l+1)} + c\Delta t \sum_{i=0}^{n-1} \|\tilde{R}_H^i\|_0^2, 1 \leqslant n \leqslant N. \tag{65}$$

Applying Lemma 1 to (65) procures

$$\|\tilde{R}_H^n\|_0^2 \leqslant cH^{2(l+1)} \exp(cn\Delta t)$$
$$\leqslant cH^{2(l+1)}, \quad 1 \leqslant n \leqslant N. \tag{66}$$

Substituting (66) into (62) procures

$$\|\nabla R_H^n\|_0 \leqslant cH^l, \quad 1 \leqslant n \leqslant N. \tag{67}$$

With the Nitsche technique in [9, Theorem 1.38] and (67), we can procure the next estimates of errors

$$\|w^n - w_H^n\|_0 + H\|\nabla(w^n - w_H^n)\|_0$$
$$\leqslant cH^{l+1}, \quad 1 \leqslant n \leqslant N. \tag{68}$$

Combining (67) and (68) with Theorem 1 yields (50).

(2) *Estimate the errors of the TGCNMFE solutions on the fine grid subdivision $\mathcal{J}_h$.*

By subtracting (48) from (5)–(7), taking $\upsilon = \upsilon_h$ and $\vartheta = \vartheta_h$, and setting $\varepsilon_h^n = w^n - T_h w^n$, $R_h^n = w^n - w_h^n$, $\epsilon_h^n = T_h w^n - w_h^n$, $\tilde{R}_h^n = \varpi^n - \varpi_h^n$, $\tilde{\varepsilon}_h^n = \varpi^n - T_h\varpi^n$, and $\tilde{\epsilon}_h^n = T_h\varpi^n - \varpi_h^n$, and by the LMVTDC, we obtain

$$\frac{1}{\Delta t}\left(R_h^n - R_h^{n-1}, \upsilon_h\right) + \frac{1}{2}\left(\nabla(R_h^n + R_h^{n-1}), \nabla\upsilon_h\right)$$
$$= \frac{1}{2}(f'(\chi_n)R_H^n + f'(\zeta_{n-1})R_h^{n-1}, \upsilon_h)$$
$$-\frac{1}{2}(f'(w_H^n)(w_h^n - w_H^n), \upsilon_h)$$
$$-\frac{1}{2}(\tilde{R}_h^n + \tilde{R}_h^{n-1}, \upsilon_h), \ \forall \upsilon_h \in \mathbb{W}_h, \ 1 \leqslant n \leqslant N, \tag{69}$$
$$\frac{2}{\Delta t}(\tilde{R}_h^n - \tilde{R}_h^{n-1}, \vartheta_h) + (\tilde{R}_h^n + \tilde{R}_h^{n-1}, \vartheta_h)$$
$$= (R_h^n + R_h^{n-1}, \vartheta_h), \ \forall \vartheta_H \in \mathbb{W}_h, 1 \leqslant n \leqslant N, \tag{70}$$
$$R_h^0 = w_0 - T_h w_0, \ \tilde{R}_h^0 = \varpi_0 - P_h\varpi_0, \ \text{ in } \Omega, \tag{71}$$

where ζ_{n-1} is located between w^{n-1} with w_h^{n-1}.

By (69), (70), (43), the Hölder and Cauchy inequalities, Taylor's formula, and (50), when $\Delta t = O(h) = O(H^{1+1/l})$, we get

$$\frac{1}{\Delta t}[\|R_h^n - R_h^{n-1}\|_0^2 + \frac{1}{2}\left(\|\nabla R_h^n\|_0^2 - \|\nabla R_h^{n-1}\|_0^2\right)$$

$$
\begin{aligned}
&= \frac{1}{\Delta t}\left(R_h^n - R_h^{n-1}, \varepsilon_h^n - \varepsilon_h^{n-1}\right) \\
&\quad + \frac{1}{\Delta t}\left(R_h^n - R_h^{n-1}, \epsilon_h^n - \epsilon_h^{n-1}\right) \\
&\quad + \frac{1}{2}\left(\nabla(\varepsilon_h^n - \varepsilon_h^{n-1}), \nabla(\varepsilon_h^n - \varepsilon_h^{n-1})\right) \\
&+ \frac{1}{2}\left(\nabla(\varepsilon_h^n + \varepsilon_h^{n-1}), \nabla(\epsilon_h^n - \epsilon_h^{n-1})\right) \\
&= \frac{1}{\Delta t}\left(R_h^n - R_h^{n-1}, \varepsilon_h^n - \varepsilon_h^{n-1}\right) \\
&\quad + \frac{1}{2}\left(\nabla(\varepsilon_h^n - \varepsilon_h^{n-1}), \nabla(\varepsilon_h^n - \varepsilon_h^{n-1})\right) \\
&\quad - \frac{1}{2}\left(\tilde{R}_h^n + \tilde{R}_h^{n-1}, \epsilon^n - \epsilon_h^{n-1}\right) \\
&\quad + \frac{1}{2}(f'(\chi_n)R_H^n + f'(\zeta_{n-1})R_h^{n-1}, \epsilon^n - \epsilon_h^{n-1}) \\
&- \frac{1}{2}(f'(w_H^n)(w_h^n - w_H^n), \epsilon^n - \epsilon_h^{n-1}) \\
&\leqslant \frac{1}{2\Delta t}\|R_h^n - R_h^{n-1}\|_0^2 + c\Delta t h^{2l} \\
&+ c\Delta t(\|R_h^n\|_0^2 + \|R_h^{n-1}\|_0^2) \\
&+ c\Delta t(\|\tilde{R}_h^n\|_0^2 + \|\tilde{R}_h^{k-1}\|_0^2), \quad 1 \leqslant n \leqslant N. \qquad (72)
\end{aligned}
$$

Thus, from (72) we obtain

$$
\begin{aligned}
&\|\nabla R_h^n\|_0^2 - \|\nabla R_h^{n-1}\|_0^2 \\
&\leqslant c\Delta t h^{2l} + c\Delta t(\|R_h^n\|_0^2 + \|R_h^{n-1}\|_0^2) \\
&\quad + c\Delta t(\|\tilde{R}_h^n\|_0^2 + \|\tilde{R}_h^{k-1}\|_0^2), \ 1 \leqslant n \leqslant N. \qquad (73)
\end{aligned}
$$

When Δt is sufficiently tiny, by summating (73) from 1 until n $(n \leqslant N)$, we procure

$$
\begin{aligned}
\|\nabla R_h^n\|_0^2 &\leqslant cn\Delta t h^{2l} + ch^{2l} + c\Delta t \sum_{i=0}^{n-1} \|\tilde{R}_h^i\|_0^2 \\
&+ c\Delta t \sum_{i=0}^{n} \|R_h^i\|_0^2, \quad 1 \leqslant n \leqslant N. \qquad (74)
\end{aligned}
$$

Applying Lemma 1 to (74) yields

$$
\begin{aligned}
\|\nabla R_h^n\|_0^2 &\leqslant \left(ch^{2l} + c\Delta t \sum_{i=0}^{n} \|R_h^i\|_0^2\right) \exp(cn\Delta t) \\
&\leqslant ch^{2l} + c\Delta t \sum_{i=0}^{n} \|R_h^i\|_0^2, \quad 1 \leqslant n \leqslant N. \qquad (75)
\end{aligned}
$$

By (70), (43), (44), and (75), using the same processes as proving (63)-(66), we procure

$$\|\tilde{R}_h^n\|_0 \leqslant ch^{l+1}, \quad 1 \leqslant n \leqslant N. \qquad (76)$$

Substituting (76) into (75) produces

$$\|\nabla R_h^n\|_0 \leqslant ch^l, \quad 1 \leqslant n \leqslant N. \qquad (77)$$

By the Nitsche method in [9, Theorem 1.38 or Remark 3.1]) and (77), we can procure the next estimates of errors

$$
\begin{aligned}
&\|w^n - w_h^n\|_0 + h\|\nabla(w^n - w_h^n)\|_0 \\
&\leqslant ch^{l+1}, \ 1 \leqslant n \leqslant N. \qquad (78)
\end{aligned}
$$

Thereupon, (51) is gotten by combining Theorem 1 with (76) and (78). This completes the poorf of Theorem 2. □

Remark 2 *The results of Theorem 2 show that the theory errors for the TGCNMFE solutions achieve optimal order and the TGCNMFE solutions are unconditionally stabilized. In the following section, we will resort to numeric tests to explain that the theory errors accord with the numerical errors.*

4 Numerical Tests

In this section, the rightness for the procured theory results and the effectiveness for the TGCNMFE method are attested through some numerical tests of the convective FitzHugh-Nagumo equation that has a set of analytical solutions, but it has no analytical solution usually.

To do so, in the convective FitzHugh-Nagumo equation, we take $\bar{\Omega} = [0,\ \pi] \times [0,\ \pi]$, the initial value functions $w_0(\boldsymbol{x}) = 0.5\sin(x_1)\sin(x_2)$ and $\varpi_0(\boldsymbol{x}) = \sin(x_1)\sin(x_2)$, and the source term as follows:

$$
\begin{aligned}
g(\boldsymbol{x}, t) &= \sin(x_1)\sin(x_2)\exp(-0.5t) \\
&+ 0.125\sin^3(x_1)\sin^3(x_2)\exp(-1.5t) \\
&+ 0.25(a-1)\sin(x_1)\sin(x_2)\exp(-t) \\
&- 0.25a\sin^2(x_1)\sin^2(x_2)\exp(-t), \qquad (79)
\end{aligned}
$$

then it has a set of analytical solution as follows:

$$u(\boldsymbol{x}, t) = 0.5\sin(x_1)\sin(x_2)\exp(-0.5t), \quad (80)$$

$$\varpi(\boldsymbol{x}, t) = \sin(x_1)\sin(x_2)\exp(-0.5t), \qquad (81)$$

to meet $u(\boldsymbol{x}, t) = \varpi(\boldsymbol{x}, t) = 0$ on $\partial\Omega$.

Suppose that the fine mesh subdivision $\mathcal{J}_h$ is composed of the squares with equal side length $(100\pi)^{-1}$. In order to reach optimal order estimates of errors, the coarse mesh subdivision $\mathcal{J}_H$ is composed of the squares with equal side length $(10\pi)^{-1}$. Thus, when $l = 1$ and $\Delta t = 10^{-2}$, the estimates of theory errors for the TGCNMFE solutions can achieve $O(10^{-4})$.

We reckoned the errors of the TGCNMFE solutions when $a = 1$, and $t = 0.2, 0.4, 0.6, 0.8$, and 1.0, shown as Table 1.

Table 1. The errors of the TGCNMFE solutions when $a = 1$, and $t = 0.2, 0.4, 0.6, 0.8$, and 1.0.

t	n	$\|u(t_n) - u_h^n\|$	$\|\varpi(t_n) - \varpi_h^n\|$
0.2	20	2.2134×10^{-4}	1.3217×10^{-4}
0.4	40	2.4463×10^{-4}	1.4545×10^{-4}
0.6	60	2.6792×10^{-4}	1.5873×10^{-4}
0.8	80	2.9121×10^{-4}	1.7211×10^{-4}
1.0	100	3.0452×10^{-4}	1.8459×10^{-4}

The data in Table 1 imply that the numerical simulation errors of the TGCNMFE solutions can also reach $O(10^{-4})$, which is accorded with the theory errors. It is showed that the procured theory results is correct and the TGCNMFE method is effective to calculate the TGCNMFE solutions of the convective FitzHugh-Nagumo equation.

5 Conclusion and Future Prospects

In this article, we have addressed a new TSDCNM method and a new TGCNMFE method for the convective FitzHugh-Nagumo equation, and have strictly attested the existence, stableness, and error estimates of the TSDCNM and TGCNMFE solutions, theoretically. We have also given some numeric tests to attest the correctness of theoretic results and the effectiveness of the TGCNMFE method. The TSDCNM and TGCNMFE methods for the convective FitzHugh-Nagumo equation are created for first-times in this article and are completely distinct from the existing time semi-discrete scheme and the MFE method, including that in [6–8]. Therefore, the TSDCNM and TGCNMFE methods in this article are original and bran-new.

Although the TGCNMFE method in this article is effective to calculate the TGCNMFE solutions of the convective FitzHugh-Nagumo equation, when it is applied to calculating the convective FitzHugh-Nagumo equation in the practical engineering, it usually involves a great number of (often more than tens of millions) unknowns and requires expending a long time to compute the results on a typical computer. Thus, after the computer has been operating for a long time, owe to the accumulation for the computation errors, the procured TGCNMFE solutions will have a significant deviation from the genuine solutions, and there is even a possibility of floating-point number overflow, resulting in some incorrect computation results. Hence, in future study, we will resort to a proper orthogonal decomposition (POD) to lower the dimensionality of the unknown solution coefficient vectors in the TGCNMFE method, and create some new POD dimension reduction methods for the convective FitzHugh-Nagumo equation.

Data Availability Statement

Data will be made available on request.

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Conflicts of Interest

The author declares no conflicts of interest.

Ethical Approval and Consent to Participate

Not applicable.

References

[1] Panfilov, A. & Hogeweg, P. (1993). Spiral breakup in a modified FitzHugh-Nagumo model. *Physics Letters A, 176*(5), 295-299. [CrossRef]

[2] Al-Juaifri, G. A. & Harfash, A. J. (2023). Finite element analysis of nonlinear reaction-diffusion system of Fitzhugh-Nagumo type with Robin boundary conditions. *Mathematics and Computers in Simulation*, 203, 486-517. [CrossRef]

[3] Liu, X., Liu, N., Liu, Y., & Li, H. (2024). Analysis of variable-time-step BDF2 combined with the fast two-grid finite element algorithm for the FitzHugh-Nagumo model. *Computers & Mathematics with Applications, 170*, 186-203. [CrossRef]

[4] Namjoo, M. & Zibaei, S. (2018). Numerical solutions of FitzHugh-Nagumo equation by exact finite-difference and NSFD schemes. *Computational and Applied Mathematics*, 37, 1395-1411. [CrossRef]

[5] Xu, J. (1996). Two-grid discretization techniques for linear and nonlinear PDEs. *SIAM journal on numerical analysis, 33*(5), 1759-1777. [CrossRef]

[6] Liu, Y., Du, Y., Li, H., Li, J., & He, S. (2015). A two-grid mixed finite element method for a nonlinear fourth-order reaction–diffusion problem with time-fractional derivative. *Computers & Mathematics with Applications, 70*(10), 2474-2492. [CrossRef]

[7] Shi, D. & Liu, Q. (2019). An efficient nonconforming finite element two-grid method for Allen-Cahn equation. *Applied Mathematics Letters, 98*, 374-380. [CrossRef]

[8] Shi, D. & Wang, R. (2020). Unconditional superconvergence analysis of a two-grid finite element method for nonlinear wave equations. *Applied Numerical Mathematics, 150*, 38-50. [CrossRef]

[9] Luo, Z. (2024). *Finite element and reduced dimension methods for partial differential equations*. Springer Nature Singapore. [CrossRef]

[10] Teng, F. & Luo, Z. D. (2024). A natural boundary element reduced-dimension model for uniform high-voltage transmission line problem in an unbounded outer domain. *Computational and Applied Mathematics, 43*(3), 106. [CrossRef]

[11] Zhang, G. & Lin, Y. (2011). *Notes on Functional Analysis* (in Chinese). Peking University Press, Beijing. [CrossRef]

[12] Li, K. & Tan, Z. (2023). A two-grid fully discrete Galerkin finite element approximation for fully nonlinear time-fractional wave equations. *Nonlinear Dynamics, 111*(9), 8497-8521. [CrossRef]

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