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RESEARCH ARTICLE

Bipolar Spherical Fuzzy Sets and Their Application in Multi-Attribute Decision Making Problems

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Abstract

In this paper, the concept of the bipolar spherical fuzzy set (BSFS) and its operations is developed. It is proposed as a generalization of fuzzy sets, bipolar fuzzy sets, picture fuzzy sets, and spherical fuzzy sets, allowing uncertain information to be handled more flexibly in the decision-making process. The key objective of this research paper is to introduce a new version of the spherical fuzzy set so called bipolar spherical fuzzy set (BSFS). The bipolar spherical fuzzy set is a new extension of spherical fuzzy sets and picture fuzzy sets. In bipolar spherical fuzzy sets, membership degrees are satisfying the condition $0 \leq (s^+)^2(x) + (i^+)^2(x) + (d^+)^2(x) \leq 1$ and $0 \leq (s^-)^2(x) + (i^-)^2(x) + (d^-)^2(x) \leq 1$ instead of $0 \leq s^2(x) + i^2(x) + d^2(x) \leq 1$ as is in spherical fuzzy sets and $0 \leq s(x) + i(x) + d(x) \leq 1$ as is in the picture fuzzy set. Here, negative membership degree denotes the implicit counter-property corresponding to a bipolar spherical fuzzy set. Also, the BSF-set weighted average operator and BSF-set weighted geometric operator are given to aggregate the BSF-sets. Further a multi attribute decision making (MADM) technique is developed and the proposed aggregation operators are utilized. Finally, a numerical approach for implementation of the proposed technique is developed.

Keywords: fuzzy sets, picture fuzzy sets, spherical fuzzy sets, bipolar spherical fuzzy sets, multi attribute decision making.

1 Introduction

Zadeh [1] assigned membership grades to elements of a set in the interval $[0, 1]$ by offering the idea of FSs. Atanassov [2] worked on IFS as a generalization of FS. He extended the concept of FSs by assigning nonmembership grade say d along with a membership grade say s to elements with a constraint $0 \leq s + d \leq 1$. Strengthening the concept of IFS, Yager [3] suggests Pythagorean fuzzy sets, which somehow enlarged the space of membership and non-membership by introducing some new constraint $0 \leq s^2 + d^2 \leq 1$.

The ordinary FS and IFS cannot be applied in situations because of their limited structure. To deal with such kinds of situations and to develop a concept which is sufficiently close to human nature, Cuong [4] proposed PFS of the form (s, i, d) where the elements in triplet represent satisfaction, abstinence and dissatisfaction degrees, respectively, under the condition $0 \leq s + i + d \leq 1$ and with refusal degree

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defined as $r = 1 - (s + i + d)$. This new structure is considerably closer to human nature than that of earlier concepts and is one of the richest research areas now as it can be used in medical image processing and diagnosis, radar image analysis, biometrics and iris recognition, decision making and medical diagnostic assessments, etc.

At the present, some authors have investigated the problems under the PFS environment. Atanassov [5] presented a decision-making method based on the picture fuzzy weighted cross-entropy and utilized it to rank the different alternatives. Wei [6] also develops some similarity measures for picture fuzzy sets and their applications. Garg [7] defined some picture fuzzy aggregation operators and their applications to multicriteria decision-making.

Mahmood et al. [8] introduced the concept of the spherical fuzzy set (SFS) and T-spherical fuzzy set (T-SFS) as a generalization of FS, IFS and PFS of the form (s, i, d) where the elements in triplet represent satisfaction, abstinence and dissatisfaction degrees, respectively, under the condition $0 \leq s^2 + i^2 + d^2 \leq 1$ and $0 \leq s^n + i^n + d^n \leq 1$ respectively. Ashraf et al. [9] further developed some properties of spherical fuzzy sets and their applications in multi-attribute decision making problems. The application of fuzzy set theory to multi-attribute decision making has been extensively studied since the pioneering work of Chen and Hwang [10]. In this paper, we propose a novel MADM method based on bipolar spherical fuzzy aggregation operators to handle uncertain information more effectively.

The objectives of this paper are: (1) to introduce the new concept that is bipolar spherical fuzzy set (BSF-set) (2) to define the bipolar spherical fuzzy numbers (BSFNs) and related basic, operational, (3) to suggest score, accuracy and certainty functions for comparison, (4) to propose bipolar spherical weighted average aggregation operators, weighted geometric aggregation operator and discussed some on their properties, (5) to present a MADM method based on these aggregation operators based on bipolar spherical fuzzy environment.

The remainder of this paper is organized as follows. In Section 2, the basic notion of PFSs sets, SFSs, BFSs and their some basic properties are presented. In Section 3, the concept of BSF-sets and their operational properties are discussed. In Section 4, the study of weighted average aggregation and weighted geometric aggregation operators to the bipolar spherical fuzzy information is studied. In Section 5, a method for multiple attribute decision making (MADM) based on the bipolar spherical fuzzy environment is developed. A real-valued example is also presented to show the effectiveness of the proposed decision making approach. In Section 6, a comparative analysis is given to show that the proposed approach is much better in comparison to others MADM approaches. Section 7, is the last section which contains the conclusion of the paper.

2 Preliminaries

2.1 Picture Fuzzy Sets

Definition 1. [4] *A picture fuzzy set A on a universe X is an object in the form of:*

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}, \tag{1}$$

where $\mu_A(x) \in [0,1]$ is called the degree of positive membership of x in A, $\eta_A(x) \in [0,1]$ is called the degree of neutral membership of x in A and $\nu_A(x) \in [0,1]$ is called the degree of negative membership of x in A, and where μ_A, η_A and ν_A satisfy the following condition:

$$(\forall x \in X)(\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1). \tag{2}$$

Definition 2. [11] *For any universal set X, a SFSs is of the form:*

$$A = \{(x, s_A(x), i_A(x), d_A(x)) \mid x \in X\}. \tag{3}$$

Here $s_A, i_A, d_A : X \to [0,1]$ represents the degree of membership, degree of abstinence and degree of non-membership, respectively, provided that

$$0 \leq s_A^2(x) + i_A^2(x) + d_A^2(x) \leq 1$$

and

$$r(x) = \sqrt{1 - (s_A^2(x) + i_A^2(x) + d_A^2(x))}$$

are known as the degree of refusal of x in S.

Note 1. *For SFS $= \{(x, s_A(x), i_A(x), d_A(x)) \mid x \in X\}$, which is triple components $(s_A(x), i_A(x), d_A(x))$ are said to SFN and each SFN can be denoted by $A = \langle s_A, i_A, d_A \rangle$, where s_A, i_A and $d_A \in [0,1]$, with condition that*

$$0 \leq s_A^2 + i_A^2 + d_A^2 \leq 1.$$

Consider two spherical fuzzy sets, $A = \{(x, s_A(x), i_A(x), d_A(x)) \mid x \in \xi\}$ and $B = \{(x, s_B(x), i_B(x), d_B(x)) \mid x \in \xi\}$.

Definition 3. *A is subset of B $(A \subseteq B)$ iff:*

$$\begin{aligned} s_A(x) \leq s_B(x), \quad i_A(x) \leq i_B(x) \\ \text{and} \quad d_A(x) \geq d_B(x) \quad \forall x \in X. \end{aligned} \tag{4}$$

Definition 4. *A is equal to B $(A = B)$ iff:*

$$A \subseteq B \quad and \quad d_A(x) \geq B \subseteq A. \tag{5}$$

Definition 5. *The complement of a SFSs A is denoted by A^c and is defined by:*

$$A^c = \{(x, d_A(x), i_A(x), s_A(x)) \mid x \in \xi\}. \tag{6}$$

Definition 6. *The intersection of two SFSs A and B is a SFS set denoted by $A \cap B$ and defined by:*

$$\begin{aligned} A \cap B =& \{(x, \lambda(s_A(x), s_B(x)), \lambda(i_A(x), i_B(x)), \\ & \gamma(d_A(x), d_B(x))) \mid x \in \xi\}. \end{aligned} \tag{7}$$

Definition 7. *The union of two SFSs A and B is a SFS denoted by $A \cup B$ and defined by:*

$$\begin{aligned} A \cup B =& \{(x, \max(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \\ & \min(F_A(x), F_B(x))) \mid x \in \xi\}. \end{aligned} \tag{8}$$

Definition 8. *Let $a = \langle s_a, i_a, d_a \rangle$, $a_1 = \langle s_{a_1}, i_{a_1}, d_{a_1} \rangle$, $a_2 = \langle s_{a_2}, i_{a_2}, d_{a_2} \rangle$ be three SFSs, and $\lambda > 0$, then their operations are defined as follows:*

i)

$$\lambda a = \left(\sqrt{1 - (1 - (s_a)^2)^\lambda},\ (i_a)^\lambda,\ (d_a)^\lambda \right)$$

ii)

$$a^\lambda = \left((s_a)^\lambda,\ (i_a)^\lambda,\ \sqrt{1 - (1 - (d_a)^2)^\lambda} \right)$$

Theorem 1. *Let $a = \langle s_a, i_a, d_a \rangle$, $a_1 = \langle s_{a_1}, i_{a_1}, d_{a_1} \rangle$, $a_2 = \langle s_{a_2}, i_{a_2}, d_{a_2} \rangle$ be three SFNs, and $\lambda > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, then their operations are defined as follows:*

i) $a_1 \oplus a_2 = a_2 \oplus a_1$

ii) $a_1 \otimes a_2 = a_2 \otimes a_1$

iii) $\lambda(a_1 \oplus a_2) = \lambda a_1 \oplus \lambda a_2$

iv) $\lambda_1 a \oplus \lambda_2 a = (\lambda_1 \oplus \lambda_2) a$

v) $(a_1 \otimes a_2)^\lambda = a_1^\lambda \otimes a_2^\lambda$

vi) $a^{\lambda_1} \otimes a^{\lambda_2} = a^{(\lambda_1 + \lambda_2)}$

Proposition 1. *Let A, B & C are three SFNs. Then*

i) $A \subseteq B$ *and* $B \subseteq C \Rightarrow A \subseteq C$

ii) *(Commutativity)* $A \cup B = B \cup A$, $A \cap B = B \cap A$, $A \times B = B \times A$

iii) *(Associativity)* $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$, $A \times (B \times C) = (A \times B) \times C$

iv) *(Distributivity)* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

v) *(Idempotency)* $A \cup A = A$, $A \cap A = A$, $\Delta\Delta A = \Delta A$, $\nabla\nabla A = \nabla A$

vi) *(Absorption)* $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

vii) *(De Morgan's Laws)* $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

viii) *(Involution)* $(A^c)^c = A$

2.2 Score Function and Accuracy Function of SFSs

Let A be a SFS. Then, the score function and accuracy function of a BSFS are defined as follows:

Definition 9. [3] *A score function on a SFN $A = \langle s_a, i_a, d_a \rangle$ is denoted by $Sc(A)$ and defined as follows:*

$$Sc(A) = \frac{1}{3}(2 + s_a - i_a - d_a). \tag{8}$$

Here, $Sc(A) \in [-1, 1]$.

Definition 10. [3] *An accuracy function on a SFN $A = \langle s_a, i_a, d_a \rangle$ is denoted by $Ac(A)$ and defined as follows:*

$$Ac(A) = s_a - d_a. \tag{9}$$

Here, $Ac(A) \in [0, 1]$.

Definition 11. [13] *(Ranking of SFSs) Let A and B be two SFSs. The ranking of A and B by score function and accuracy function is defined as follows:*

i) If $Sc(A) < Sc(B)$, then $A < B$.

ii) If $Sc(A) = Sc(B)$ and if:

a) $Ac(A) < Ac(B)$, then $A < B$.

b) $Ac(A) > Ac(B)$, then $A > B$.

c) $Ac(A) = Ac(B)$, then $A = B$.

Bipolar fuzzy sets were first introduced by Zhang [12] to represent positive and negative information simultaneously, reflecting the bipolar nature of human decision-making. The concept was further developed and formalized by Lee [14] as bipolar-valued fuzzy sets.

Definition 12. [14] *A bipolar-valued fuzzy set in the sense of Lee [14] (BVFSL) on X is a mapping:*

$$A : X \to [-1, 1]. \tag{10}$$

AIFSs are also a particular case of bipolar-valued fuzzy set. Even more, if we have an Atanassov intuitionistic pair (μ, ν), with $\mu, \nu \in [0,1]$ and $\mu + \nu \leq 1$, since we can consider the linear transformation $F : [0,1] \to [-1,0]$ given by $F(t) = t - 1$, we see that the pair (μ, ν) becomes a new pair $(p, q) \in [0,1] \times [-1,0]$ with $p + q = \mu + \nu - 1 \leq 0$.

For every two bipolar fuzzy sets $A = (\mu_A^+, \mu_A^-)$ and $B = (\mu_B^+, \mu_B^-)$ on X,

$$(A \cap B)(x) = \{\min(\mu_A^+, \mu_B^+), \max(\mu_A^-, \mu_B^-)\},$$

$$(A \cup B)(x) = \{\max(\mu_A^+, \mu_B^+), \min(\mu_A^-, \mu_B^-)\}.$$

3 Bipolar Spherical Fuzzy Set

In this section the concept of Bipolar Spherical Fuzzy Sets (BSFSs) is proposed. Also discuss BFSNs score, accuracy and certainty function.

Definition 13. *A bipolar spherical fuzzy set A in X is defined as an object of the form*

$$\begin{aligned} A =&\{(x, s_A^+(x), i_A^+(x), d_A^+(x), \\ & s_A^-(x), i_A^-(x), d_A^-(x)) \mid x \in X\} \end{aligned} \tag{11}$$

where $s_A^+, i_A^+, d_A^+ : X \to [0,1]$, $s_A^-, i_A^-, d_A^- : X \to [-1,0]$ and

$$0 \leq (s_A^+(x))^2 + (i_A^+(x))^2 + (d_A^+(x))^2 \leq 1$$

and

$$0 \leq (s_A^-(x))^2 + (i_A^-(x))^2 + (d_A^-(x))^2 \leq 1.$$

where the positive membership degree $s_A^+(x), i_A^+(x), d_A^+(x)$ denotes the positive membership, neutral membership and negative membership of an element $x \in X$ corresponding to a bipolar spherical fuzzy set A and the negative membership degree $s_A^-(x), i_A^-(x), d_A^-(x)$ denotes the positive membership, neutral membership and negative membership of an element $x \in X$ to some implicit counter-property corresponding to a bipolar spherical fuzzy set A.

For BSFS $A = \{(x, s_A^+(x), i_A^+(x), d_A^+(x), s_A^-(x), i_A^-(x), d_A^-(x)) \mid x \in X\}$, which has six components,

$$\langle s_A^+(x), i_A^+(x), d_A^+(x), s_A^-(x), i_A^-(x), d_A^-(x)\rangle$$

are said to BSFN and each BSFN can be denoted by

$$\langle s_A^+, i_A^+, d_A^+, s_A^-, i_A^-, d_A^-\rangle$$

where $s_A^+, i_A^+, d_A^+ \in [0,1]$, $s_A^-, i_A^-, d_A^- \in [-1,0]$ and

$$0 \leq (s_A^+)^2 + (i_A^+)^2 + (d_A^+)^2 \leq 1$$

and

$$0 \leq (s_A^-)^2 + (i_A^-)^2 + (d_A^-)^2 \leq 1.$$

Example 1. *Let $X = \{x_1, x_2, x_3\}$, then*

$$A = \begin{Bmatrix} \langle x_1, 0.6, 0.3, 0.2, -0.6, -0.4, -0.3\rangle \\ \langle x_2, 0.3, 0.3, 0.6, -0.2, -0.3, -0.6\rangle \\ \langle x_3, 0.2, 0.5, 0.6, -0.4, -0.5, -0.2\rangle \end{Bmatrix}$$

is a bipolar spherical fuzzy sets in X.

Consider two bipolar spherical fuzzy sets, $A = \{\langle x, s_A^+(x), i_A^+(x), d_A^+(x), s_A^-(x), i_A^-(x), d_A^-(x)\rangle \mid x \in X\}$ and $B = \{\langle x, s_B^+(x), i_B^+(x), d_B^+(x), s_B^-(x), i_B^-(x), d_B^-(x)\rangle \mid x \in X\}$.

Definition 14. *A is subset of B $(A \subseteq B)$ iff:*

$$s_A^+(x) \leq s_B^+(x),\ i_A^+(x) \leq i_B^+(x),\ d_A^+(x) \geq d_B^+(x)$$

and

$$s_A^-(x) \geq s_B^-(x),\ i_A^-(x) \geq i_B^-(x),\ d_A^-(x) \leq d_B^-(x) \quad \forall x \in X. \tag{12}$$

Definition 15. *A is equal to B $(A = B)$ iff:*

$$A \subseteq B \quad \text{and} \quad B \subseteq A. \tag{13}$$

Definition 16. *The complement of a bipolar SFSs A is denoted by A^c and is defined by:*

$$A^c = \{\langle x, d_A^+(x), i_A^+(x), s_A^+(x), d_A^-(x), i_A^-(x), s_A^-(x)\rangle \mid x \in X\}. \tag{14}$$

Example 2. *Let $X = \{x_1, x_2, x_3\}$, then*

$$A = \begin{Bmatrix} \langle x_1, 0.5, 0.3, 0.2, -0.6, -0.4, -0.3\rangle \\ \langle x_2, 0.3, 0.3, 0.6, -0.2, -0.3, -0.6\rangle \\ \langle x_3, 0.2, 0.5, 0.5, -0.4, -0.5, -0.2\rangle \end{Bmatrix}$$

is a bipolar spherical fuzzy sets in X.

Then the complement of A as follows:

$$A^c = \begin{Bmatrix} \langle x_1, 0.2, 0.3, 0.5, -0.3, -0.4, -0.6\rangle \\ \langle x_2, 0.6, 0.3, 0.3, -0.6, -0.3, -0.2\rangle \\ \langle x_3, 0.5, 0.5, 0.2, -0.2, -0.5, -0.4\rangle \end{Bmatrix}$$

Definition 17. *The intersection of two bipolar SFSs A and B is a bipolar SFS set denoted by $A \cap B$ and defined by*

$$A \cap B = \begin{cases} \langle(x, \min(s_A^+(x), s_B^+(x)), \min(i_A^+(x), i_B^+(x)), \\ \quad \max(d_A^+(x), d_B^+(x))), \max(s_A^-(x), s_B^-(x)), \\ \quad \min(i_A^-(x), i_B^-(x)), \min(d_A^-(x), d_B^-(x)\rangle \quad \forall x \in X. \end{cases}$$

Example 3. *Let $X = \{x_1, x_2, x_3\}$, then*

$$A = \begin{Bmatrix} (x_1, 0.6, 0.3, 0.2, -0.6, -0.4, -0.3) \\ (x_2, 0.3, 0.4, 0.6, -0.2, -0.3, -0.6) \\ (x_3, 0.2, 0.4, 0.6, -0.4, -0.5, -0.2) \end{Bmatrix}$$

and

$$B = \begin{Bmatrix} (x_1, 0.4, 0.3, 0.3, -0.5, -0.5, -0.3) \\ (x_2, 0.3, 0.3, 0.7, -0.4, -0.3, -0.6) \\ (x_3, 0.3, 0.5, 0.5, -0.2, -0.6, -0.2) \end{Bmatrix}$$

is two bipolar spherical fuzzy sets in X.

Then their intersection as follows:

$$A \cap B = \begin{Bmatrix} (x_1, 0.4, 0.3, 0.3, -0.6, -0.5, -0.3) \\ (x_2, 0.3, 0.3, 0.7, -0.4, -0.3, -0.6) \\ (x_3, 0.2, 0.4, 0.6, -0.4, -0.5, -0.2) \end{Bmatrix}$$

Definition 18. *The union of two bipolar SFSs A and B is a bipolar SFS denoted by* $A \cup B$ *and defined by:*

$$A \cup B = \begin{cases} \langle x, \max(s_A^+(x), s_B^+(x)), \min(i_A^+(x), i_B^+(x)), \\ \min(d_A^+(x), d_B^+(x))\rangle, \langle x, \min(s_A^-(x), s_B^-(x)), \\ \min(i_A^-(x), i_B^-(x)), \max(d_A^-(x), d_B^-(x))\rangle \quad \forall x \in X. \end{cases}$$

Example 4. *Let* $X = \{x_1, x_2, x_3\}$, *then:*

$$A = \begin{cases} \langle x_1, 0.7, 0.3, 0.2, -0.6, -0.3, -0.3\rangle \\ \langle x_2, 0.3, 0.4, 0.5, -0.2, -0.3, -0.6\rangle \\ \langle x_3, 0.2, 0.3, 0.6, -0.4, -0.6, -0.1\rangle \end{cases}$$

and

$$B = \begin{cases} \langle x_1, 0.4, 0.3, 0.3, -0.5, -0.6, -0.3\rangle \\ \langle x_2, 0.5, 0.3, 0.7, -0.4, -0.3, -0.5\rangle \\ \langle x_3, 0.3, 0.5, 0.7, -0.2, -0.6, -0.2\rangle \end{cases}$$

is two bipolar spherical fuzzy sets in X.

Then their union as follows:

$$A \cup B = \begin{cases} \langle x_1, 0.7, 0.3, 0.2, -0.6, -0.6, -0.3\rangle \\ \langle x_2, 0.5, 0.3, 0.5, -0.4, -0.3, -0.5\rangle \\ \langle x_3, 0.3, 0.3, 0.6, -0.4, -0.6, -0.1\rangle \end{cases}$$

Definition 19. *Consider two bipolar spherical fuzzy numbers (BSFNs),* $A = \langle s_A^+, i_A^+, d_A^+, s_A^-, i_A^-, d_A^-\rangle$ *and* $B = \langle s_B^+, i_B^+, d_B^+, s_B^-, i_B^-, d_B^-\rangle$. *Then the operations for these numbers are defined as below:*

i)

$$\lambda A = \Big\langle \sqrt{1-(1-(s_A^+)^2)^\lambda},\ (i_A^+)^\lambda,\ (d_A^+)^\lambda,\ (-s_A^-)^\lambda,\ (-i_A^-)^\lambda,\ -\sqrt{1-(1-(-(d_A^-)^2))^\lambda} \Big\rangle$$

ii)

$$A^\lambda = \Big\langle (s_A^+)^\lambda,\ (i_A^+)^\lambda,\ \sqrt{1-(1-(d_A^+)^2)^\lambda},\ -\sqrt{1-(1-(-(s_A^-)^2))^\lambda},\ -(i_A^-)^\lambda,\ -(-d_A^-)^\lambda \Big\rangle$$

iii)

$$A + B = \Big\langle \sqrt{(s_A^+)^2 + (s_B^+)^2 - (s_A^+)^2 \cdot (s_B^+)^2},\ (i_A^+ \cdot i_B^+),\ (d_A^+ \cdot d_B^+),\ -(s_A^- \cdot s_B^-),\ -(i_A^- \cdot i_B^-),\ -\sqrt{(d_A^-)^2 + (d_B^-)^2 - (d_A^-)^2 \cdot (d_B^-)^2} \Big\rangle$$

iv)

$$A \times B = \big\langle (s_A^+ \cdot s_B^+),\ (i_A^+ \cdot i_B^+),\ \sqrt{(d_A^+)^2 + (d_B^+)^2 - (d_A^+)^2 \cdot (d_B^+)^2},\ -\sqrt{(s_A^-)^2 + (s_B^-)^2 - (s_A^-)^2 \cdot (s_B^-)^2},\ -(i_A^- \cdot i_B^-),\ -(d_A^- \cdot d_B^-)\big\rangle$$

Theorem 2. *Let* $A = \langle s_A^+, i_A^+, d_A^+, s_A^-, i_A^-, d_A^-\rangle$ *and* $B = \langle s_B^+, i_B^+, d_B^+, s_B^-, i_B^-, d_B^-\rangle$ *be two bipolar SFNs, and* $\lambda > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, *then their operations are defined as follows:*

i) $A \oplus B = B \oplus A$

ii) $A \otimes B = B \otimes A$

iii) $\lambda(A \oplus B) = \lambda A \oplus \lambda B$

iv) $\lambda_1 A \oplus \lambda_2 A = (\lambda_1 \oplus \lambda_2)A$

v) $(A \otimes B)^\lambda = A^\lambda \otimes B^\lambda$

vi) $A^{\lambda_1} \otimes B^{\lambda_2} = A^{(\lambda_1+\lambda_2)}$

Proof. i)

$$A \oplus B = \langle s_A^+, i_A^+, d_A^+, s_A^-, i_A^-, d_A^-\rangle + \langle s_B^+, i_B^+, d_B^+, s_B^-, i_B^-, d_B^-\rangle$$

$$= \big\langle (s_A^+)^2 + (s_B^+)^2 - (s_A^+)^2 \cdot (s_B^+)^2,\ (i_A^+ \cdot i_B^+),\ (d_A^+ \cdot d_B^+),\ -(s_A^- \cdot s_B^-),\ -(i_A^- \cdot i_B^-)\big\rangle - \left(\sqrt{-(d_A^+)^2 - (d_B^+)^2} - (d_A^+)^2 \cdot (d_B^+)^2\right)$$

$$= \big\langle (s_B^+)^2 + (s_A^+)^2 - (s_B^+)^2 \cdot (s_A^+)^2,\ (i_B^+ \cdot i_A^+),\ (d_B^+ \cdot d_A^+),\ -(s_B^- \cdot s_A^-),\ -(i_B^- \cdot i_A^-)\big\rangle - \left(\sqrt{-(d_B^+)^2 - (d_A^+)^2} - (d_B^+)^2 \cdot (d_A^+)^2\right)$$

$$= B \oplus A.$$

ii)

$$A \otimes B = \langle s_A^+, i_A^+, d_A^+, s_A^-, i_A^-, d_A^- \rangle \otimes \langle s_B^+, i_B^+, d_B^+, s_B^-, i_B^-, d_B^- \rangle$$
$$= \Big\langle (s_A^+ \cdot s_B^+),\ (i_A^+ \cdot i_B^+),\ \sqrt{(d_A^+)^2 + (d_B^+)^2 - (d_A^+)^2 \cdot (d_B^+)^2},$$
$$-\sqrt{-(s_A^+)^2 - (s_B^+)^2 - (s_A^+)^2 \cdot (s_B^+)^2},\ -(i_A^- \cdot i_B^-),\ -(d_A^- \cdot d_B^-) \Big\rangle$$
$$= \Big\langle (s_B^+ \cdot s_A^+),\ (i_B^+ \cdot i_A^+),\ \sqrt{(d_B^+)^2 + (d_A^+)^2 - (d_B^+)^2 \cdot (d_A^+)^2},$$
$$-\sqrt{-(s_B^+)^2 - (s_A^+)^2 - (s_B^+)^2 \cdot (s_A^+)^2},\ -(i_B^- \cdot i_A^-),\ -(d_B^- \cdot d_A^-) \Big\rangle$$
$$= B \otimes A$$

The proofs of iii) and iv) are straightforward as i) and ii).

v)

$$(A \otimes B)^\lambda = \langle (s_A^+ \cdot s_B^+),\ (i_A^+ \cdot i_B^+),$$
$$\sqrt{(d_A^+)^2 + (d_B^+)^2 - (d_A^+)^2 \cdot (d_B^+)^2},$$
$$-\sqrt{-(s_A^+)^2 - (s_B^+)^2 - (s_A^+)^2 \cdot (s_B^+)^2},$$
$$-(i_A^- \cdot i_B^-),\ -(d_A^- \cdot d_B^-) \rangle$$
$$= \langle (s_A^+ \cdot s_B^+),\ (i_A^+ \cdot i_B^+),$$
$$\sqrt{1 - \left(1 - \left((d_A^+)^2 + (d_B^+)^2 - (d_A^+)^2 \cdot (d_B^+)^2\right)\right)^\lambda},$$
$$-\left((i_A^- \cdot i_B^-)\right)^\lambda,\ -(d_A^- \cdot d_B^-) \Big\rangle$$

vi) Proof can easily as v.

□

3.1 Comparison rules for bipolar SFNs

In this section we introduce some functions which play an important role for the ranking of BSFNs are

Definition 20. *For any BSFN $A = \langle s_A^+, i_A^+, d_A^+, s_A^-, i_A^-, d_A^- \rangle$, the score function of A is defined as follows:*

$$Sc(A) = \frac{1}{6}(s_A^+ + 1 - i_A^+ + 1 - d_A^+ + 1 + s_A^- - i_A^- - d_A^-). \quad (16)$$

For any two BSFNs A, B,

- *if $Sc(A) < Sc(B)$, then $A < B$;*
- *if $Sc(A) > Sc(B)$, then $A > B$;*
- *if $Sc(A) = Sc(B)$, then further study needed as follows.*

Definition 21. *For any BSFN $A = \langle s_A^+, i_A^+, d_A^+, s_A^-, i_A^-, d_A^- \rangle$, the accuracy function of A is defined as follows:*

$$Ac(A) = s_A^+ - d_A^+ + s_A^- - d_A^-. \quad (17)$$

For any two BSFNs A, B,

(i) if $Sc(A) > Sc(B)$, then $A > B$;

(ii) if $Sc(A) = Sc(B)$, then

(ii_1) if $Ac(A) > Ac(B)$, then $A > B$;

(ii_2) if $Ac(A) = Ac(B)$, then further study needed as follows.

Definition 22. *For any BSFN $A = \langle s_A^+, i_A^+, d_A^+, s_A^-, i_A^-, d_A^- \rangle$, the certainty function of A is defined as follows:*

$$Ce(A) = s_A^+ - d_A^-$$

where $Ce(A) \in [0, 1]$.

For any two BSFNs A, B,

(i) if $Sc(A) > Sc(B)$, then $A > B$;

(ii) if $Sc(A) = Sc(B)$, then

(ii_1) if $Ac(A) > Ac(B)$, then $A > B$;

(ii_2) if $Sc(A) = Sc(B)$, $Ac(A) = Ac(B)$, $Ce(A) = Ce(B)$, then $A \sim B$.

Example 5. *Let $X = \{x_1\}$, then*

$$A = \langle x_1, 0.8, 0.4, 0.3, -0.6, -0.3, -0.3 \rangle$$

and

$$B = \langle x_1, 0.4, 0.3, 0.1, -0.8, -0.5, -0.3 \rangle$$

be two bipolar spherical fuzzy sets. Here,

$$Sc(A) = 0.517 \quad and \quad Sc(B) = 0.5, \quad i.e., A > B.$$

Example 6. *Let $X = \{x_1\}$, then*

$$A = \langle x_1, 0.7, 0.4, 0.3, -0.6, -0.3, -0.3 \rangle$$

and

$$B = \langle x_1, 0.4, 0.3, 0.1, -0.8, -0.5, -0.3 \rangle$$

be two bipolar spherical fuzzy sets. Here,

$$Sc(A) = 0.5 \quad and \quad Sc(B) = 0.5.$$

Now,

$$Ac(A) = 0.1 \quad and \quad Ac(B) = -0.2.$$

Example 7. *Let $X = \{x_1\}$, then*

$$A = \langle x_1, 0.65, 0.6, 0.05, -0.8, -0.5, -0.3\rangle$$

and

$$B = \langle x_1, 0.7, 0.4, 0.3, -0.6, -0.3, -0.3\rangle$$

be two bipolar spherical fuzzy sets. Here,

$$Sc(A) = 0.5 \quad \text{and} \quad Sc(B) = 0.5.$$

Also,

$$Ac(A) = 0.1 \quad \text{and} \quad Ac(B) = 0.1.$$

Now,

$$Ce(A) = 0.79 \quad \text{and} \quad Ce(B) = 1.0, \quad \text{i.e., } B > A.$$

4 Bipolar spherical fuzzy aggregation operators

Definition 23. *Let $s_j = (s_j^+, i_j^+, d_j^+, s_j^-, i_j^-, d_j^-)$ $(j = 1, 2, \ldots, n)$ be a collection of BSFNs, then we define the bipolar spherical fuzzy weighted average (BSFWA) operator as below:*

$$BSFWA_w(s_1, s_2, \ldots, s_n) = \bigoplus_{j=1}^{n}(w_j s_j) \tag{19}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of s_j $(j = 1, 2, \ldots, n)$, and $w_j > 0$, $\sum_{j=1}^{n} w_j = 1$.

According to the operational laws of BSFNs, we can get the following theorem.

Theorem 3. *Let $s_j = (s_j^+, i_j^+, d_j^+, s_j^-, i_j^-, d_j^-)$ $(j = 1, 2, \ldots, n)$ be a collection of BSFNs, then their aggregated value by using the BSFWA operator is also a BSFN, and*

$$BSFWA_w(s_1, s_2, \ldots, s_n)$$
$$= \Bigg\langle \sqrt{1 - \prod_{j=1}^{n}\left(1 - (s_j^+)^2\right)^{w_j}}, \prod_{j=1}^{n}(i_j^+)^{w_j}, \prod_{j=1}^{n}(d_j^+)^{w_j},$$
$$-\prod_{j=1}^{n}(-s_j^-)^{w_j}, -\prod_{j=1}^{n}(-i_j^-)^{w_j}, -\sqrt{1 - \prod_{j=1}^{n}\left(1 - (-d_j^-)^2\right)^{w_j}} \Bigg\rangle$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of s_j $(j = 1, 2, \ldots, n)$, and $w_j > 0$, $\sum_{j=1}^{n} w_j = 1$.

Proof. We can prove the theorem by utilizing the technique of mathematical induction. Therefore, we proceed as follows.

(a) For $j = 2$, since

$$w_1 s_1 = \Bigg\langle \sqrt{1 - (1 - (s_1^+)^2)^{w_1}}, (i_1^+)^{w_1}, (d_1^+)^{w_1},$$
$$-(s_1^-)^{w_1}, -(i_1^-)^{w_1}, -\sqrt{1 - (1 - (d_1^-)^2)^{w_1}} \Bigg\rangle$$

and

$$w_2 s_2 = \Bigg\langle \sqrt{1 - (1 - (s_2^+)^2)^{w_2}}, (i_2^+)^{w_2}, (d_2^+)^{w_2},$$
$$-(s_2^-)^{w_2}, -(i_2^-)^{w_2}, -\sqrt{1 - (1 - (d_2^-)^2)^{w_2}} \Bigg\rangle$$

Then,

$$BSFWA_w(s_1, s_2) = w_1 s_1 + w_2 s_2$$
$$= \Bigg\langle \sqrt{1 - (1 - (s_1^+)^2)^{w_1}}, (i_1^+)^{w_1}, (d_1^+)^{w_1},$$
$$-(s_1^-)^{w_1}, -(i_1^-)^{w_1}, -\sqrt{1 - (1 - (d_1^-)^2)^{w_1}} \Bigg\rangle$$
$$+ \Bigg\langle \sqrt{1 - (1 - (s_2^+)^2)^{w_2}}, (i_2^+)^{w_2}, (d_2^+)^{w_2},$$
$$-(s_2^-)^{w_2}, -(i_2^-)^{w_2}, -\sqrt{1 - (1 - (d_2^-)^2)^{w_2}} \Bigg\rangle$$
$$= \Bigg\langle \sqrt{1 - (1 - (s_1^+)^2)^{w_1}(1 - (s_2^+)^2)^{w_2}}, (i_1^+)^{w_1}(i_2^+)^{w_2},$$
$$(d_1^+)^{w_1}(d_2^+)^{w_2}, -(s_1^-)^{w_1}(s_2^-)^{w_2}, -(i_1^-)^{w_1}(i_2^-)^{w_2},$$
$$-\sqrt{1 - (1 - (d_1^-)^2)^{w_1}(1 - (d_2^-)^2)^{w_2}} \Bigg\rangle$$
$$= \Bigg\langle \sqrt{1 - \prod_{j=1}^{2}(1 - (s_j^+)^2)^{w_j}}, \prod_{j=1}^{2}(i_j^+)^{w_j},$$
$$\prod_{j=1}^{2}(d_j^+)^{w_j}, -\prod_{j=1}^{2}(-s_j^-)^{w_j}, -\prod_{j=1}^{2}(-i_j^-)^{w_j},$$
$$-\sqrt{1 - \prod_{j=1}^{2}(1 - (-d_j^-)^2)^{w_j}} \Bigg\rangle$$

(b) Suppose that outcome is true for $j = m$, that is,

$$BSFWA_w(s_1, s_2, \ldots, s_m)$$
$$= \Bigg\langle \sqrt{1 - \prod_{j=1}^{m}(1 - (s_j^+)^2)^{w_j}}, \prod_{j=1}^{m}(i_j^+)^{w_j},$$
$$\prod_{j=1}^{m}(d_j^+)^{w_j}, -\prod_{j=1}^{m}(-s_j^-)^{w_j}, -\prod_{j=1}^{m}(-i_j^-)^{w_j},$$
$$-\sqrt{1 - \prod_{j=1}^{m}(1 - (-d_j^-)^2)^{w_j}} \Bigg\rangle$$

(c) Now we have to prove that outcome is true for $j = m + 1$. By utilizing (a) and (b) we have:

$$\begin{aligned}
&BSFWA_w(s_1, s_2, \dots, s_{m+1}) \\
&= BSFWA_w(s_1, s_2, \dots, s_m) + w_{m+1}s_{m+1} \\
&= \Bigg\langle \sqrt{1 - \prod_{j=1}^{m}(1 - (s_j^+)^2)^{w_j}},\ \prod_{j=1}^{m}(i_j^+)^{w_j}, \\
&\prod_{j=1}^{m}(d_j^+)^{w_j},\ -\prod_{j=1}^{m}(-s_j^-)^{w_j}, \\
&-\prod_{j=1}^{m}(-i_j^-)^{w_j},\ -\sqrt{1 - \prod_{j=1}^{m}(1 - (-d_j^-)^2)^{w_j}} \Bigg\rangle \\
&+ \Big\langle \sqrt{1 - (1 - (s_{m+1}^+)^2)^{w_{m+1}}},\ (i_{m+1}^+)^{w_{m+1}}, \\
&(d_{m+1}^+)^{w_{m+1}},\ -(-s_{m+1}^-)^{w_{m+1}}, \\
&-(-i_{m+1}^-)^{w_{m+1}},\ -\sqrt{1 - (1 - (-d_{m+1}^-)^2)^{w_{m+1}}} \Big\rangle \\
&= \Bigg\langle \sqrt{1 - \prod_{j=1}^{m+1}(1 - (s_j^+)^2)^{w_j}},\ \prod_{j=1}^{m+1}(i_j^+)^{w_j}, \\
&\prod_{j=1}^{m+1}(d_j^+)^{w_j},\ -\prod_{j=1}^{m+1}(-s_j^-)^{w_j}, \\
&-\prod_{j=1}^{m+1}(-i_j^-)^{w_j},\ -\sqrt{1 - \prod_{j=1}^{m+1}(1 - (-d_j^-)^2)^{w_j}} \Bigg\rangle
\end{aligned}$$

Thus, the theorem holds for $j = m + 1$. Therefore, the theorem is satisfied for all j. Hence,

$$\begin{aligned}
&BSFWA_w(s_1, s_2, \dots, s_n) \\
&= \Bigg\langle \sqrt{1 - \prod_{j=1}^{n}(1 - (s_j^+)^2)^{w_j}},\ \prod_{j=1}^{n}(i_j^+)^{w_j}, \\
&\prod_{j=1}^{n}(d_j^+)^{w_j},\ -\prod_{j=1}^{n}(-s_j^-)^{w_j}, \\
&-\prod_{j=1}^{n}(-i_j^-)^{w_j},\ -\sqrt{1 - \prod_{j=1}^{n}(1 - (-d_j^-)^2)^{w_j}} \Bigg\rangle
\end{aligned}$$

which completes the proof. □

4.1 Properties of the BSFWA Operator

There are some properties which are fulfilled by the BSFWA operator obviously.

4.1.1 Idempotency

Let $s_j = (s_j^+, i_j^+, d_j^+, s_j^-, i_j^-, d_j^-)$ $(j = 1, 2, \dots, n)$ be any collection of BSFNs. If all s_j are identical to s, then we have:

$$BSFWA_w(s_1, s_2, \dots, s_n) = s.$$

4.1.2 Boundedness

Let $s_j = (s_j^+, i_j^+, d_j^+, s_j^-, i_j^-, d_j^-)$ $(j = 1, 2, \dots, n)$ be any collection of BSFNs. Then,

$$\min_{j=1,2,\dots,n} s_j \le BSFWA_w(s_1, s_2, \dots, s_n) \le \max_{j=1,2,\dots,n} s_j.$$

4.1.3 Monotonicity

Let $s_j = (s_j^+, i_j^+, d_j^+, s_j^-, i_j^-, d_j^-)$ $(j = 1, 2, \dots, n)$ be any collection of BSFNs. If it satisfies that $s_j \subseteq S_j$ for all $j \in N$, then

$$BSFWA_w(s_1, s_2, \dots, s_n) \subseteq BSFWA_w(S_1, S_2, \dots, S_n).$$

Definition 24. *Let $s_j = (s_j^+, i_j^+, d_j^+, s_j^-, i_j^-, d_j^-)$ $(j = 1, 2, \dots, n)$ be a collection of BSFNs, then we define the bipolar spherical fuzzy weighted geometric (BSFWG) operator as below:*

$$BSFWG_w(s_1, s_2, \dots, s_n) = \bigoplus_{j=1}^{n}(w_j s_j)$$

where $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of s_j $(j = 1, 2, \dots, n)$, and $w_j > 0$, $\sum_{j=1}^{n} w_j = 1$.

According to the operational laws of BSFNs, we can get the following theorem.

Theorem 4. *Let $s_j = (s_j^+, i_j^+, d_j^+, s_j^-, i_j^-, d_j^-)$ $(j = 1, 2, \dots, n)$ be a collection of BSFNs, then their aggregated value by using the BSFWG operator is also a BSFN, and*

$$\begin{aligned}
&BSFWG_w(s_1, s_2, \dots, s_n) \\
&= \Bigg\langle \prod_{j=1}^{n}(s_j^+)^{w_j},\ \prod_{j=1}^{n}(i_j^+)^{w_j},\ \sqrt{1 - \prod_{j=1}^{n}(1 - (d_j^+)^2)^{w_j}}, \\
&-\sqrt{1 - \prod_{j=1}^{n}(1 - (-s_j^-)^2)^{w_j}}, \\
&-\sqrt{1 - \prod_{j=1}^{n}(1 - (-i_j^-)^2)^{w_j}},\ -\prod_{j=1}^{n}(-d_j^-)^{w_j} \Bigg\rangle,
\end{aligned}$$

where $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of s_j $(j = 1, 2, \dots, n)$, and $w_j > 0$, $\sum_{j=1}^{n} w_j = 1$.

Proof. We can prove the theorem by utilizing the technique of mathematical induction. Therefore, we proceed as follows.

(**a**) For $j=2$, since

$$w_1 s_1 = \left\langle (s_1^+)^{w_1},\ (i_1^+)^{w_1},\ \sqrt{1-(1-(d_1^+)^2)^{w_1}},\ -\sqrt{1-(1-(-s_1^-)^2)^{w_1}},\ -(i_1^-)^{w_1},\ -(d_1^-)^{w_1} \right\rangle$$

and

$$w_2 s_2 = \left\langle (s_2^+)^{w_2},\ (i_2^+)^{w_2},\ \sqrt{1-(1-(d_2^+)^2)^{w_2}},\ -\sqrt{1-(1-(-s_2^-)^2)^{w_2}},\ -(i_2^-)^{w_2},\ -(d_2^-)^{w_2} \right\rangle$$

Then,

$$\begin{aligned}
&BSFWG_w(s_1,s_2) = w_1 s_1 + w_2 s_2\\
&= \left\langle (s_1^+)^{w_1},\ (i_1^+)^{w_1},\ \sqrt{1-(1-(d_1^+)^2)^{w_1}},\ -\sqrt{1-(1-(-s_1^-)^2)^{w_1}},\ -(i_1^-)^{w_1},\ -(d_1^-)^{w_1} \right\rangle\\
&+ \left\langle (s_2^+)^{w_2},\ (i_2^+)^{w_2},\ \sqrt{1-(1-(d_2^+)^2)^{w_2}},\ -\sqrt{1-(1-(-s_2^-)^2)^{w_2}},\ -(i_2^-)^{w_2},\ -(d_2^-)^{w_2} \right\rangle\\
&= \langle (s_1^+)^{w_1}(s_2^+)^{w_2},\ (i_1^+)^{w_1}(i_2^+)^{w_2},\\
&\sqrt{1-(1-(d_1^+)^2)^{w_1}(1-(d_2^+)^2)^{w_2}},\\
&-\sqrt{1-(1-(-s_1^-)^2)^{w_1}(1-(-s_2^-)^2)^{w_2}},\\
&-(i_1^-)^{w_1}(i_2^-)^{w_2},\ -(d_1^-)^{w_1}(d_2^-)^{w_2}\rangle\\
&= \left\langle \prod_{j=1}^{2}(s_j^+)^{w_j},\ \prod_{j=1}^{2}(i_j^+)^{w_j},\ \sqrt{1-\prod_{j=1}^{2}(1-(d_j^+)^2)^{w_j}},\right.\\
&-\sqrt{1-\prod_{j=1}^{2}(1-(-s_j^-)^2)^{w_j}},\\
&\left. -\prod_{j=1}^{2}(i_j^-)^{w_j},\ -\prod_{j=1}^{2}(d_j^-)^{w_j} \right\rangle
\end{aligned}$$

(**b**) Suppose that outcome is true for $j=m$, that is,

$$\begin{aligned}
&BSFWG_w(s_1,s_2,\ldots,s_m)\\
&= \left\langle \prod_{j=1}^{m}(s_j^+)^{w_j},\ \prod_{j=1}^{m}(i_j^+)^{w_j},\ \sqrt{1-\prod_{j=1}^{m}(1-(d_j^+)^2)^{w_j}},\right.\\
&\left. -\sqrt{1-\prod_{j=1}^{m}(1-(-s_j^-)^2)^{w_j}},\ -\prod_{j=1}^{m}(i_j^-)^{w_j},\ -\prod_{j=1}^{m}(d_j^-)^{w_j} \right\rangle
\end{aligned}$$

(**c**) Now we have to prove that outcome is true for $j=m+1$. By utilizing (a) and (b) we have:

$$\begin{aligned}
&BSFWG_w(s_1,s_2,\ldots,s_{m+1})\\
&= BSFWG_w(s_1,s_2,\ldots,s_m) + w_{m+1}s_{m+1}\\
&= \left\langle \prod_{j=1}^{m}(s_j^+)^{w_j},\ \prod_{j=1}^{m}(i_j^+)^{w_j},\ \sqrt{1-\prod_{j=1}^{m}(1-(d_j^+)^2)^{w_j}},\right.\\
&\left. -\sqrt{1-\prod_{j=1}^{m}(1-(-s_j^-)^2)^{w_j}},\ -\prod_{j=1}^{m}(i_j^-)^{w_j},\ -\prod_{j=1}^{m}(d_j^-)^{w_j} \right\rangle\\
&+ \left\langle (s_{m+1}^+)^{w_{m+1}},\ (i_{m+1}^+)^{w_{m+1}},\ \sqrt{1-(1-(d_{m+1}^+)^2)^{w_{m+1}}},\right.\\
&\left. -\sqrt{1-(1-(-s_{m+1}^-)^2)^{w_{m+1}}},\ -(i_{m+1}^-)^{w_{m+1}},\ -(d_{m+1}^-)^{w_{m+1}} \right\rangle\\
&= \left\langle \prod_{j=1}^{m+1}(s_j^+)^{w_j},\ \prod_{j=1}^{m+1}(i_j^+)^{w_j},\ \sqrt{1-\prod_{j=1}^{m+1}(1-(d_j^+)^2)^{w_j}},\right.\\
&\left. -\sqrt{1-\prod_{j=1}^{m+1}(1-(-s_j^-)^2)^{w_j}},\ -\prod_{j=1}^{m+1}(i_j^-)^{w_j},\ -\prod_{j=1}^{m+1}(d_j^-)^{w_j} \right\rangle
\end{aligned}$$

Thus, the theorem holds for $j=m+1$. Therefore, the theorem is satisfied for all j. Hence,

$$\begin{aligned}
&BSFWG_w(s_1,s_2,\ldots,s_n)\\
&= \left\langle \prod_{j=1}^{n}(s_j^+)^{w_j},\ \prod_{j=1}^{n}(i_j^+)^{w_j},\ \sqrt{1-\prod_{j=1}^{n}(1-(d_j^+)^2)^{w_j}},\right.\\
&\left. -\sqrt{1-\prod_{j=1}^{n}(1-(-s_j^-)^2)^{w_j}},\ -\prod_{j=1}^{n}(i_j^-)^{w_j},\ -\prod_{j=1}^{n}(d_j^-)^{w_j} \right\rangle
\end{aligned}$$

which completes the proof. □

4.2 Properties of the BSFWG Operator

There are some properties which are fulfilled by the BSFWG operator obviously.

4.2.1 *Idempotency*

Let $s_j = (s_j^+, i_j^+, d_j^+, s_j^-, i_j^-, d_j^-)$ $(j = 1,2,\ldots,n)$ be any collection of BSFNs. If all s_j are identical to s, then we have:

$$BSFWG_w(s_1,s_2,\ldots,s_n) = s.$$

4.2.2 *Boundedness*

Let $s_j = (s_j^+, i_j^+, d_j^+, s_j^-, i_j^-, d_j^-)$ $(j = 1,2,\ldots,n)$ be any collection of BSFNs. Then,

$$\min_{j=1,2,\ldots,n} s_j \le BSFWG_w(s_1,s_2,\ldots,s_n) \le \max_{j=1,2,\ldots,n} s_j.$$

4.2.3 Monotonicity

Let $s_j = (s_j^+, i_j^+, d_j^+, s_j^-, i_j^-, d_j^-)$ $(j = 1, 2, \ldots, n)$ be any collection of BSFNs. If it satisfies that $s_j \subseteq S_j$ for all $j \in N$, then

$$BSFWG_w(s_1, s_2, \ldots, s_n) \subseteq BSFWG_w(S_1, S_2, \ldots, S_n).$$

5 An Approach to MADM with bipolar spherical fuzzy aggregation operators

The bipolar spherical fuzzy decision matrix is presented in Table 1, where c_i $(i = 1, 2, \ldots, m)$ denote the alternatives and g_j $(j = 1, 2, \ldots, n)$ denote the attributes.

In this section, we develop an algorithm to utilize the proposed operators to multiple attribute decision making (MADM) problem under bipolar spherical fuzzy environment. As the procedures of BSFWA and BSFWG are similar, we only consider the BSFWA operator here.

For a MADM problem, as shown in Table 1, assuming that $C = \{c_1, c_2, \ldots, c_m\}$ is any finite collection of m alternatives, $G = \{g_1, g_2, \ldots, g_n\}$ is any finite collection of n attributes and $T = \{t_1, t_2, \ldots, t_p\}$ is any collection of p DMs. If the z-th $(z = 1, 2, \ldots, p)$ DM delivers the assessment of the alternative c_i $(i = 1, 2, \ldots, m)$ on the attribute g_j $(j = 1, 2, \ldots, n)$ under any discrete term set. Let

$$D = [s_{jk}] = \left[(s_{jk}^+, i_{jk}^+, d_{jk}^+, s_{jk}^-, i_{jk}^-, d_{jk}^-)\right]_{m\times n}$$

be the decision matrix, where $(s_{jk}^+, i_{jk}^+, d_{jk}^+, s_{jk}^-, i_{jk}^-, d_{jk}^-)$ is a BSFN representing the evaluation information of alternative c_i $(i = 1, 2, \ldots, m)$ on attribute g_j $(j = 1, 2, \ldots, n)$. Let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of attributes, with $w_j > 0$, $\sum_{j=1}^n w_j = 1$. Then, the main steps of handling the MADM problems are listed below:

Step 1

We utilize the decision information given in matrix D, and the BSFWA operator:

$$s_i = BSFWA_w(s_{i1}, s_{i2}, \ldots, s_{in}) = \prod_{j=1}^n s_{ij}^{w_j}$$

$$= \left\langle \sqrt{1 - \prod_{j=1}^n (1 - (s_{ij}^+)^2)^{w_j}},\ \prod_{j=1}^n (i_{ij}^+)^{w_j},\ \prod_{j=1}^n (d_{ij}^+)^{w_j},\right.$$

$$\left. -\prod_{j=1}^n (-s_{ij}^-)^{w_j},\ -\prod_{j=1}^n (-i_{ij}^-)^{w_j},\ -\sqrt{1 - \prod_{j=1}^n (1 - (-d_{ij}^-)^2)^{w_j}} \right\rangle$$

Step 2

We collect the BSFNs for every decision maker which are given, then use the BSFWA aggregation and BSFWG aggregation operators discussed in Section 4 to aggregate the bipolar spherical fuzzy information.

Step 3

Calculate the score, accuracy and certainty function values respectively of the overall bipolar spherical fuzzy sets by using Definition 20, Definition 21 and Definition 22.

Step 4

Rank all the alternatives C_i $(i = 1, 2, \ldots, m)$ in accordance with the values of score, accuracy and certainty functions and select the best one(s).

Step 5

End.

5.1 Numerical Example

This part describes a MADM problem, which is utilized to illustrate the pertinence and effectiveness for the procedure of decision making problems. There are three manageable alternatives:

(a) c_1 is a car company;

Table 1. Bipolar spherical fuzzy decision matrix.

$C \setminus G$	g_1	g_2	$\cdots$	g_n
c_1	$\langle s_{11}^+, i_{11}^+, d_{11}^+, s_{11}^-, i_{11}^-, d_{11}^-\rangle$	$\langle s_{12}^+, i_{12}^+, d_{12}^+, s_{12}^-, i_{12}^-, d_{12}^-\rangle$	$\cdots$	$\langle s_{1n}^+, i_{1n}^+, d_{1n}^+, s_{1n}^-, i_{1n}^-, d_{1n}^-\rangle$
c_2	$\langle s_{21}^+, i_{21}^+, d_{21}^+, s_{21}^-, i_{21}^-, d_{21}^-\rangle$	$\langle s_{22}^+, i_{22}^+, d_{22}^+, s_{22}^-, i_{22}^-, d_{22}^-\rangle$	$\cdots$	$\langle s_{2n}^+, i_{2n}^+, d_{2n}^+, s_{2n}^-, i_{2n}^-, d_{2n}^-\rangle$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
c_m	$\langle s_{m1}^+, i_{m1}^+, d_{m1}^+, s_{m1}^-, i_{m1}^-, d_{m1}^-\rangle$	$\langle s_{m2}^+, i_{m2}^+, d_{m2}^+, s_{m2}^-, i_{m2}^-, d_{m2}^-\rangle$	$\cdots$	$\langle s_{mn}^+, i_{mn}^+, d_{mn}^+, s_{mn}^-, i_{mn}^-, d_{mn}^-\rangle$

Table 2. Decision Matrix.

	g_1	g_2	g_3
c_1	$\langle 0.6, 0.4, 0.3, -0.3, -0.4, -0.7\rangle$	$\langle 0.5, 0.4, 0.3, -0.3, -0.2, -0.6\rangle$	$\langle 0.3, 0.3, 0.6, -0.3, -0.3, -0.7\rangle$
c_2	$\langle 0.4, 0.2, 0.7, -0.5, -0.2, -0.7\rangle$	$\langle 0.6, 0.2, 0.6, -0.4, -0.4, -0.5\rangle$	$\langle 0.6, 0.4, 0.5, -0.6, -0.4, -0.4\rangle$
c_3	$\langle 0.7, 0.1, 0.3, -0.1, -0.7, -0.2\rangle$	$\langle 0.8, 0.1, 0.2, -0.7, -0.2, -0.7\rangle$	$\langle 0.2, 0.6, 0.4, -0.6, -0.4, -0.7\rangle$

Table 3. Aggregated decision matrix using BSFWG operator.

Alternatives	Aggregated BSFN
c_1	$\langle 0.45, 0.37, 0.43, -0.30, -0.30, -0.66\rangle$
c_2	$\langle 0.53, 0.25, 0.61, -0.50, -0.35, -0.52\rangle$
c_3	$\langle 0.51, 0.17, 0.30, -0.58, -0.49, -0.48\rangle$

(b) c_2 is a food company;

(c) c_3 is a computer company.

According to the attributes the company takes the decision:

(a) g_1 is the risk;

(b) g_2 is the growth;

(c) g_3 is the environmental impact.

The weight vector of the attributes is $\omega = (0.3, 0.4, 0.3)$. The decision matrix $[s_{jk}]$ is presented in Table 2, where s_{jk} $(j = 1, 2, 3; k = 1, 2, 3)$ are in the form of BSFNs. Now we can calculate the following bipolar spherical fuzzy number decision matrix as follows.

Step 1
Construct the decision matrix provided by the customer as:

Step 2
Compute $s_j = BSFWA_w(s_{j1}, s_{j2}, s_{j3})$ for each $j = 1, 2, 3$:

$$s_1 = \langle 0.49, 0.37, 0.37, -0.30, -0.28, -0.66\rangle,$$
$$s_2 = \langle 0.55, 0.25, 0.59, -0.48, -0.32, -0.56\rangle,$$
$$s_3 = \langle 0.68, 0.17, 0.28, -0.37, -0.36, -0.62\rangle.$$

Step 3
Calculate the score values $Sc(s_j)$ $(j = 1, 2, 3)$ for the collective overall bipolar spherical fuzzy numbers s_j $(j = 1, 2, \ldots, n)$ as:

$$Sc(s_1) = 0.565, \quad Sc(s_2) = 0.518, \quad Sc(s_3) = 0.640.$$

Note 2. *Here all the score values are different, i.e., it is not necessary to calculate the accuracy and certainty values.*

Step 4
Rank all the alternatives s_j $(j = 1, 2, 3)$ in accordance with the score values as:

$$s_3 > s_1 > s_2.$$

Thus, s_3 is the most desirable alternative.

Now, considering the same weights, we use the weighted geometric average operators as follows.

Step 1
Construct the decision matrix provided by the customer as shown in Table 3.

Step 2
Compute $s_j = BSFWG_w(s_{j1}, s_{j2}, s_{j3})$ for each $j = 1, 2, 3$:

$$s_1 = \langle 0.45, 0.37, 0.43, -0.30, -0.30, -0.66\rangle,$$
$$s_2 = \langle 0.53, 0.25, 0.61, -0.50, -0.35, -0.52\rangle,$$
$$s_3 = \langle 0.51, 0.17, 0.30, -0.58, -0.49, -0.48\rangle.$$

Step 3
Calculate the score values $Sc(s_j)$ $(j = 1, 2, 3)$ for the collective overall bipolar spherical fuzzy numbers s_j $(j = 1, 2, \ldots, 3)$:

$$Sc(s_1) = 0.552, \quad Sc(s_2) = 0.507, \quad Sc(s_3) = 0.572.$$

Step 4
Rank all the alternatives s_j $(j = 1, 2, 3)$ in accordance with the score values as:

$$s_3 > s_1 > s_2.$$

Thus, s_3 is the most desirable alternative.

6 Comparative analysis

The proposed method has several advantages as below.

6.1 Comparison analysis with the Pythagorean fuzzy TOPSIS

In order to verify the accuracy and effectiveness of the proposed techniques, a comparative approach is conducted using the Pythagorean fuzzy TOPSIS method [16], aggregation operators of PFS [8] and aggregation operators on SFS [3]. These two techniques are valid to solve the MADM problems, TOPSIS method which demonstrated by Hwang et al. [15] simple and valuable strategy to solve the MADM problems with crisp numbers, goes for picking the option with the shortest distance from the positive ideal solution (PIS) and the farthest distance from the negative ideal solution (NIS). As per Zhang et al. [16], the initial step is to distinguish the Pythagorean fuzzy positive ideal solution (PFPIS) and the Pythagorean fuzzy negative ideal solution (PFNIS) of the decision matrix and the bigger is the most desirable alternative.

The decision matrix used in this paper doesn't solve at this above technique because in this paper I use positive, neutral, and negative membership degrees that's square sum is less or equal to 1. Pythagorean fuzzy TOPSIS method fails to solve such problems where the neutral membership grade is considered. So, my proposed technique is more reliable and valid against the Pythagorean fuzzy TOPSIS technique.

6.2 Comparison analysis with the Picture fuzzy Sets

Picture fuzzy set demonstrated by Cuong [4] and Picture fuzzy aggregation operators which were presented by Garg [7]are recorded as, (1) Picture fuzzy weighted averaging operator, (2) Picture fuzzy ordered weighted averaging operator, (3) Picture fuzzy hybrid averaging operator. By using these aggregation operators, a decision making system proposed in the picture fuzzy condition. At last picked the best one alternative from the decision matrices.

The above proposed aggregation operators for PFS manages the real-valued problem where the sum of membership degrees equals or less than 1. In circumstances where the sum of membership degrees surpasses 1, PFS fails to give attractive outcomes in such circumstance. Then again, a procedure proposed in this paper can give satisfactory outcomes in such circumstances where the sum of the membership degrees is greater than 1, and picked the best option from the decision making problem.

6.3 Comparison analysis with the Spherical fuzzy Sets

Spherical fuzzy set initialized by Mahmood et al. [8] and Spherical fuzzy aggregation operators which were introduced by Ashraf et al. [9] are listed as, (1) Spherical fuzzy number weighted averaging aggregation operators, (2) Spherical fuzzy number weighted geometric aggregation operators, (3) Spherical fuzzy hybrid weighted averaging operator. By using these aggregation operators, a decision making technique proposed for the Spherical fuzzy condition. At last picked the best one alternative from decision matrices.

But as of human nature in the time of decision making people make decisions on the basis of positive and negative effects. Chen et al. [17] said that "Bipolarity refers to the propensity of the human mind to reason and make decisions on the basis of positive and negative effects. Positive information states what is possible, satisfactory, permitted, desired, or considered as being acceptable. On the other hand, negative statements express what is impossible, rejected, or forbidden. Negative preferences correspond to the constraints, since they specify which values or objects have to be rejected (i.e., those that do not satisfy the constraints), while positive preferences correspond to wishes, as they specify which objects are more desirable than others (i.e., satisfy user wishes) without rejecting those that do not meet the wishes." Considering in this statement, we can say that the space of bipolar spherical fuzzy sets is greater than the space of spherical fuzzy sets. This indicates that the bipolar spherical fuzzy set takes much more information and our method is meaningful and realistic.

7 Conclusion

In this paper, the concept of the bipolar spherical fuzzy set (BSF-set) and its operations, which is a generalization of fuzzy sets, bipolar fuzzy set, picture fuzzy set and spherical fuzzy set so that it can handle uncertain information more flexibly in the process of decision making is proposed and developed. Information aggregation procedure plays a dynamic job through the decision making procedure and subsequently in this track, the comparative importance of the criteria remains unchanged in modified problems, at that point I proposed BSFWA aggregation and BSFWG aggregation operators has been executed to identify the best alternative. In the past almost all the researchers have worked the intuitionistic fuzzy sets and Pythagorean fuzzy sets by considering the positive and negative membership degrees only. However, it has been identified that in a few conditions, as in the circumstance of casting a ballot, human thoughts contemplations including more degrees as, positive, neutral, negative, refusal, and in circumstance where the sum of the membership degree is greater than 1, which can't be actually described in Intuitionistic fuzzy sets or Pythagorean fuzzy sets or Picture fuzzy sets respectively. In this circumstance, Spherical fuzzy set, which is an extension of the Picture fuzzy sets, which is more realistic. But as of human nature in the time of decision making people makes decisions on the basis of positive and negative effects. In this context Bipolar spherical fuzzy set which is an extension of the Spherical fuzzy set, has been used in the paper and correspondingly aggregation operators have been defined and used. A few required properties of these operators have likewise been investigated. Finally, in light of these operators, a multi attribute decision-making (MADM) method has been built up for ranking the dissimilar alternatives by using the bipolar spherical fuzzy environment. The recommended technique has been demonstrated with a descriptive example for viewing validity and applicability of the new proposed approach. Therefore, the recommended operators give another direction to the decision making process and give a new easier track to hold the vulnerabilities throughout the decision-making procedure.

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The author declares no conflicts of interest.

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