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RESEARCH ARTICLE

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Semi-Analytical Solution of the Fractional Wazwaz–Benjamin–Bona–Mahony (WBBM) System via the Laplace Transform Adomian Decomposition Method

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Abstract

This study explores the application of the Adomian Decomposition Method (ADM), combined with the Laplace transform, to obtain approximate solutions of the time–fractional Wazwaz–Benjamin–Bona–Mahony (WBBM) equations. These equations, which describe wave phenomena in fluid dynamics within a fractional–time framework, present significant analytical challenges. By integrating the Laplace transform with ADM, a modified technique—referred to as the Laplace Transform Adomian Decomposition Method (LTADM)—is introduced. The time–fractional WBBM equations are solved using LTADM, and three–dimensional solution plots are generated for six different values of the fractional order δ (0.5, 0.6, 0.7, 0.8, 0.9, and 1). The results demonstrate that LTADM is an efficient and reliable method for solving nonlinear fractional differential equations such as the WBBM model.

Keywords: fractional calculus, laplace transform, adomian decomposition method, WBBM system, nonlinear evolution equations.

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1 Introduction

Time-dependent phenomena in various domains are modeled by evolutionary differential equations (EDEs), which describe the temporal evolution of systems through ordinary differential equations (ODEs) and partial differential equations (PDEs). Complex behaviors such as turbulence and chaos are represented by nonlinear evolutionary differential equations (NLEDEs), which arise from the incorporation of nonlinear terms. Alongside substantial advancements in analytical and numerical methods for NLEDEs, fractional-order nonlinear PDEs have recently enhanced our understanding of physical, chemical, and biological processes [1].

The study of surface wave dynamics and nonlinear wave propagation involves nonlinear PDEs such as the Korteweg–de Vries (KdV), Boussinesq, Kadomtsev–Petviashvili (KP), Benjamin–Bona–Mahony (BBM), and Sawada–Kotera (SK) equations. While the KdV equation successfully models shallow water wave phenomena, it exhibits limitations when applied to long-wavelength waves.

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To overcome this issue, Benjamin *et al.* [2] proposed the BBM equation in 1972, which accurately models long waves of small amplitude:

$$\mathcal{Y}_t(x,t) + \delta_1 \mathcal{Y}_x(x,t) + \delta_2 \mathcal{Y}\mathcal{Y}_x(x,t) - \delta_3 \mathcal{Y}_{xxt}(x,t) = 0. \quad (1)$$

Here, δ_1, δ_2, and δ_3 are constant coefficients. This equation is a modified version of the classical KdV equation and is suitable for modeling one-dimensional long waves of small amplitude:

$$\mathcal{Y}_t(x,t) + \mathcal{Y}_x(x,t) + \mathcal{Y}\mathcal{Y}_x(x,t) + \mathcal{Y}_{xxx}(x,t) = 0. \quad (2)$$

The generalized BBM equation extends this model as

$$\begin{aligned} \mathcal{Y}_t(x,y,z,t) + \delta_1 \mathcal{Y}_x(x,y,z,t) + \delta_2 \mathcal{Y}^n \mathcal{Y}_x(x,y,z,t) \\ - \delta_3 \mathcal{Y}_{xxt}(x,y,z,t) = 0, \quad n \geq 1, \end{aligned} \quad (3)$$

which reduces to the standard BBM equation for $n = 1$ and to the modified BBM (mBBM) equation for $n = 2$, analogous to the modified KdV (mKdV) equation.

Both the BBM and modified BBM equations can be transformed into ODEs that are solvable using elliptic functions [3] and possess the Painlevé property [4]. Numerous analytical techniques have been developed to obtain traveling-wave solutions of generalized BBM-type equations, including the tanh–sech method [3, 5], the sine–cosine method [3, 5, 6], elliptic function techniques [7], symbolic computation methods [8], and several other approaches [9–12]. Further studies have also considered the BBM equation with dual-power-law nonlinearity [9, 12]:

$$\begin{aligned} \mathcal{Y}_t(x,t) + \delta_1 \mathcal{Y}_x(x,t) + (\delta_2 \mathcal{Y}^n + \delta_3 \mathcal{Y}^{2n}) \mathcal{Y}_x(x,t) \\ - \delta_4 \mathcal{Y}_{xxt}(x,t) = 0, \end{aligned} \quad (4)$$

where δ_2 and δ_3 denote the dual-power-law nonlinearity coefficients, δ_1 and δ_4 represent dispersion parameters, n is the nonlinearity index, and $\mathcal{Y}$ denotes the wave profile. Exact solutions for $n = 1$ were obtained by Johnpillai *et al.* [13].

Although the BBM equation has been extensively analyzed in the integer-order case, its fractional-order extension provides deeper insight into wave propagation in complex media, including shallow water, viscoelastic materials, and stratified fluids. Fractional calculus captures memory and nonlocal effects, which are significant in coastal engineering, geophysical flows, and nonlinear optics. Kolebaje and Popoola [14] introduced the fractional BBM equation

$$\mathcal{Y}_t^{\alpha}(x,t) + \mathcal{Y}_x^{\beta}(x,t) + \mathcal{Y}\mathcal{Y}_x^{\beta} - \mathcal{Y}_t^{\alpha}\mathcal{Y}_{xx}^{\beta} = 0, \quad (5)$$

which was solved using the $\frac{G}{G'}$-expansion method. Subsequent studies employed techniques such as the extended trial equation method [15], fractional sub-equation methods [16–18], and generalized fractional sub-equation approaches [19].

The BBM–Burgers equation introduces dissipation and is defined as [20]

$$\mathcal{Y}_t + \delta_1 \mathcal{Y}_x + \delta_2 \mathcal{Y}^n \mathcal{Y}_x - \delta_3 \mathcal{Y}_{xxt} - \delta_4 \mathcal{Y}_{xx} = 0, \quad n \geq 1, \quad (6)$$

reducing to the Burgers equation for $n = 1$, $\delta_1 = 0$, and $\delta_3 = 0$, and to the generalized BBM equation for $\delta_4 = 0$. The fractional BBM–Burgers (BBMB) equation is given by

$$\mathcal{Y}_t^{\alpha} + \delta_1 \mathcal{Y}_x^{\beta} + \delta_2 \mathcal{Y}^n \mathcal{Y}_x^{\beta} - \delta_3 \mathcal{Y}_t^{\alpha} \mathcal{Y}_{xx}^{2\beta} - \delta_4 \mathcal{Y}_{xx}^{2\beta} = 0, \quad n \geq 1. \quad (7)$$

In the context of wave propagation, Wazwaz [21] introduced a system of WBBM equations describing multi-component wave interactions:

$$\begin{cases} \mathcal{Y}_t + \alpha \mathcal{Y}_x + \beta \mathcal{Y}^2 \mathcal{Y}_y - \gamma \mathcal{Y}_{xzt} = 0, \\ \mathcal{Y}_t + \alpha \mathcal{Y}_z + \beta \mathcal{Y}^2 \mathcal{Y}_x - \gamma \mathcal{Y}_{xyt} = 0, \\ \mathcal{Y}_t + \alpha \mathcal{Y}_y + \beta \mathcal{Y}^2 \mathcal{Y}_z - \gamma \mathcal{Y}_{xxt} = 0, \end{cases} \quad (8)$$

which has been investigated using various analytical approaches [22–24]. The fractional WBBM system was later presented by Seadawy *et al.* [25], followed by traveling-wave and numerical solutions [26–29].

Integral transforms such as the Laplace and Fourier transforms are essential tools for solving both fractional and classical differential equations, as they convert differential equations into algebraic forms. Dhaigude *et al.* [30] applied the Laplace–Adomian decomposition method to a fractional BBM–Burgers equation. Motivated by this approach, we apply the Laplace Transform Adomian Decomposition Method (LTADM) to solve the fractional WBBM equations in the present study.

2 Basic Definitions

2.1 Fractional Derivatives

Fractional calculus extends the concept of integer-order differentiation to non-integer orders. Among several existing definitions, the Caputo and Riemann–Liouville derivatives are the most widely used in theory and applications.

2.1.1 Caputo Fractional Derivative

The Caputo fractional derivative is particularly suitable for physical and engineering models, as it permits the use of classical initial conditions expressed in terms of integer-order derivatives.

Let $f(t) \in C^n[0,T]$ and let $\alpha > 0$. The Caputo fractional derivative of order α is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau, \quad n = \lceil \alpha \rceil, \tag{9}$$

where $\Gamma(\cdot)$ denotes the Gamma function and $\lceil \alpha \rceil$ represents the smallest integer greater than or equal to α.

2.1.2 *Riemann–Liouville Fractional Derivative*

The Riemann–Liouville fractional derivative of order $\alpha > 0$ is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau, \ n = \lceil \alpha \rceil. \tag{10}$$

Unlike the Caputo definition, the Riemann–Liouville derivative involves fractional-order initial conditions, which may complicate the formulation of physical problems.

2.2 Laplace Transform of the Caputo Derivative

The Laplace transform of the Caputo fractional derivative plays a fundamental role in solving fractional differential equations. If $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{D^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n = \lceil \alpha \rceil. \tag{11}$$

This property allows fractional differential equations with classical initial conditions to be transformed into algebraic equations in the Laplace domain, thereby simplifying their analytical treatment.

3 Proposed Algorithm

Consider the fractional WBBM system with initial condition

$$\mathcal{Y}(x,y,z,0) = \mathcal{Y}_0(x,y,z),$$

where $0 < \delta \leq 1$. The system is given by

$$\begin{cases} {}^C D_t^\delta \mathcal{Y} + \alpha D_x^{\delta_1} \mathcal{Y} + \beta \mathcal{Y}^2 D_y^{\delta_1} \mathcal{Y} - \gamma D_x^{\delta_1} D_z^{\delta_1} {}^C D_t^\delta \mathcal{Y} = 0, \\ {}^C D_t^\delta \mathcal{Y} + \alpha D_z^{\delta_1} \mathcal{Y} + \beta \mathcal{Y}^2 D_z^{\delta_1} \mathcal{Y} - \gamma D_x^{\delta_1} D_y^{\delta_1} {}^C D_t^\delta \mathcal{Y} = 0, \\ {}^C D_t^\delta \mathcal{Y} + \alpha D_y^{\delta_1} \mathcal{Y} + \beta \mathcal{Y}^2 D_x^{\delta_1} \mathcal{Y} - \gamma D_x^{2\delta_1} {}^C D_t^\delta \mathcal{Y} = 0. \end{cases}$$

Laplace Transform in Time

Applying the Laplace transform $\mathcal{L}\{\cdot\}$ with respect to t and using the Caputo property,

$$\mathcal{L}\{{}^C D_t^\delta \mathcal{Y}\} = s^\delta \mathcal{Y}(x,y,z,s) - s^{\delta-1} \mathcal{Y}_0(x,y,z),$$

we obtain

$$\begin{aligned} &s^\delta \mathcal{Y}(x,y,z,s) - s^{\delta-1}\mathcal{Y}_0(x,y,z) + \alpha\, \mathcal{L}\{D_x^{\delta_1}\mathcal{Y}\} \\ &\quad + \beta\, \mathcal{L}\{\mathcal{Y}^2 D_y^{\delta_1}\mathcal{Y}\} - \gamma\, \mathcal{L}\{D_x^{\delta_1} D_z^{\delta_1} {}^C D_t^\delta \mathcal{Y}\} = 0. \end{aligned} \tag{12}$$

Solving for $\mathcal{Y}(x,y,z,s)$ yields

$$\begin{aligned} \mathcal{Y}(x,y,z,s) = {} & \frac{1}{s}\mathcal{Y}_0(x,y,z) \\ & - \alpha s^{-\delta} \mathcal{L}\{D_x^{\delta_1}\mathcal{Y}\} - \beta s^{-\delta} \mathcal{L}\{\mathcal{Y}^2 D_y^{\delta_1}\mathcal{Y}\} \\ & + \gamma s^{-\delta} \mathcal{L}\{D_x^{\delta_1} D_z^{\delta_1} {}^C D_t^\delta \mathcal{Y}\}. \end{aligned} \tag{13}$$

Inverse Laplace Transform

Applying the inverse Laplace transform,

$$\begin{aligned} \mathcal{Y}(x,y,z,t) = {} & \mathcal{Y}_0(x,y,z) \\ & - \alpha \mathcal{L}^{-1}\Big[s^{-\delta}\mathcal{L}\{D_x^{\delta_1}\mathcal{Y}\}\Big] \\ & - \beta \mathcal{L}^{-1}\Big[s^{-\delta}\mathcal{L}\{\mathcal{Y}^2 D_y^{\delta_1}\mathcal{Y}\}\Big] \\ & + \gamma \mathcal{L}^{-1}\Big[s^{-\delta}\mathcal{L}\{D_x^{\delta_1} D_z^{\delta_1} {}^C D_t^\delta \mathcal{Y}\}\Big]. \end{aligned}$$

Define the operator

$$\mathcal{I} := \mathcal{L}^{-1} \circ (s^{-\delta} \cdot) \circ \mathcal{L}.$$

Then the equation becomes

$$\mathcal{Y} = \mathcal{Y}_0 - \alpha \mathcal{I}[\mathbf{L}^1] - \beta \mathcal{I}[\mathbf{N}] + \gamma \mathcal{I}[\mathbf{L}^2], \tag{14}$$

where

$$\mathbf{L}^1 = D_x^{\delta_1}\mathcal{Y}, \qquad \mathbf{L}^2 = D_x^{\delta_1} D_z^{\delta_1} {}^C D_t^\delta \mathcal{Y}, \qquad \mathbf{N} = \mathcal{Y}^2 D_y^{\delta_1}\mathcal{Y}.$$

Series Decomposition

Assume a series solution

$$\mathcal{Y} = \sum_{n=0}^\infty \mathcal{Y}_n, \qquad \mathcal{Y}_0(x,y,z,t) = \mathcal{Y}_0(x,y,z).$$

The recursive scheme becomes

$$\begin{aligned} \sum_{n=1}^\infty \mathcal{Y}_n = {} & -\alpha \mathcal{I}\left[\sum_{n=1}^\infty \mathbf{L}_{n-1}^1\right] - \beta \mathcal{I}\left[\sum_{n=1}^\infty \mathbf{N}_{n-1}\right] \\ & + \gamma \mathcal{I}\left[\sum_{n=1}^\infty \mathbf{L}_{n-1}^2\right]. \end{aligned}$$

Adomian Decomposition of Nonlinear Term

Let

$$\mathbf{N} = \mathcal{Y}^2 D_y^{\delta_1}\mathcal{Y}, \qquad \mathcal{Y} = \sum_{n=0}^{\infty}\mathcal{Y}_n.$$

Introduce Adomian polynomials

$$\mathbf{N} = \sum_{n=0}^{\infty}\mathcal{A}_n,$$

where

$$\boxed{\mathcal{A}_n = \sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \mathcal{Y}_i\,\mathcal{Y}_j\,D_y^{\delta_1}\mathcal{Y}_k, \qquad n = 0,1,2,\dots}$$

which follows directly from expanding $\mathcal{Y}^2 D_y^{\delta_1}\mathcal{Y}$ under the series representation.

4 Implementation of Numerical Example Problems

4.1 Time–Fractional First WBBM Equation and Its Solution

Consider the time–fractional WBBM equation

$$\begin{aligned} {}^{C}D_t^{\delta}\mathcal{Y} + \alpha\,\mathcal{Y}_x + \beta\,\mathcal{Y}^2\,\mathcal{Y}_y \\ -\gamma\,\mathcal{Y}_{xzt} = 0, \qquad 0<\delta\leq 1, \end{aligned} \tag{15}$$

with parameters

$$\alpha = \beta = \gamma = 1, \tag{16}$$

and initial condition

$$\mathcal{Y}(x,y,z,0) = e^{-(x+z)}\sin y. \tag{17}$$

Solution via LTADM

Applying the fractional integral operator

$$\mathcal{I}[\,\cdot\,] = \mathcal{L}^{-1}\left[s^{-\delta}\mathcal{L}\{\cdot\}\right], \tag{18}$$

we obtain

$$\begin{aligned} \mathcal{Y} = \mathcal{Y}_0 - \mathcal{I}[\mathcal{Y}_x] - \mathcal{I}[\mathcal{Y}^2\mathcal{Y}_y] \\ + \mathcal{I}[\mathcal{Y}_{xzt}], \end{aligned} \tag{19}$$

where

$$\mathcal{Y}_0(x,y,z) = e^{-(x+z)}\sin y. \tag{20}$$

Assume the series representation

$$\mathcal{Y} = \sum_{n=0}^{\infty}\mathcal{Y}_n, \qquad \mathcal{Y}_0 = e^{-(x+z)}\sin y. \tag{21}$$

The recursive scheme becomes

$$\begin{aligned} \mathcal{Y}_{n+1} = & -\mathcal{I}[(\mathcal{Y}_n)_x] \\ & -\mathcal{I}[\mathcal{A}_n] \\ & +\mathcal{I}[(\mathcal{Y}_n)_{xzt}]\,. \end{aligned} \tag{22}$$

Adomian Polynomials

For the nonlinear term

$$\mathcal{N} = \mathcal{Y}^2\mathcal{Y}_y = \sum_{n=0}^{\infty}\mathcal{A}_n, \tag{23}$$

where

$$\mathcal{A}_0 = \mathcal{Y}_0^2(\mathcal{Y}_0)_y, \tag{24}$$

$$\begin{aligned} \mathcal{A}_1 = & 2\mathcal{Y}_0\mathcal{Y}_1(\mathcal{Y}_0)_y \\ & + \mathcal{Y}_0^2(\mathcal{Y}_1)_y. \end{aligned} \tag{25}$$

First Approximation

Since $(\mathcal{Y}_0)_{xzt} = 0$, we obtain

$$\begin{aligned} \mathcal{Y}_1 = & -\mathcal{I}[(\mathcal{Y}_0)_x] \\ & -\mathcal{I}[\mathcal{A}_0]\,. \end{aligned} \tag{26}$$

Define

$$\begin{aligned} f_1(x,y,z) = & \sin y\, e^{-(x+z)} \\ & \times\left(1 - e^{-2(x+z)}\cos y\,\sin y\right). \end{aligned} \tag{27}$$

Thus,

$$\mathcal{Y}_1 = \frac{t^{\delta}}{\Gamma(\delta+1)}\,f_1(x,y,z). \tag{28}$$

Second Approximation

Using the recursive relation,

$$\begin{aligned} \mathcal{Y}_2 = & -\mathcal{I}[(\mathcal{Y}_1)_x] \\ & -\mathcal{I}[\mathcal{A}_1] \\ & +\mathcal{I}[(\mathcal{Y}_1)_{xzt}]\,. \end{aligned} \tag{29}$$

Since

$$\mathcal{Y}_1 = \frac{t^{\delta}}{\Gamma(\delta+1)}f_1,$$

we obtain

$$\mathcal{Y}_2 = \frac{t^{2\delta}}{\Gamma(2\delta+1)}\,f_2(x,y,z), \tag{30}$$

where

$$\begin{aligned} f_2 = & -(f_1)_x \\ & -\left(2\mathcal{Y}_0 f_1(\mathcal{Y}_0)_y + \mathcal{Y}_0^2(f_1)_y\right). \end{aligned} \tag{31}$$

Third Approximation

Similarly,

$$\mathcal{Y}_3 = \frac{t^{3\delta}}{\Gamma(3\delta+1)} f_3(x,y,z), \tag{32}$$

where f_3 is an explicitly computable spatial function.

Explicit Spatial Functions

The zeroth-order term is

$$\mathcal{Y}_0(x,y,z) = e^{-(x+z)} \sin y. \tag{33}$$

First Spatial Function

$$f_1(x,y,z) = e^{-(x+z)} \sin y - e^{-3(x+z)} \sin^2 y \cos y. \tag{34}$$

Second Spatial Function

$$\begin{aligned} f_2(x,y,z) = {}& e^{-(x+z)} \sin y \\ & - 6e^{-3(x+z)} \sin^2 y \cos y \\ & + 2e^{-5(x+z)} \sin^3 y \cos^2 y \\ & + e^{-5(x+z)} \sin^2 y \left(2 \sin y \cos^2 y - \sin^3 y\right). \end{aligned} \tag{35}$$

Third Spatial Function

$$\begin{aligned} f_3(x,y,z) = {}& -(f_2)_x \\ & - \Big(2\mathcal{Y}_0 f_2 (\mathcal{Y}_0)_y + \mathcal{Y}_0^2 (f_2)_y\Big), \end{aligned} \tag{36}$$

where

$$(\mathcal{Y}_0)_y = e^{-(x+z)} \cos y.$$

Truncated Series Solution

$$\begin{aligned} \mathcal{Y}(x,y,z,t) \approx {}& \mathcal{Y}_0 \\ & + \frac{t^{\delta}}{\Gamma(\delta+1)} f_1 \\ & + \frac{t^{2\delta}}{\Gamma(2\delta+1)} f_2 \\ & + \frac{t^{3\delta}}{\Gamma(3\delta+1)} f_3. \end{aligned} \tag{37}$$

or

$$\begin{aligned} \mathcal{Y}(x,y,z,t) \approx {}& \sum_{n=0}^{3} \mathcal{Y}_n \\ = {}& e^{-(x+z)} \sin y + \frac{t^{\delta}}{\Gamma(\delta+1)} f_1 \\ & - \frac{t^{2\delta}}{\Gamma(2\delta+1)} f_{1x} - \frac{t^{2\delta}}{\Gamma(2\delta+1)} f_2 + \frac{t^{3\delta-1}}{\Gamma(3\delta)} f_{1xz} \\ & + \frac{t^{3\delta}}{\Gamma(3\delta+1)} (f_1)_{xx} + \frac{t^{3\delta}}{\Gamma(3\delta+1)} (f_2)_x \\ & - \frac{t^{4\delta-1}}{\Gamma(4\delta)} (f_1)_{xxz} - \frac{t^{3\delta-1}}{\Gamma(3\delta)} (f_1)_{xxz} \\ & - \frac{2\delta\, t^{3\delta-1}}{\Gamma(3\delta)} (f_2)_{xz} + \frac{(3\delta-1)\, t^{4\delta-2}}{\Gamma(4\delta-1)} (f_1)_{xxzz} \\ & + \frac{\Gamma(2\delta+1)}{\Gamma(3\delta+3)} e^{-(x+z)} \sin^2 y \, (f_1)_{xy} \, t^{3\delta+2} \\ & - \frac{\Gamma(2\delta+1)}{\Gamma(3\delta+2)} f_1^2 (f_1)_y \, t^{3\delta+1} \\ & - \frac{\Gamma(2\delta+1)^2}{\Gamma(3\delta+2)} e^{-(x+z)} \cos y \left[(f_1)_x\right]^2 t^{3\delta+1}. \end{aligned} \tag{38}$$

The solution is illustrated in Figures 1 and 2 for six different values of the fractional order parameter $\delta = 0.5, 0.6, 0.7, 0.8, 0.9$, and 1.

Figure 1 represents the spatial distribution of the solution with respect to the variables x and z while fixing $y = \frac{\pi}{3}$ and $z = 5$. It is observed that the solution exhibits an exponential decay in the positive spatial direction due to the factor $e^{-(x+z)}$, whereas a sharp gradient appears in the negative region of the spatial domain. The magnitude of the solution increases significantly as the spatial variables decrease, which is consistent with the exponential structure of the series components f_1, f_2, and f_3.

Figure 2 shows the spatial distribution of the solution with respect to x and z, with fixed parameters $y = \frac{\pi}{3}$ and $t = 2$. For smaller values of δ (e.g., $\delta = 0.5$), the spatial profile is more flattened owing to the fractional memory effect. As δ increases toward 1, the solution approaches the classical behavior, exhibiting a more pronounced spatial variation and larger peak values.

It is evident from both figures that increasing δ reduces the damping effect associated with fractional dynamics, leading to higher solution amplitudes. When $\delta = 1$, the model recovers the classical WBBM

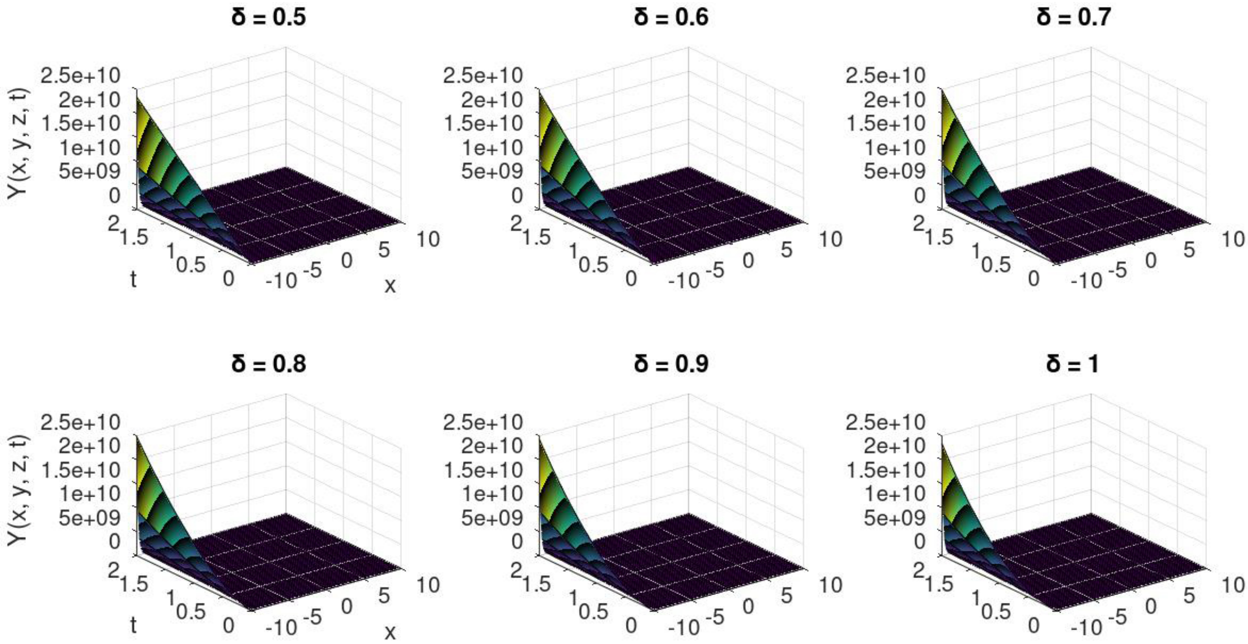

Figure 1. 3D plot for three terms series solution for example 1 for $y = \frac{\pi}{3}$ and $z = 5$.

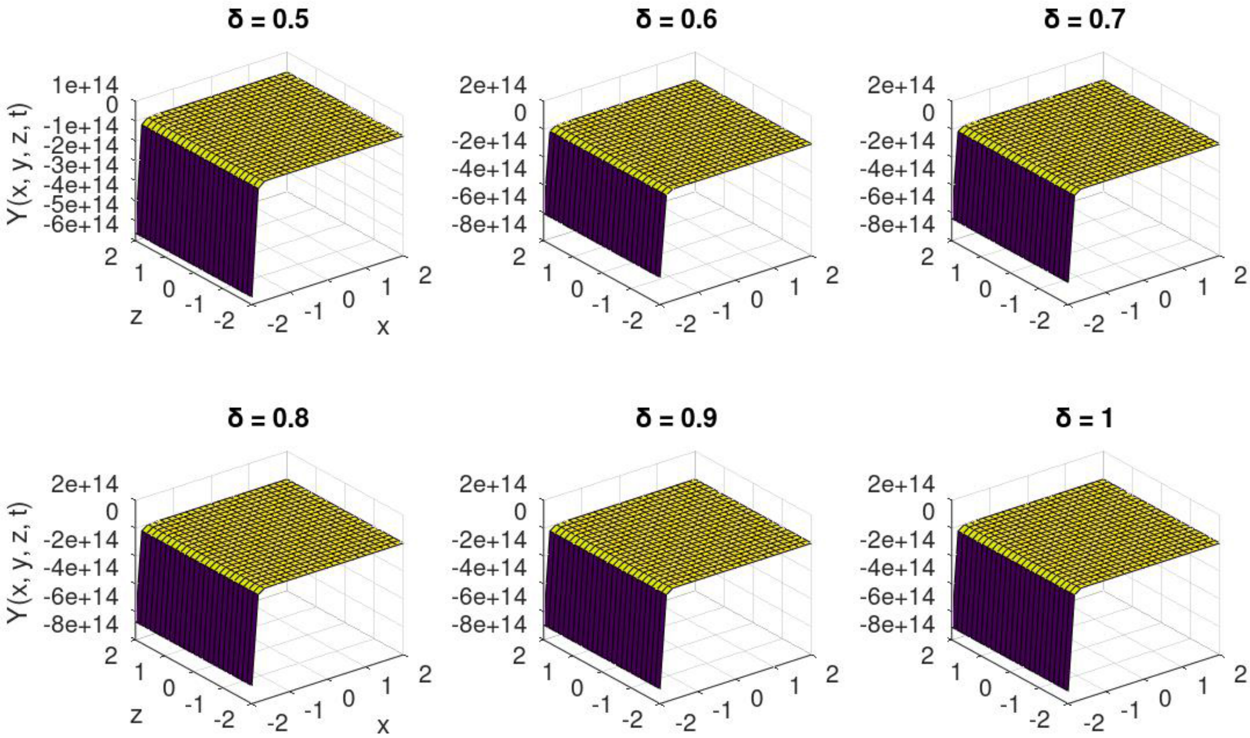

Figure 2. 3D plot for three terms series solution for example 1 for $y = \frac{\pi}{3}$ and $t = 2$.

equation, and the surface profile becomes smoother and more regular compared to the fractional cases.

Overall, the graphical results confirm that the fractional order parameter δ significantly influences both the spatial distribution and temporal evolution of the solution, highlighting the memory-dependent nature of the fractional model.

4.2 Time–fractional WBBM (second form)

Consider the time–fractional WBBM equation (second form)

$$\begin{aligned} {}^{C}D_t^{\delta}\mathcal{Y}(x,y,z,t) &+ \mathcal{Y}_z(x,y,z,t) \\ &+ \mathcal{Y}^2(x,y,z,t)\,\mathcal{Y}_x(x,y,z,t) \qquad 0<\delta<1, \\ &- \mathcal{Y}_{yzt}(x,y,z,t) = 0, \end{aligned} \tag{39}$$

with initial condition

$$\mathcal{Y}(x,y,z,0)=f(x,y,z)=e^{-(x+y+z)}. \tag{40}$$

Solution by LTADM
Let $\mathcal{I}=I_t^{\delta}$ denote the Riemann–Liouville fractional integral of order δ in t. Applying $\mathcal{I}$ and using

$$\mathcal{I}({}^{C}D_t^{\delta}\mathcal{Y})=\mathcal{Y}-\mathcal{Y}(x,y,z,0),$$

we obtain

$$\begin{aligned}\mathcal{Y}(x,y,z,t)=&f(x,y,z)-\mathcal{I}[\mathcal{Y}_z]\\&-\mathcal{I}[\mathcal{Y}^2\mathcal{Y}_x]+\mathcal{I}[\mathcal{Y}_{yzt}].\end{aligned} \tag{41}$$

We assume the decomposition

$$\mathcal{Y}=\sum_{n=0}^{\infty}\mathcal{Y}_n,\qquad \mathcal{Y}_0=e^{-(x+y+z)}. \tag{42}$$

The nonlinear term is written as

$$\mathcal{Y}^2\mathcal{Y}_x=\sum_{n=0}^{\infty}\mathcal{A}_n, \tag{43}$$

where

$$\mathcal{A}_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\Big(\sum_{k\geq 0}\lambda^k\mathcal{Y}_k\Big)^2\Big(\sum_{k\geq 0}\lambda^k\mathcal{Y}_k\Big)_x\right]_{\lambda=0}. \tag{44}$$

In particular,

$$\mathcal{A}_0=\mathcal{Y}_0^2(\mathcal{Y}_0)_x,\qquad \mathcal{A}_1=2\mathcal{Y}_0\mathcal{Y}_1(\mathcal{Y}_0)_x+\mathcal{Y}_0^2(\mathcal{Y}_1)_x. \tag{45}$$

Define

$$\mathbf{L}_n^1=(\mathcal{Y}_n)_z,\qquad \mathbf{L}_n^2=(\mathcal{Y}_n)_{yzt},\qquad \mathbf{N}_n=\mathcal{A}_n.$$

The recurrence relation becomes

$$\begin{aligned}\mathcal{Y}_{n+1}=&-\mathcal{I}[\mathbf{L}_n^1]-\mathcal{I}[\mathbf{N}_n]\\&+\mathcal{I}[\mathbf{L}_n^2],\qquad n\geq 0.\end{aligned} \tag{46}$$

First term. Let $S=x+y+z$. Then

$$(\mathcal{Y}_0)_z=-e^{-S},\qquad (\mathcal{Y}_0)_x=-e^{-S}.$$

$$\mathcal{A}_0=\mathcal{Y}_0^2(\mathcal{Y}_0)_x=e^{-2S}(-e^{-S})=-e^{-3S}. \tag{47}$$

Since $(\mathcal{Y}_0)_{yzt}=0$, we obtain

$$\mathcal{Y}_1=\frac{t^{\delta}}{\Gamma(\delta+1)}\left(e^{-S}+e^{-3S}\right). \tag{48}$$

Second term. Using the recurrence for $n=1$,

$$\begin{aligned}\mathcal{Y}_2=&-\mathcal{I}[(\mathcal{Y}_1)_z]\\&-\mathcal{I}[\mathcal{A}_1]\\&+\mathcal{I}[(\mathcal{Y}_1)_{yzt}].\end{aligned} \tag{49}$$

We compute

$$(\mathcal{Y}_1)_z=-\frac{t^{\delta}}{\Gamma(\delta+1)}\left(e^{-S}+3e^{-3S}\right), \tag{50}$$

$$(\mathcal{Y}_1)_{yzt}=\frac{t^{\delta-1}}{\Gamma(\delta)}\left(e^{-S}+9e^{-3S}\right). \tag{51}$$

After applying $\mathcal{I}=I_t^{\delta}$ termwise, we obtain

$$\begin{aligned}\mathcal{Y}_2=&\frac{t^{2\delta}}{\Gamma(2\delta+1)}\left(6e^{-3S}+e^{-S}+5e^{-5S}\right)\\&+\frac{t^{2\delta-1}}{\Gamma(2\delta)}\left(9e^{-3S}+e^{-S}\right).\end{aligned} \tag{52}$$

The three–dimensional graphical representations of the truncated LADM series solution for the second form of the time–fractional WBBM equation are displayed for different fractional orders $\delta=0.7, 0.8, 0.9$, and 1.

Figure 3 illustrates the temporal evolution of $\mathcal{Y}(x,y,z,t)$ with respect to x and t, with fixed spatial parameters $y=z=5$. The solution exhibits a sharp peak near the negative x-region while gradually decaying as x increases, consistent with the exponential structure of the initial condition $e^{-(x+y+z)}$. As the fractional order δ increases, the peak becomes more pronounced, indicating reduced fractional damping.

Figure 4 presents the temporal evolution of the solution with respect to x and t for fixed values of the remaining spatial variables. For smaller δ values (e.g., $\delta=0.7$), the surface grows more gradually in time due to the fractional memory effect represented by the term t^{δ}. As δ approaches 1, the solution behavior becomes closer to the classical integer–order case, showing a smoother and slightly faster temporal increase.

It is evident from both spatial and temporal profiles that the fractional parameter δ significantly controls the amplitude and growth rate of the solution. Lower fractional orders introduce stronger memory effects, which moderate the evolution of the system, whereas $\delta=1$ recovers the classical WBBM dynamics with

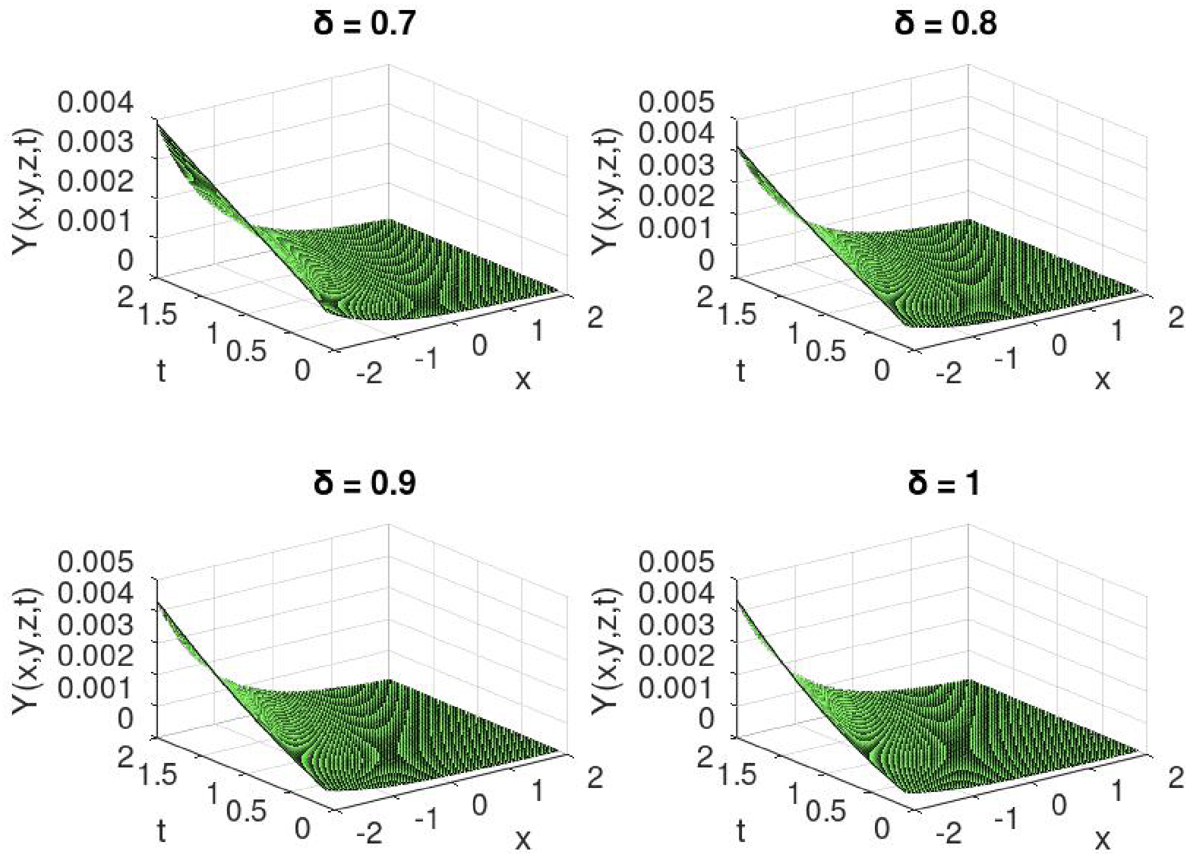

Figure 3. 3D plot for three terms series solution for example 2 for $y = z = 5$.

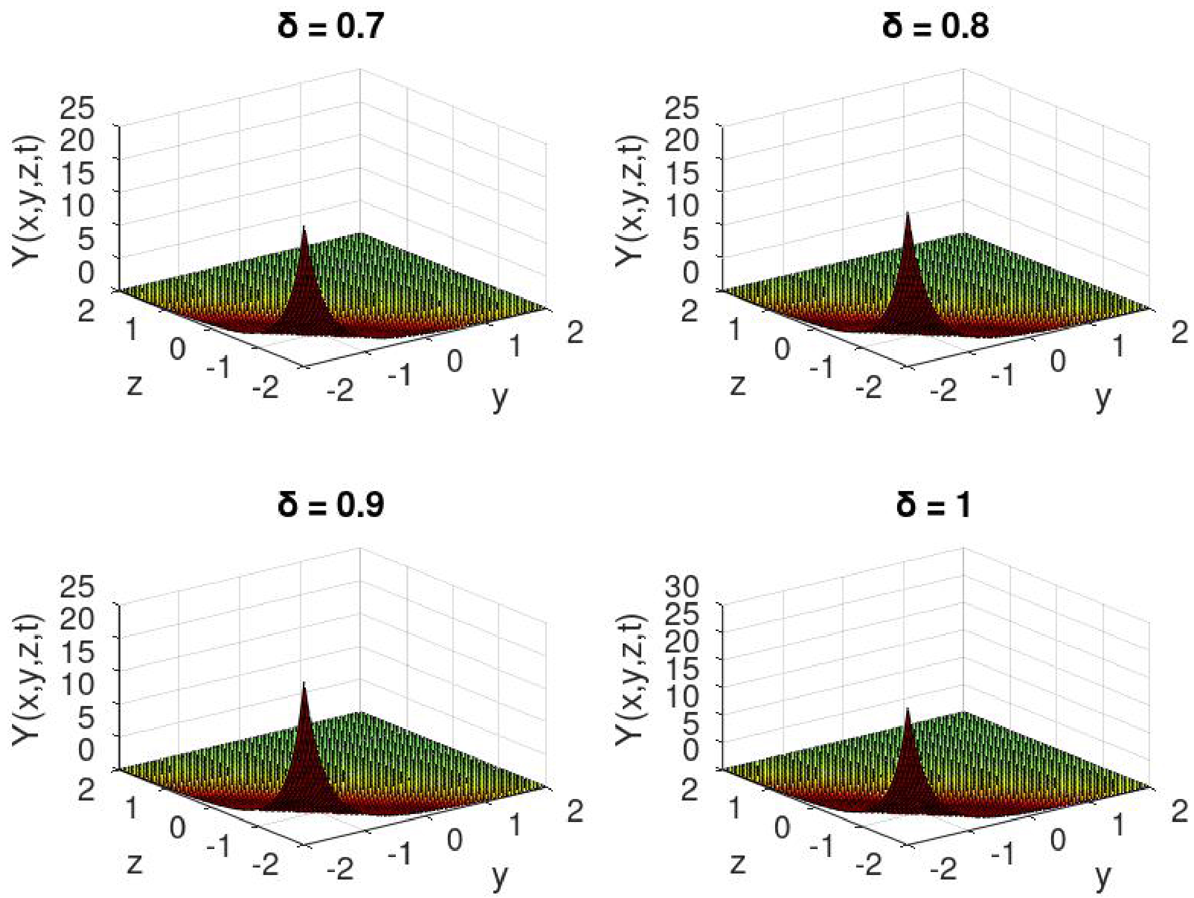

Figure 4. 3D plot for three terms series solution for example 2 for $x = 5$ and $t = 2$.

comparatively stronger propagation and larger peak values.

Overall, the graphical analysis confirms that the fractional order plays a crucial role in shaping both the spatial distribution and temporal development of the solution, highlighting the importance of fractional modeling in capturing memory–dependent wave phenomena.

5 Conclusion

In this work, the Laplace Transform Adomian Decomposition Method (LTADM) has been successfully applied to obtain approximate analytical solutions of the time–fractional Wazwaz–Benjamin–Bona–Mahony (WBBM) equations in both considered forms. The proposed approach effectively combines the Laplace transform with the Adomian decomposition technique, allowing the nonlinear fractional models to be handled in a systematic and computationally efficient manner.

The truncated series solutions were derived explicitly, and their accuracy and convergence behavior were illustrated through three–dimensional graphical representations for various fractional orders $\delta \in (0, 1]$. The numerical simulations demonstrate that the fractional parameter δ significantly influences the amplitude and propagation characteristics of the solution. As δ increases toward 1, the system gradually approaches the classical integer–order behavior, whereas smaller values of δ introduce stronger memory effects and moderated dynamics.

Overall, the results confirm that LTADM is a reliable and powerful analytical tool for solving nonlinear time–fractional partial differential equations, particularly those arising in wave propagation and fluid dynamics models.

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Data will be made available on request.

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Conflicts of Interest

The authors declare no conflicts of interest.

AI Use Statement

The authors declare that generative AI was used in the preparation of this manuscript, limited to language refinement. The AI tool employed was ChatGPT-5. No AI was used for data analysis, result interpretation, conceptual development, or generation of core scientific content. The authors take full responsibility for the accuracy, originality, and integrity of the work.

Ethical Approval and Consent to Participate

Not applicable.

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