RESEARCH ARTICLE

Numerical Simulation of Stiff Systems of Ordinary Differential Equations Using a Block Extended Backward Differentiation Scheme with a Stability Control Parameter

Buhari Alhassan[1,*], **Musa Hamisu**[2], **Abubakar Sadiq Saadu**[1] **and Kabiru Abubakar Kantsi**[1]

[1] Department of Mathematics and Statistics, Al-Qalam University Katsina, Katsina State, Nigeria
[2] Department of Mathematics, Umaru Musa Yar'adua University, Katsina, Katsina State, Nigeria

Abstract

Stiff systems of ordinary differential equations (ODEs) arise frequently in several areas of applied mathematics, science, and engineering, and their numerical integration requires methods with strong stability characteristics. In this paper, a block extended backward differentiation formula (BEBDF) with a stability control parameter is developed for the numerical solution of stiff systems of ODEs. The proposed method is constructed within the framework of block multistep methods, allowing the simultaneous computation of solution values at several grid points. A free parameter ρ is incorporated into the formulation of the method to regulate and improve its stability behavior. The presence of this parameter provides additional flexibility in controlling the stability properties of the scheme, which is particularly beneficial when dealing with stiff problems. The nonlinear system arising from the implementation of the proposed method is solved using Newton's iteration technique to ensure efficient and reliable convergence. The computational algorithm for the method is implemented in the Dev-C++ compiler environment, where the block structure of the scheme facilitates efficient numerical computation. Furthermore, the fundamental properties of the method, including consistency, zero-stability, and convergence, are analyzed to establish its theoretical reliability. Numerical experiments performed on selected stiff test problems demonstrate that the proposed method produces accurate and stable results. The findings indicate that the inclusion of the free parameter ρ significantly enhances the stability control and overall performance of the method in the numerical solution of stiff systems of ODEs.

Keywords: block extended backward differentiation formula, stiff ordinary differential equations, stability control parameter ρ, Newton's iteration method, numerical

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***Corresponding author:**
Buhari Alhassan
buharialhassan@auk.edu.ng

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solution, block methods.

1 Introduction

Stiff systems of ordinary differential equations (ODEs) constitute an important class of initial value problems (IVPs) that frequently arise in the mathematical modeling of complex physical, chemical, biological, and engineering processes. These systems are typically characterized by the presence of widely separated time scales among the dependent variables, resulting in solution components that evolve at significantly different rates. In such situations, certain components of the solution may vary rapidly while others change slowly over time [1, 2]. Stiff problems arise in diverse areas including chemical kinetics, biological systems, and control engineering, and their numerical treatment requires specialized methods [3–5]. This disparity in rates of variation leads to the phenomenon commonly referred to as stiffness. The presence of stiffness presents considerable challenges for numerical computation because conventional numerical methods often fail to maintain stability unless extremely small step sizes are employed [6, 7].

In this paper, we consider the following first-order stiff IVPs of the form:

$$y' = f(x, y), \quad y(a) = y_0, \quad a \leq x \leq b, \tag{1}$$

where $y, f \in \mathbb{R}^n$.

However, when standard explicit numerical integration techniques are applied to such problems, the requirement of maintaining numerical stability forces the use of prohibitively small step sizes. This restriction leads to excessive computational cost and reduces the overall efficiency of the numerical procedure. For this reason, implicit numerical methods have become the preferred approach for solving stiff systems, as they possess stronger stability characteristics and allow the use of comparatively larger integration steps. However, the implementation of implicit schemes generally requires the solution of nonlinear algebraic systems at each integration step, thereby increasing the computational complexity of the algorithm [8–10].

Over the past several decades, substantial effort has been devoted to the development of numerical methods capable of efficiently solving stiff IVPs. Among the most widely used techniques are the backward differentiation formulas (BDF), which belong to the class of implicit linear multistep methods (LMMs). These methods are well known for their favorable stability properties and have been extensively employed in many scientific computing environments designed for stiff differential equations. The effectiveness of BDF methods in handling stiff problems stems from their ability to maintain stability even when relatively large step sizes are used. Nevertheless, classical BDF methods may exhibit certain limitations when higher accuracy, improved stability control, or enhanced flexibility in numerical implementation is required [8].

An important theoretical limitation in the construction of multistep methods was established through the well-known Dahlquist stability barrier. This result states that no explicit linear multistep method can possess A-stability and that the maximum order attainable by an A-stable implicit LMM is two [11]. This restriction has motivated considerable research aimed at developing numerical schemes capable of achieving improved stability properties while maintaining higher orders of accuracy. Several strategies have been proposed in the literature to overcome this limitation. These include the incorporation of higher derivatives of the solution, the introduction of additional stages, and the use of off-step or super future points, which lead to the broader class of general linear methods [9, 12].

One notable development in this direction is the Extended Backward Differentiation Formula (EBDF), which modifies the classical BDF framework through the introduction of an additional future point. The inclusion of this super future point improves the stability behavior of the method and allows the construction of schemes with higher orders of accuracy while maintaining desirable stability properties. The EBDF formulation has demonstrated improved numerical performance compared with conventional BDF methods when applied to stiff differential systems [7].

In recent years, block numerical methods have attracted significant attention as efficient alternatives for solving systems of differential equations. Unlike traditional multistep methods such as BDF, EBDF and superclass of extended backward differentiation formulas (SEBDF) that compute a single solution value at each integration step, block methods generate multiple solution approximations simultaneously. This feature improves computational efficiency and reduces the reliance on predictor-corrector strategies. Furthermore, the block formulation naturally supports parallel computation and provides additional flexibility in controlling numerical stability and accuracy [13, 14, 16–18]. Block methods have also been extended to higher-order differential equations, further demonstrating the versatility of this framework [15].

One important contribution in this area is the block backward differentiation formula (BBDF), which extends the classical BDF framework by computing several solution values within a single step. The BBDF method has been successfully applied to stiff ODEs and has demonstrated strong stability properties [1, 19, 20]. Further improvements led to the development of the Superclass of block backward differentiation formula (SBDF), which introduces additional flexibility through the inclusion of a free parameter ρ. This parameter acts as a stability control parameter that allows the method to generate a family of numerical schemes with varying stability characteristics. The classical BBDF methods can be recovered as a special case when the parameter ρ is set to zero, while different numerical behaviors are obtained when the parameter varies within a specified interval $(-1, 1)$ [21–26].

Despite the progress achieved in the development of block multistep techniques, the number of available implicit block methods capable of efficiently solving stiff IVPs remains relatively limited. Consequently, further research aimed at constructing more effective and reliable numerical algorithms continues to be of significant interest. Motivated by previous strategies for circumventing the Dahlquist stability barrier, the present study incorporates an additional future point into the block formulation.

Specifically, by introducing a super future point into the existing formulation of fully implicit fifth order three-point SBDF, a new block implicit scheme referred to as the three-point super class of block extended backward differentiation formula (3SBEBDF) is developed. The resulting method combines the advantages of block multistep techniques with the improved stability characteristics provided by the extended formulation.

2 Mathematical Formulation of the Method

This section develops the mathematical formulation of the 3SBEBDF method, obtained through a modification of the existing three-point superclass of block backward differentiation formula (3SBBDF) scheme [26]. The modification introduces a super future point into the formulation of 3SBBDF scheme to enhance its stability properties when solving stiff IVPs. The classical 3SBBDF method is expressed as:

$$\sum_{j=0}^{5} \alpha_{j,i} y_{n+j-2} = h\beta_{k,i}(f_{n+k} - \rho f_{n+k-1}), \; k = i = 1, 2, 3, \tag{2}$$

which was derived using Taylor series expansion and leads to the following difference equations expressed as

$$\begin{cases} y_{n+1} = \dfrac{-2-3\rho}{20(3\rho-1)} y_{n-2} + \dfrac{3(1+2\rho)}{4(3\rho-1)} y_{n-1} + \dfrac{\rho-3}{3\rho-1} y_n \\ \qquad + \dfrac{3(2+\rho)}{4(3\rho-1)} y_{n+2} + \dfrac{-2-3\rho}{20(3\rho-1)} y_{n+3} \\ \qquad + h\left(-\dfrac{3}{3\rho-1} f_{n+1} + \dfrac{3\rho}{3\rho-1} f_n\right), \\ y_{n+2} = \dfrac{3+2\rho}{5(-13+6\rho)} y_{n-2} + \dfrac{-4-3\rho}{-13+6\rho} y_{n-1} + \dfrac{12(1+\rho)}{-13+6\rho} y_n \\ \qquad - \dfrac{4(6+\rho)}{-13+6\rho} y_{n+1} + \dfrac{3(4+\rho)}{5(-13+6\rho)} y_{n+3} \\ \qquad + h\left(\dfrac{12\rho}{-13+6\rho} f_{n+1} - \dfrac{12}{-13+6\rho} f_{n+2}\right), \\ y_{n+3} = -\dfrac{3(4+\rho)}{-137+12\rho} y_{n-2} + \dfrac{5(15+4\rho)}{-137+12\rho} y_{n-1} \\ \qquad - \dfrac{20(10+3\rho)}{-137+12\rho} y_n + \dfrac{60(5+\rho)}{-13+6\rho} y_{n+1} - \dfrac{5(60+13\rho)}{-137+12\rho} y_{n+2} \\ \qquad + h\left(\dfrac{6\rho}{-137+12\rho} f_{n+2} - \dfrac{60}{-137+12\rho} f_{n+3}\right). \end{cases} \tag{3}$$

The scheme in (3) has been shown to possess fifth-order accuracy, and it satisfies consistency, zero-stability, and convergence conditions. In addition, it is A-stable for $\rho = 1/2$ and almost A-stable for $\rho = -3/4$, making it suitable for the numerical integration of stiff ODEs [26]. The block BDF framework underlying this construction has been further analyzed for stability in related formulations [27].

The goal of this section is to construct a block method capable of producing three solution values simultaneously for stiff IVPs. The formulation is inspired by the EBDF scheme introduced by Cash in [7], but with an important extension: the inclusion of a super future point. This idea is incorporated into 3SBBDF scheme. Based on this framework, we define our proposed method as 3SBEBDF which is given by:

$$\sum_{j=0}^{5} \alpha_{j,i} y_{n+j-2} = h\beta_{k,i}(f_{n+k} - \rho f_{n+k-1}) + h\beta_{k+1,i} f_{n+k+1}, \quad k = i = 1, 2, 3, \tag{4}$$

Here, $k = i = 1, 2, 3$ correspond to the first, second, and third point formulae, respectively. Using the known values at x_{n-2}, x_{n-1} and x_n, the method simultaneously computes approximations at x_{n+1}, x_{n+2} and x_{n+3}. The introduction of the additional node x_{n+4}, referred to as the super-future point, distinguishes the present formulation from the earlier methods developed in [26, 27], as it provides a mechanism for enhancing the stability of the scheme.

The super future point x_{n+4} is defined as a grid point beyond the current computational block, given by

$$x_{n+4} = x_n + 4h,$$

where h denotes the step size. In contrast to the standard nodes x_{n+1}, x_{n+2} and x_{n+3}, which lie within the block, x_{n+4} extends the computational stencil forward.

From a numerical standpoint, the inclusion of this forward point enriches the structure of the method by incorporating additional information from ahead of the integration interval. This extended stencil leads to improved derivative approximation, enhanced accuracy, and a wider stability region. Consequently, the method exhibits stronger damping of spurious high-frequency components, which is essential for stable integration of stiff systems.

From a physical perspective, x_{n+4} may be viewed as providing a predictive insight into the future behavior of the solution. This forward-looking feature enables better control of rapidly varying solution components commonly encountered in stiff problems.

Thus, the incorporation of the super future point x_{n+4} significantly strengthens both the stability and accuracy of the method, contributing to its robustness and effectiveness in solving stiff IVPs. The interpolation points involved in the 3SBEBDF method are illustrated in Figure 1.

To derive the first, second and third point formulae, the linear operator L_i associated with the first, second and third point formulae of the 3SBEBDF scheme is defined as:

$$\begin{aligned} L_i[y(x_n), h] :=\ & \alpha_{0,i}y_{n-2} + \alpha_{1,i}y_{n-1} + \alpha_{2,i}y_n + \alpha_{3,i}y_{n+1} \\ & + \alpha_{4,i}y_{n+2} + \alpha_{5,i}y_{n+3} - h\beta_{k,i}f_{n+k} \\ & + h\rho\beta_{k,i}f_{n+k-1} - h\beta_{k+1,i}f_{n+k+1} = 0, \\ & k = i = 1, 2, 3, \end{aligned} \tag{5}$$

To determine the first point y_{n+1}, we set $k = i = 1$ in equation (5) and redefine the linear operator L_i as follows:

$$\begin{aligned} & \alpha_{0,1}y_{n-2} + \alpha_{1,1}y_{n-1} + \alpha_{2,1}y_n + \alpha_{3,1}y_{n+1} + \alpha_{4,1}y_{n+2} \\ & + \alpha_{5,1}y_{n+3} - h\beta_{1,1}f_{n+1} + h\rho\beta_{1,1}f_n - h\beta_{2,1}f_{n+2} = 0, \end{aligned} \tag{6}$$

Equation (6) is expanded in a Taylor series about x_n, and upon grouping like terms, this yields:

$$\begin{aligned} & C_{0,1}y(x_n) + C_{1,1}hy'(x_n) + C_{2,1}h^2y''(x_n) \\ & + C_{3,1}h^3y'''(x_n) + \cdots = 0. \end{aligned} \tag{7}$$

where,

$$\begin{cases} C_{0,1} = \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} + \alpha_{4,1} + \alpha_{5,1} = 0 \\ C_{1,1} = -2\alpha_{0,1} - \alpha_{1,1} + \alpha_{3,1} + 2\alpha_{4,1} + 3\alpha_{5,1} \\ \qquad - \beta_{1,1}(1-\rho) - \beta_{2,1} = 0 \\ C_{2,1} = 2\alpha_{0,1} + \frac{1}{2}\alpha_{1,1} + \frac{1}{2}\alpha_{3,1} + 2\alpha_{4,1} + \frac{9}{2}\alpha_{5,1} - \beta_{1,1} \\ \qquad - 2\beta_{2,1} = 0 \\ C_{3,1} = -\frac{4}{3}\alpha_{0,1} - \frac{1}{6}\alpha_{1,1} + \frac{1}{6}\alpha_{3,1} + \frac{4}{3}\alpha_{4,1} + \frac{9}{2}\alpha_{5,1} \\ \qquad - \frac{1}{2}\beta_{1,1} - 2\beta_{2,1} = 0 \\ C_{4,1} = \frac{2}{3}\alpha_{0,1} + \frac{1}{24}\alpha_{1,1} + \frac{1}{24}\alpha_{3,1} + \frac{2}{3}\alpha_{4,1} + \frac{27}{8}\alpha_{5,1} \\ \qquad - \frac{1}{6}\beta_{1,1} - \frac{4}{3}\beta_{2,1} = 0 \\ C_{5,1} = -\frac{4}{15}\alpha_{0,1} - \frac{1}{120}\alpha_{1,1} + \frac{1}{120}\alpha_{3,1} + \frac{4}{15}\alpha_{4,1} \\ \qquad + \frac{81}{40}\alpha_{5,1} - \frac{1}{120}\beta_{1,1} - \frac{4}{15}\beta_{2,1} = 0 \\ C_{6,1} = \frac{4}{45}\alpha_{0,1} + \frac{1}{720}\alpha_{1,1} + \frac{1}{720}\alpha_{3,1} + \frac{4}{45}\alpha_{4,1} \\ \qquad + \frac{81}{80}\alpha_{5,1} - \frac{1}{120}\beta_{1,1} - \frac{4}{15}\beta_{2,1} = 0 \end{cases} \tag{8}$$

In the derivation of the first point y_{n+1}, the coefficient $\alpha_{3,1}$ is normalized to unity. The resulting system of simultaneous equations in (8) is solved using Maple environment to determine the coefficients $\alpha_{j,i}$ and $\beta_{j,i}$ leading to the following formula for the first point:

$$\begin{aligned} y_{n+1} =\ & -\frac{1}{80}y_{n-2} + \frac{1}{8}\frac{4\rho+1}{3\rho+1}y_{n-1} + \frac{1}{4}\frac{5\rho-3}{3\rho+1}y_n \\ & + \frac{1}{16}\frac{19\rho+25}{3\rho+1}y_{n+2} + \frac{1}{40}\frac{4\rho+3}{3\rho+1}y_{n+3} \\ & - \frac{3}{6\rho+2}hf_{n+1} + \frac{3\rho}{6\rho+2}hf_n - \frac{3(\rho+1)}{4(3\rho+1)}hf_{n+2}, \end{aligned} \tag{9}$$

The second and third points are derived using a similar procedure as for the first point, and their expressions

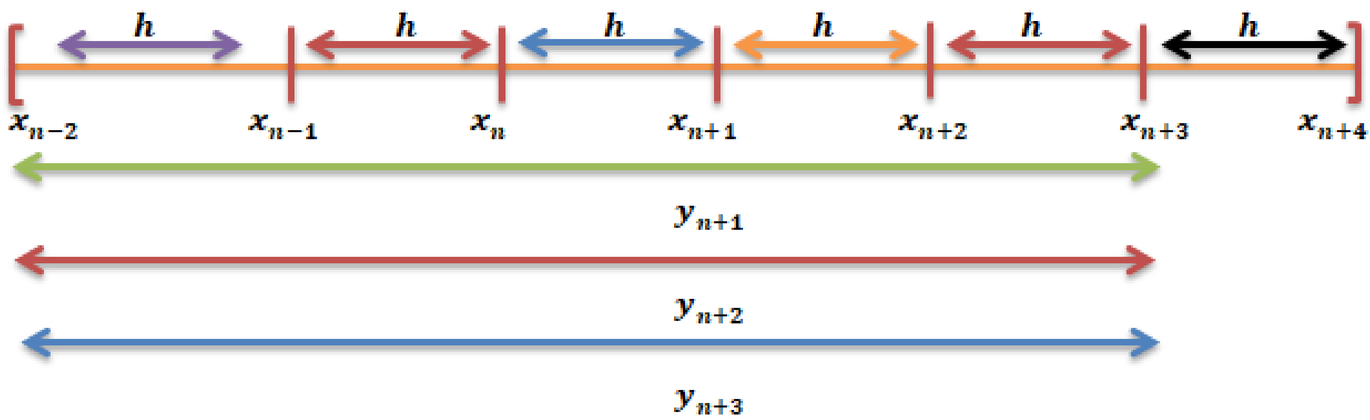

Figure 1. Interpolation points involved in the 3SBEBDF method.

are given by:

$$\begin{cases} y_{n+2} = \frac{1}{25}\frac{4\rho+3}{12\rho-1}y_{n-2} - \frac{1}{2}\frac{3\rho+2}{12\rho-1}y_{n-1} + \frac{4(2\rho+1)}{12\rho-1}y_n \\ \quad + \frac{2(\rho-6)}{12\rho-1}y_{n+1} + \frac{1}{50}\frac{167\rho+394}{12\rho-1}y_{n+3} \\ \quad - \frac{12}{12\rho-1}hf_{n+2} + \frac{12h\rho}{12\rho-1}f_{n+1} - \frac{6}{5}\frac{\rho+2}{12\rho-1}hf_{n+3}, \\ y_{n+3} = -\frac{1}{5}\frac{167\rho+394}{936\rho-4973}y_{n-2} + \frac{5(88\rho+195)}{936\rho-4973}y_{n-1} \\ \quad - \frac{20(81\rho+160)}{936\rho-4973}y_n + \frac{20}{3}\frac{628\rho+935}{936\rho-4973}y_{n+1} \\ \quad - \frac{5(403\rho+1770)}{936\rho-4973}y_{n+2} - \frac{2940h}{936\rho-4973}f_{n+3} \\ \quad + \frac{2940h\rho}{936\rho-4973}f_{n+2} + \frac{40(\rho+5)h}{936\rho-4973}f_{n+4}. \end{cases} \tag{10}$$

Combining equations (9) and (10), the 3SBEBDF scheme is thus obtained as:

$$\begin{cases} y_{n+1} = -\frac{1}{80}y_{n-2} + \frac{1}{8}\frac{4\rho+1}{3\rho+1}y_{n-1} + \frac{1}{4}\frac{5\rho-3}{3\rho+1}y_n \\ \quad + \frac{1}{16}\frac{19\rho+25}{3\rho+1}y_{n+2} + \frac{1}{40}\frac{4\rho+3}{3\rho+1}y_{n+3} \\ \quad - \frac{3}{6\rho+2}hf_{n+1} + \frac{3h\rho}{6\rho+2}hf_n - \frac{3}{4}\frac{\rho+1}{3\rho+1}hf_{n+2}, \\ y_{n+2} = \frac{1}{25}\frac{4\rho+3}{12\rho-1}y_{n-2} - \frac{1}{2}\frac{3\rho+2}{12\rho-1}y_{n-1} + \frac{4(2\rho+1)}{12\rho-1}y_n \\ \quad + \frac{2(\rho-6)}{12\rho-1}y_{n+1} + \frac{1}{50}\frac{167\rho+394}{12\rho-1}y_{n+3} \\ \quad - \frac{12}{12\rho-1}hf_{n+2} + \frac{12h\rho}{12\rho-1}f_{n+1} - \frac{6}{5}\frac{\rho+2}{12\rho-1}hf_{n+3}, \\ y_{n+3} = -\frac{1}{3}\frac{167\rho+394}{936\rho-4973}y_{n-2} + \frac{5(88\rho+195)}{936\rho-4973}y_{n-1} \\ \quad - \frac{20(81\rho+160)}{936\rho-4973}y_n + \frac{20}{3}\frac{628\rho+935}{936\rho-4973}y_{n+1} \\ \quad - \frac{5(403\rho+1770)}{936\rho-4973}y_{n+2} - \frac{2940h}{936\rho-4973}f_{n+3} \\ \quad + \frac{2940h\rho}{936\rho-4973}f_{n+2} + \frac{40(\rho+5)h}{936\rho-4973}f_{n+4}. \end{cases} \tag{11}$$

The parameter ρ in the 3SBEBDF method functions as a stability control parameter, significantly influencing both the stability and accuracy of the scheme. By varying ρ within the interval $(-1, 1)$, the absolute stability region of the method can be adjusted, thereby enhancing its ability to effectively handle stiff IVPs and suppress spurious oscillations in the numerical solution. In general, larger absolute values of ρ improve the damping of unwanted components without affecting the order of accuracy.

A suitable choice of ρ ensures that the method preserves key numerical properties such as zero-stability and A-stability, leading to improved convergence and reliable error control. Notably, when $\rho = 0$, the stabilizing effect vanishes and the method reduces to the classical 3BEBDF, which is fully implicit and A-stable [28]. This indicates that the 3SBEBDF scheme is therefore a generalization of the classical 3BEBDF method, with the superclass encompassing the classical method as a special case. The incorporation of such stability-controlling parameters in block and multistep methods is a well-established technique for enhancing performance in stiff systems as it has rigorously been discussed in [23–27].

3 Derivation of Order of the Method

In this section, the order and error constant of the method are derived corresponding to the equations in (11). To establish the order of the method, formulae in (11) can be expressed in the following form:

$$\begin{cases} \frac{1}{80}y_{n-2} - \frac{1}{8}\frac{4\rho+1}{3\rho+1}y_{n-1} - \frac{1}{4}\frac{5\rho-3}{3\rho+1}y_n + y_{n+1} \\ \quad - \frac{1}{16}\frac{19\rho+25}{3\rho+1}y_{n+2} - \frac{1}{40}\frac{4\rho+3}{3\rho+1}y_{n+3} \\ \quad = \frac{3}{6\rho+2}\rho h f_n - \frac{3}{6\rho+2}hf_{n+1} - \frac{3}{4}\frac{\rho+1}{3\rho+1}hf_{n+2}, \\ -\frac{1}{25}\frac{4\rho+3}{12\rho-1}y_{n-2} + \frac{1}{2}\frac{3\rho+2}{12\rho-1}y_{n-1} - \frac{4(2\rho+1)}{12\rho-1}y_n \\ \quad - \frac{2(\rho-6)}{12\rho-1}y_{n+1} + y_{n+2} - \frac{1}{50}\frac{167\rho+394}{12\rho-1}y_{n+3} \\ \quad = \frac{12}{12\rho-1}\rho h f_{n+1} - \frac{12}{12\rho-1}hf_{n+2} - \frac{6}{5}\frac{\rho+2}{12\rho-1}hf_{n+3}, \\ \frac{1}{3}\frac{167\rho+394}{936\rho-4973}y_{n-2} - \frac{5(88\rho+195)}{936\rho-4973}y_{n-1} + \frac{20(81\rho+160)}{936\rho-4973}y_n \\ \quad - \frac{20}{3}\frac{628\rho+935}{936\rho-4973}y_{n+1} + \frac{5(403\rho+1770)}{936\rho-4973}y_{n+2} + y_{n+3} \\ \quad = \frac{2940}{936\rho-4973}\rho h f_{n+2} - \frac{2940}{936\rho-4973}hf_{n+3} + \frac{40(\rho+5)}{936\rho-4973}hf_{n+4}. \end{cases} \tag{12}$$

The general matrix form of (12) is defined and expressed as follows:

$$\sum_{j=0}^{1} D_j^* Y_{m+j-1} = h \sum_{j=0}^{2} G_j^* F_{m+j-1} \tag{13}$$

where $D_0^*, D_1^*, G_0^*, G_1^*$ and G_2^* are three by three square matrices defined by:

$$D_0^* = \begin{pmatrix} \frac{1}{80} & -\frac{1}{8}\frac{4\rho+1}{3\rho+1} & -\frac{1}{4}\frac{5\rho-3}{3\rho+1} \\ -\frac{1}{25}\frac{4\rho+3}{12\rho-1} & \frac{1}{2}\frac{3\rho+2}{12\rho-1} & -\frac{4(2\rho+1)}{12\rho-1} \\ \frac{1}{3}\frac{167\rho+394}{936\rho-4973} & -\frac{5(88\rho+195)}{936\rho-4973} & \frac{20(81\rho+160)}{936\rho-4973} \end{pmatrix},$$

$$D_1^* = \begin{pmatrix} 1 & -\frac{1}{16}\frac{19\rho+25}{3\rho+1} & -\frac{1}{40}\frac{4\rho+3}{3\rho+1} \\ -\frac{2(\rho-6)}{12\rho-1} & 1 & -\frac{1}{50}\frac{167\rho+394}{12\rho-1} \\ -\frac{20}{3}\frac{628\rho+935}{936\rho-4973} & \frac{5(403\rho+1770)}{936\rho-4973} & 1 \end{pmatrix},$$

$$G_0^* = \begin{pmatrix} 0 & 0 & \frac{3\rho}{6\rho+2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$G_1^* = \begin{pmatrix} -\frac{3\rho}{6\rho+2} & -\frac{3}{4}\frac{\rho+1}{3\rho+1} & 0 \\ \frac{12\rho}{12\rho-1} & -\frac{12}{12\rho-1} & -\frac{6}{5}\frac{\rho+2}{12\rho-1} \\ 0 & \frac{2940\rho}{936\rho-4973} & -\frac{2940\rho}{936\rho-4973} \end{pmatrix},$$

$$G_2^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{40(\rho+5)}{936\rho-4973} & 0 & 0 \end{pmatrix}$$

and $Y_{m-1}, Y_m, F_{m-1}, F_m$ and F_{m+1} are column vectors defined by:

$$Y_{m-1} = \begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}, \quad Y_m = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix},$$

$$F_{m-1} = \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}, \quad F_m = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix},$$

$$F_{m+1} = \begin{pmatrix} f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix}.$$

Let $D_0^*, D_1^*, G_0^*, G_1^*$ and G_2^* be block matrices defined by:

$$D_0^* = (D_0\ D_1\ D_2), \quad D_1^* = (D_3\ D_4\ D_5),$$
$$G_0^* = (G_0\ G_1\ G_2), \quad G_1^* = (G_3\ G_4\ G_5)$$
$$\text{and } G_2^* = (G_6\ G_7\ G_8),$$

where

$$D_0 = \begin{pmatrix} \frac{1}{80} \\ -\frac{1}{25}\frac{4\rho+3}{12\rho-1} \\ \frac{1}{3}\frac{167\rho+394}{936\rho-4973} \end{pmatrix}, \quad D_1 = \begin{pmatrix} -\frac{1}{8}\frac{4\rho+1}{3\rho+1} \\ \frac{1}{2}\frac{3\rho+2}{12\rho-1} \\ -\frac{5(88\rho+195)}{936\rho-4937} \end{pmatrix},$$

$$D_2 = \begin{pmatrix} -\frac{1}{4}\frac{5\rho-3}{3\rho+1} \\ -\frac{4(2\rho+1)}{12\rho-1} \\ \frac{20(81\rho+160)}{936\rho-4973} \end{pmatrix}, \quad D_3 = \begin{pmatrix} 1 \\ -\frac{2(\rho-6)}{12\rho-1} \\ -\frac{20}{3}\frac{628\rho+935}{936\rho-4973} \end{pmatrix},$$

$$D_4 = \begin{pmatrix} -\frac{1}{16}\frac{19\rho+25}{3\rho+1} \\ 1 \\ \frac{5(403\rho+1770)}{936\ rho-4973} \end{pmatrix}, \quad D_5 = \begin{pmatrix} -\frac{1}{40}\frac{4\rho+3}{3\rho+1} \\ -\frac{1}{50}\frac{167\rho+394}{12\rho-1} \\ 1 \end{pmatrix},$$

$$G_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} \frac{3\rho}{6\rho+2} \\ 0 \\ 0 \end{pmatrix},$$

$$G_3=\begin{pmatrix}-\frac{3\rho}{6\rho+2}\\ \frac{12\rho}{12\rho-1}\\ 0\end{pmatrix},\quad G_4=\begin{pmatrix}-\frac{3}{4}\frac{\rho+1}{3\rho+1}\\ -\frac{12}{12\rho-1}\\ \frac{2940\rho}{936\rho-4973}\end{pmatrix},$$

$$G_5=\begin{pmatrix}0\\ -\frac{6}{5}\frac{\rho+2}{12\rho-1}\\ -\frac{2940}{936\rho-4973}\end{pmatrix},G_6=\begin{pmatrix}0\\ 0\\ \frac{40(\rho+5)}{936\rho-4973}\end{pmatrix},G_7=\begin{pmatrix}0\\0\\0\end{pmatrix},$$

$$G_8=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Definition 1 (Order):

The order of the block method (11) and its associated linear operator given by:

$$L[y(x);h]=\sum_{j=0}^{k=5}[D_jy(x+jh)]-h\sum_{j=0}^{k+1}[G_jy'(x+jh)], \tag{14}$$

is a unique P such that $E_0=E_1=E_2=\ldots=E_P=0$ and $E_{P+1}\neq 0$, where E_{P+1} is a column vector which is called the error constant [29]. The constant column vectors $E_0,E_1,\ldots,E_q$ are defined by:

$$\left.\begin{aligned}
&E_0=D_0+D_1+\ldots+D_k\\
&E_1=D_1+2D_2+\ldots+kD_k-(G_0+G_1+\ldots+G_k)\\
&\quad\vdots\\
&E_q=\frac{1}{q!}(D_1+2^qD_2+\ldots+k^qD_k)\\
&\qquad-\frac{1}{(q-1)!}(G_1+2^{q-1}G_2+\ldots+(k)^{q-1}G_k)\\
&\qquad q=2,3,\ldots
\end{aligned}\right\} \tag{15}$$

where

$$E_0=\sum_{j=0}^{5}D_j=D_0+D_1+D_2+D_3+D_4+D_5$$

$$=\begin{pmatrix}\frac{1}{80}\\ -\frac{1}{25}\frac{4\rho+3}{12\rho-1}\\ \frac{1}{3}\frac{167\rho+394}{936\rho-4973}\end{pmatrix}+\begin{pmatrix}-\frac{1}{8}\frac{4\rho+1}{3\rho+1}\\ \frac{1}{2}\frac{3\rho+2}{12\rho-1}\\ -\frac{5(88\rho+195)}{936\rho-4973}\end{pmatrix}$$

$$+\begin{pmatrix}-\frac{5\rho-3}{4(3\rho+1)}\\ -\frac{4(2\rho+1)}{12\rho-1}\\ \frac{20(81\rho+160)}{936\rho-4973}\end{pmatrix}+\begin{pmatrix}1\\ -\frac{2(\rho-6)}{12\rho-1}\\ -\frac{20(628\rho+935)}{3(936\rho-4973)}\end{pmatrix}$$

$$+\begin{pmatrix}-\frac{19\rho+25}{16(3\rho+1)}\\ 1\\ \frac{5(403\rho+1770)}{936\rho-4973}\end{pmatrix}+\begin{pmatrix}-\frac{4\rho+3}{40(3\rho+1)}\\ -\frac{167\rho+394}{50(12\rho-1)}\\ 1\end{pmatrix}$$

$$=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

$$\begin{aligned}E_1&=\sum_{j=0}^{5}jD_j-\sum_{j=0}^{6}G_j\\
&=(D_1+2D_2+3D_3+4D_4+5D_5)\\
&\quad-(G_2+G_3+G_4+G_5+G_6)\end{aligned}$$

$$=\left(\begin{pmatrix}-\frac{4\rho+1}{8(3\rho+1)}\\ \frac{3\rho+2}{2(12\rho-1)}\\ -\frac{5(88\rho+195)}{936\rho-4973}\end{pmatrix}+2\begin{pmatrix}-\frac{5\rho-3}{4(3\rho+1)}\\ -\frac{4(2\rho+1)}{12\rho-1}\\ \frac{20(81\rho+160)}{936\rho-4973}\end{pmatrix}\right.$$

$$+3\begin{pmatrix}1\\ -\frac{2(\rho-6)}{12\rho-1}\\ -\frac{20(628\rho+935)}{3(936\rho-4973)}\end{pmatrix}+4\begin{pmatrix}-\frac{19\rho+25}{16(3\rho+1)}\\ 1\\ \frac{5(403\rho+1770)}{936\rho-4973}\end{pmatrix}$$

$$\left.+5\begin{pmatrix}-\frac{4\rho+3}{40(3\rho+1)}\\ -\frac{167\rho+394}{50(12\rho-1)}\\ 1\end{pmatrix}\right)$$

$$-\left(\begin{pmatrix}\frac{3\rho}{6\rho+2}\\ 0\\ 0\end{pmatrix}+\begin{pmatrix}-\frac{3}{6\rho+2}\\ \frac{12\rho}{12\rho-1}\\ 0\end{pmatrix}+\begin{pmatrix}-\frac{3(\rho+1)}{4(3\rho+1)}\\ -\frac{12}{12\rho-1}\\ \frac{2940\rho}{936\rho-4973}\end{pmatrix}\right.$$

$$\left.+\begin{pmatrix}0\\ -\frac{6(\rho+2)}{5(12\rho-1)}\\ -\frac{2940}{936\rho-4973}\end{pmatrix}+\begin{pmatrix}0\\ 0\\ \frac{40(\rho+5)}{936\rho-4973}\end{pmatrix}\right)$$

$$=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

$$
\begin{aligned}
E_2 &= \frac{1}{2!}\sum_{j=0}^{5} j^2 D_j - \frac{1}{1!}\sum_{j=0}^{6} j G_j \\
&= \frac{1}{2!}\left(D_1 + 2^2 D_2 + 3^2 D_3 + 4^2 D_4 + 5^2 D_5\right) \\
&\quad - \frac{1}{1!}\left(2G_2 + 3G_3 + 4G_4 + 5G_5 + 6G_6\right) \\
&= \frac{1}{2!}\left(\begin{pmatrix} -\frac{4\rho+1}{8(3\rho+1)} \\ \frac{3\rho+2}{2(12\rho-1)} \\ -\frac{5(88\rho+195)}{936\rho-4973} \end{pmatrix} + 2^2\begin{pmatrix} -\frac{5\rho-3}{4(3\rho+1)} \\ -\frac{4(2\rho+1)}{12\rho-1} \\ \frac{20(81\rho+160)}{936\rho-4973} \end{pmatrix}\right. \\
&\quad + 3^2\begin{pmatrix} 1 \\ -\frac{2(\rho-6)}{12\rho-1} \\ -\frac{20(628\rho+935)}{3(936\rho-4973)} \end{pmatrix} + 4^2\begin{pmatrix} -\frac{19\rho+25}{16(3\rho+1)} \\ 1 \\ \frac{5(403\rho+1770)}{936\rho-4973} \end{pmatrix} \\
&\quad \left. +5^2\begin{pmatrix} -\frac{4\rho+3}{40(3\rho+1)} \\ -\frac{167\rho+394}{50(12\rho-1)} \\ 1 \end{pmatrix}\right) \\
&\quad - \left(2\begin{pmatrix} \frac{3\rho}{6\rho+2} \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} -\frac{3}{6\rho+2} \\ \frac{12\rho}{12\rho-1} \\ 0 \end{pmatrix}\right. \\
&\quad + 4\begin{pmatrix} -\frac{3(\rho+1)}{4(3\rho+1)} \\ -\frac{12}{12\rho-1} \\ \frac{2940\rho}{936\rho-4973} \end{pmatrix} + 5\begin{pmatrix} 0 \\ -\frac{6(\rho+2)}{5(12\rho-1)} \\ -\frac{2940}{936\rho-4973} \end{pmatrix} \\
&\quad \left. +6\begin{pmatrix} 0 \\ 0 \\ \frac{40(\rho+5)}{936\rho-4973} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
E_3 &= \frac{1}{3!}\sum_{j=0}^{5} j^3 D_j - \frac{1}{2!}\sum_{j=0}^{6} j^2 G_j \\
&= \frac{1}{3!}\left(D_1 + 2^3 D_2 + 3^3 D_3 + 4^3 D_4 + 5^3 D_5\right) \\
&\quad - \frac{1}{2!}\left(2^2G_2 + 3^2G_3 + 4^2G_4 + 5^2G_5 + 6^2G_6\right) \\
&= \frac{1}{3!}\left(\begin{pmatrix} -\frac{4\rho+1}{8(3\rho+1)} \\ \frac{3\rho+2}{2(12\rho-1)} \\ -\frac{5(88\rho+195)}{936\rho-4973} \end{pmatrix} + 2^3\begin{pmatrix} -\frac{5\rho-3}{4(3\rho+1)} \\ -\frac{4(2\rho+1)}{12\rho-1} \\ \frac{20(81\rho+160)}{936\rho-4973} \end{pmatrix}\right. \\
&\quad + 3^3\begin{pmatrix} 1 \\ -\frac{2(\rho-6)}{12\rho-1} \\ -\frac{20(628\rho+935)}{3(936\rho-4973)} \end{pmatrix} + 4^3\begin{pmatrix} -\frac{19\rho+25}{16(3\rho+1)} \\ 1 \\ \frac{5(403\rho+1770)}{936\rho-4973} \end{pmatrix} \\
&\quad \left. +5^3\begin{pmatrix} -\frac{4\rho+3}{40(3\rho+1)} \\ -\frac{167\rho+394}{50(12\rho-1)} \\ 1 \end{pmatrix}\right) \\
&\quad - \left(2^2\begin{pmatrix} \frac{3\rho}{6\rho+2} \\ 0 \\ 0 \end{pmatrix} + 3^2\begin{pmatrix} -\frac{3}{6\rho+2} \\ \frac{12\rho}{12\rho-1} \\ 0 \end{pmatrix}\right. \\
&\quad + 4^2\begin{pmatrix} -\frac{3(\rho+1)}{4(3\rho+1)} \\ -\frac{12}{12\rho-1} \\ \frac{2940\rho}{936\rho-4973} \end{pmatrix} + 5^2\begin{pmatrix} 0 \\ -\frac{6(\rho+2)}{5(12\rho-1)} \\ -\frac{2940}{936\rho-4973} \end{pmatrix} \\
&\quad \left. +6^2\begin{pmatrix} 0 \\ 0 \\ \frac{40(\rho+5)}{936\rho-4973} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
E_4 &= \frac{1}{4!}\sum_{j=0}^{5} j^4 D_j - \frac{1}{3!}\sum_{j=0}^{6} j^3 G_j \\
&= \frac{1}{4!}\left(D_1 + 2^4 D_2 + 3^4 D_3 + 4^4 D_4 + 5^4 D_5\right) \\
&- \frac{1}{3!}\left(2^3 G_2 + 3^3 G_3 + 4^3 G_4 + 5^3 G_5 + 6^3 G_6\right) \\
&= \frac{1}{4!}\left(\begin{pmatrix} -\frac{4\rho+1}{8(3\rho+1)} \\ \frac{3\rho+2}{2(12\rho-1)} \\ -\frac{5(88\rho+195)}{936\rho-4973} \end{pmatrix} + 2^4 \begin{pmatrix} -\frac{5\rho-3}{4(3\rho+1)} \\ -\frac{4(2\rho+1)}{12\rho-1} \\ \frac{20(81\rho+160)}{936\rho-4973} \end{pmatrix}\right. \\
&\quad + 3^4 \begin{pmatrix} 1 \\ -\frac{2(\rho-6)}{12\rho-1} \\ -\frac{20(628\rho+935)}{3(936\rho-4973)} \end{pmatrix} + 4^4 \begin{pmatrix} -\frac{19\rho+25}{16(3\rho+1)} \\ 1 \\ \frac{5(403\rho+1770)}{936\rho-4973} \end{pmatrix} \\
&\quad \left. + 5^4 \begin{pmatrix} -\frac{4\rho+3}{40(3\rho+1)} \\ -\frac{167\rho+394}{50(12\rho-1)} \\ 1 \end{pmatrix}\right) \\
&\quad - \frac{1}{3!}\left(2^3 \begin{pmatrix} \frac{3\rho}{6\rho+2} \\ 0 \\ 0 \end{pmatrix} + 3^3 \begin{pmatrix} -\frac{3}{6\rho+2} \\ \frac{12\rho}{12\rho-1} \\ 0 \end{pmatrix}\right. \\
&\quad + 4^3 \begin{pmatrix} -\frac{3(\rho+1)}{4(3\rho+1)} \\ -\frac{12}{12\rho-1} \\ \frac{2940\rho}{936\rho-4973} \end{pmatrix} + 5^3 \begin{pmatrix} 0 \\ -\frac{6(\rho+2)}{5(12\rho-1)} \\ -\frac{2940}{936\rho-4973} \end{pmatrix} \\
&\quad \left. + 6^3 \begin{pmatrix} 0 \\ 0 \\ \frac{40(\rho+5)}{936\rho-4973} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
E_5 &= \frac{1}{5!}\sum_{j=0}^{5} j^5 D_j - \frac{1}{4!}\sum_{j=0}^{6} j^4 G_j \\
&= \frac{1}{5!}\left(D_1 + 2^5 D_2 + 3^5 D_3 + 4^5 D_4 + 5^5 D_5\right) \\
&- \frac{1}{4!}\left(2^4 G_2 + 3^4 G_3 + 4^4 G_4 + 5^4 G_5 + 6^4 G_6\right) \\
&= \frac{1}{5!}\left(\begin{pmatrix} -\frac{4\rho+1}{8(3\rho+1)} \\ \frac{3\rho+2}{2(12\rho-1)} \\ -\frac{5(88\rho+195)}{936\rho-4973} \end{pmatrix} + 2^5 \begin{pmatrix} -\frac{5\rho-3}{4(3\rho+1)} \\ -\frac{4(2\rho+1)}{12\rho-1} \\ \frac{20(81\rho+160)}{936\rho-4973} \end{pmatrix}\right. \\
&\quad + 3^5 \begin{pmatrix} 1 \\ -\frac{2(\rho-6)}{12\rho-1} \\ -\frac{20(628\rho+935)}{3(936\rho-4973)} \end{pmatrix} + 4^5 \begin{pmatrix} -\frac{19\rho+25}{16(3\rho+1)} \\ 1 \\ \frac{5(403\rho+1770)}{936\rho-4973} \end{pmatrix} \\
&\quad \left. + 5^5 \begin{pmatrix} -\frac{4\rho+3}{40(3\rho+1)} \\ -\frac{167\rho+394}{50(12\rho-1)} \\ 1 \end{pmatrix}\right) \\
&\quad - \frac{1}{4!}\left(2^4 \begin{pmatrix} \frac{3\rho}{6\rho+2} \\ 0 \\ 0 \end{pmatrix} + 3^4 \begin{pmatrix} -\frac{3}{6\rho+2} \\ \frac{12\rho}{12\rho-1} \\ 0 \end{pmatrix}\right. \\
&\quad + 4^4 \begin{pmatrix} -\frac{3(\rho+1)}{4(3\rho+1)} \\ -\frac{12}{12\rho-1} \\ \frac{2940\rho}{936\rho-4973} \end{pmatrix} + 5^4 \begin{pmatrix} 0 \\ -\frac{6(\rho+2)}{5(12\rho-1)} \\ -\frac{2940}{936\rho-4973} \end{pmatrix} \\
&\quad \left. + 6^4 \begin{pmatrix} 0 \\ 0 \\ \frac{40(\rho+5)}{936\rho-4973} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
E_6 &= \frac{1}{6!}\sum_{j=0}^{5} j^6 D_j - \frac{1}{5!}\sum_{j=0}^{6} j^5 G_j \\
&= \frac{1}{6!}\left(D_1 + 2^6 D_2 + 3^6 D_3 + 4^6 D_4 + 5^6 D_5\right) \\
&- \frac{1}{5!}\left(2^5 G_2 + 3^5 G_3 + 4^5 G_4 + 5^5 G_5 + 6^5 G_6\right) \\
&= \frac{1}{6!}\left(\begin{pmatrix} -\frac{4\rho+1}{8(3\rho+1)} \\ \frac{3\rho+2}{2(12\rho-1)} \\ -\frac{5(88\rho+195)}{936\rho-4973} \end{pmatrix} + 2^6 \begin{pmatrix} -\frac{5\rho-3}{4(3\rho+1)} \\ -\frac{4(2\rho+1)}{12\rho-1} \\ \frac{20(81\rho+160)}{936\rho-4973} \end{pmatrix}\right. \\
&\qquad + 3^6 \begin{pmatrix} 1 \\ -\frac{2(\rho-6)}{12\rho-1} \\ -\frac{20(628\rho+935)}{3(936\rho-4973)} \end{pmatrix} + 4^6 \begin{pmatrix} -\frac{19\rho+25}{16(3\rho+1)} \\ 1 \\ \frac{5(403\rho+1770)}{936\rho-4973} \end{pmatrix} \\
&\qquad \left. + 5^6 \begin{pmatrix} -\frac{4\rho+3}{40(3\rho+1)} \\ -\frac{167\rho+394}{50(12\rho-1)} \\ 1 \end{pmatrix}\right) \\
&\quad - \frac{1}{5!}\left(2^5 \begin{pmatrix} \frac{3\rho}{6\rho+2} \\ 0 \\ 0 \end{pmatrix} + 3^5 \begin{pmatrix} -\frac{3}{6\rho+2} \\ \frac{12\rho}{12\rho-1} \\ 0 \end{pmatrix}\right. \\
&\qquad + 4^5 \begin{pmatrix} -\frac{3(\rho+1)}{4(3\rho+1)} \\ -\frac{12}{12\rho-1} \\ \frac{2940\rho}{936\rho-4973} \end{pmatrix} + 5^5 \begin{pmatrix} 0 \\ -\frac{6(\rho+2)}{5(12\rho-1)} \\ -\frac{2940}{936\rho-4973} \end{pmatrix} \\
&\qquad \left. + 6^5 \begin{pmatrix} 0 \\ 0 \\ \frac{40(\rho+5)}{936\rho-4973} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
E_7 &= \frac{1}{7!}\sum_{j=0}^{5} j^7 D_j - \frac{1}{6!}\sum_{j=0}^{6} j^6 G_j \\
&= \frac{1}{7!}\left(D_1 + 2^7 D_2 + 3^7 D_3 + 4^7 D_4 + 5^7 D_5\right) \\
&- \frac{1}{6!}\left(2^6 G_2 + 3^6 G_3 + 4^6 G_4 + 5^6 G_5 + 6^6 G_6\right) \\
&= \frac{1}{7!}\left(\begin{pmatrix} -\frac{4\rho+1}{8(3\rho+1)} \\ \frac{3\rho+2}{2(12\rho-1)} \\ -\frac{5(88\rho+195)}{936\rho-4973} \end{pmatrix} + 2^7 \begin{pmatrix} -\frac{5\rho-3}{4(3\rho+1)} \\ -\frac{4(2\rho+1)}{12\rho-1} \\ \frac{20(81\rho+160)}{936\rho-4973} \end{pmatrix}\right. \\
&\qquad + 3^7 \begin{pmatrix} 1 \\ -\frac{2(\rho-6)}{12\rho-1} \\ -\frac{20(628\rho+935)}{3(936\rho-4973)} \end{pmatrix} + 4^7 \begin{pmatrix} -\frac{19\rho+25}{16(3\rho+1)} \\ 1 \\ \frac{5(403\rho+1770)}{936\rho-4973} \end{pmatrix} \\
&\qquad \left. + 5^7 \begin{pmatrix} -\frac{4\rho+3}{40(3\rho+1)} \\ -\frac{167\rho+394}{50(12\rho-1)} \\ 1 \end{pmatrix}\right) \\
&\quad - \frac{1}{6!}\left(2^6 \begin{pmatrix} \frac{3\rho}{6\rho+2} \\ 0 \\ 0 \end{pmatrix} + 3^6 \begin{pmatrix} -\frac{3}{6\rho+2} \\ \frac{12\rho}{12\rho-1} \\ 0 \end{pmatrix}\right. \\
&\qquad + 4^6 \begin{pmatrix} -\frac{3(\rho+1)}{4(3\rho+1)} \\ -\frac{12}{12\rho-1} \\ \frac{2940\rho}{936\rho-4973} \end{pmatrix} + 5^6 \begin{pmatrix} 0 \\ -\frac{6(\rho+2)}{5(12\rho-1)} \\ -\frac{2940}{936\rho-4973} \end{pmatrix} \\
&\qquad \left. + 6^6 \begin{pmatrix} 0 \\ 0 \\ \frac{40(\rho+5)}{936\rho-4973} \end{pmatrix}\right) \\
&= \begin{pmatrix} -\frac{2\rho+1}{280(3\rho+1)} \\ \frac{2(\rho+1)}{35(12\rho-1)} \\ \frac{2(118\rho+345)}{21(936\rho-4973)} \end{pmatrix}.
\end{aligned}
$$

Thus, the 3SBEBDF is of order six, with error constant given by

$$
E_7 = \begin{pmatrix} -\frac{2\rho+1}{280(3\rho+1)} \\ \frac{2(\rho+1)}{35(12\rho-1)} \\ \frac{2(118\rho+345)}{7(936\rho-4973)} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
$$

4 Stability and Convergence Analysis of the Method

The stability properties of the proposed 3SBEBDF scheme are investigated in terms of zero-stability and A-stability by applying the Dahlquist linear test equation. Before proceeding with the stability analysis, we first recall the definitions of the root condition, zero-stability, and A-stability as follows.

Definition 1 (Root Condition). The first characteristic polynomial of a linear multistep method (LMM) is said to satisfy the root condition if all its roots lie inside or on the unit circle, and every root on the unit circle is simple. In other words, if the roots r_i satisfy

$$|r_i| \leq 1, \qquad i = 1, 2, \ldots, k,$$

and every root satisfying $|r_i| = 1$ has multiplicity one, then the method satisfies the root condition [30].

Definition 2 (Zero-Stability). A linear multistep method is said to be zero-stable if its first characteristic polynomial satisfies the root condition. Equivalently, as the step size $h \to 0$, the numerical solution remains bounded under arbitrarily small perturbations in the starting values [6, 31].

Definition 3 (A-Stability). A linear multistep method is said to be A-stable if its region of absolute stability contains the entire left-half complex plane, namely,

$$\{z \in \mathbb{C} : \mathrm{Re}(z) \leq 0\},$$

or equivalently, when applied to the Dahlquist test equation

$$y' = \lambda y, \qquad \mathrm{Re}(\lambda) \leq 0,$$

the numerical solution satisfies

$$|R(z)| \leq 1, \qquad z = h\lambda,$$

where $R(z)$ denotes the stability function of the method [32].

The matrix representation of (11) is expressed as:

$$\begin{pmatrix} 1 & -\frac{19\rho+25}{16(3\rho+1)} & -\frac{4\rho+3}{40(3\rho+1)} \\ -\frac{2(\rho-6)}{12\rho-1} & 1 & -\frac{167\rho+394}{50(12\rho-1)} \\ -\frac{20(628\rho+935)}{3(936\rho-4973)} & \frac{5(403\rho+1770)}{936\rho-4973} & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{1}{80} & \frac{4\rho+1}{8(3\rho+1)} & \frac{5\rho-3}{4(3\rho+1)} \\ \frac{4\rho+3}{25(12\rho-1)} & -\frac{3\rho+2}{2(12\rho-1)} & \frac{4(2\rho+1)}{12\rho-1} \\ -\frac{167\rho+394}{3(936\rho-4973)} & \frac{5(88\rho+195)}{936\rho-4973} & -\frac{20(81\rho+160)}{936\rho-4973} \end{pmatrix}$$
$$\times \begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix} + h \begin{pmatrix} 0 & 0 & \frac{3\rho}{6\rho+2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}$$
$$+ h \begin{pmatrix} -\frac{3}{6\rho+2} & -\frac{3(\rho+1)}{4(3\rho+1)} & 0 \\ \frac{12\rho}{12\rho-1} & -\frac{12}{12\rho-1} & -\frac{6(\rho+2)}{5(12\rho-1)} \\ 0 & \frac{2940\rho}{936\rho-4973} & -\frac{2940}{936\rho-4973} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix}$$
$$+ h \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{40(\rho+5)}{936\rho-4973} & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix}. \tag{16}$$

Let

$$Y_m = \left[y_{n+1}, y_{n+2}, \ldots, y_{n+r}\right]^T,$$
$$F_m = \left[f_{n+1}, f_{n+2}, \ldots, f_{n+r}\right]^T.$$

Adopting the block matrix representation framework of [33], adapted here for first-order systems, the k-block, r-point method corresponding to (1) can be written in the matrix form

$$A_0 Y_m = A_1 Y_{m-1} + h\left(B_0 F_{m-1} + B_1 F_m + B_2 F_{m+1}\right), \tag{17}$$

where $r = 3$, $n = mr$, and $m = 0, 1, 2, \ldots$.

$$Y_m = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \end{pmatrix},$$

$$Y_{m-1} = \begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_{n} \end{pmatrix} = \begin{pmatrix} y_{3m-2} \\ y_{3m-1} \\ y_{3m} \end{pmatrix} = \begin{pmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \end{pmatrix},$$

$$F_m = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} = \begin{pmatrix} f_{3m+1} \\ f_{3m+2} \\ f_{3m+3} \end{pmatrix},$$

$$F_{m-1} = \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix} = \begin{pmatrix} f_{3m-2} \\ f_{3m-1} \\ f_{3m} \end{pmatrix} = \begin{pmatrix} f_{3(m-1)+1} \\ f_{3(m-1)+2} \\ f_{3(m-1)+3} \end{pmatrix},$$

$$F_{m+1} = \begin{pmatrix} f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix} = \begin{pmatrix} f_{3m+4} \\ f_{3m+5} \\ f_{3m+6} \end{pmatrix} = \begin{pmatrix} f_{3(m+1)+1} \\ f_{3(m+1)+2} \\ f_{3(m+1)+3} \end{pmatrix}.$$

$$A_0 = \begin{pmatrix} 1 & -\dfrac{19\rho+25}{16(3\rho+1)} & -\dfrac{4\rho+3}{40(3\rho+1)} \\ -\dfrac{2(\rho-6)}{12\rho-1} & 1 & -\dfrac{167\rho+394}{50(12\rho-1)} \\ -\dfrac{20(628\rho+935)}{3(936\rho-4973)} & \dfrac{5(403\rho+1770)}{936\rho-4973} & 1 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -\dfrac{1}{80} & \dfrac{4\rho+1}{8(3\rho+1)} & \dfrac{5\rho-3}{4(3\rho+1)} \\ \dfrac{4\rho+3}{25(12\rho-1)} & -\dfrac{3\rho+2}{2(12\rho-1)} & \dfrac{4(2\rho+1)}{12\rho-1} \\ -\dfrac{167\rho+394}{3(936\rho-4973)} & \dfrac{5(88\rho+195)}{936\rho-4973} & -\dfrac{20(81\rho+160)}{936\rho-4973} \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 & 0 & \dfrac{3\rho}{6\rho+2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -\dfrac{3}{6\rho+2} & -\dfrac{3(\rho+1)}{4(3\rho+1)} & 0 \\ \dfrac{12\rho}{12\rho-1} & -\dfrac{12}{12\rho-1} & -\dfrac{6(\rho+2)}{5(12\rho-1)} \\ 0 & \dfrac{2940\rho}{936\rho-4973} & -\dfrac{2940}{936\rho-4973} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{40(\rho+5)}{936\rho-4973} & 0 & 0 \end{pmatrix}.$$

By substituting the Dahlquist linear test equation

$$y' = \lambda y$$

into equation (17) and introducing $\bar{h} = \lambda h$, we obtain

$$A_0 Y_m = A_1 Y_{m-1} + \bar{h}\left(B_0 Y_{m-1} + B_1 Y_m + B_2 Y_{m+1}\right). \tag{18}$$

where A_0, A_1, B_0, B_1, and B_2 are defined as above, and

$$Y_{m+1} = \begin{pmatrix} y_{n+4} \\ y_{n+5} \\ y_{n+6} \end{pmatrix} = \begin{pmatrix} y_{3m+4} \\ y_{3m+5} \\ y_{3m+6} \end{pmatrix} = \begin{pmatrix} y_{3(m+1)+1} \\ y_{3(m+1)+2} \\ y_{3(m+1)+3} \end{pmatrix}.$$

To investigate the A-stability of the proposed method, the corresponding characteristic polynomial is obtained from

$$R(t, \bar{h}) = \det\left(\left(A_0 - \bar{h}B_1 - \bar{h}B_2\right)t - \left(A_1 + \bar{h}B_0\right)\right) = 0, \tag{19}$$

The stability polynomial of the method is equivalently given as:

$$\begin{aligned} R(t,\bar{h}) = -\, & \frac{1}{60(3\rho+1)(12\rho-1)(936\rho-4973)} \\ & \times \Big(2160\bar{h}^3\rho^3t^3 + 3175200\bar{h}^3\rho^3t^2 - 1887840\bar{h}^3\rho^2t^3 \\ & - 712152\bar{h}^2\rho^3t^3 + 43200\bar{h}^3\rho^2t^2 - 2185920\bar{h}^3\rho t^3 \\ & + 140616\bar{h}^2\rho^3t^2 + 5146488\bar{h}^2\rho^2t^3 + 1834191\bar{h}\rho^3t^3 \\ & + 216000\bar{h}^3\rho t^2 - 3153600\bar{h}^3t^3 + 522936\bar{h}^2\rho^3t \\ & - 9185760\bar{h}^2\rho^2t^2 + 1129824\bar{h}^2\rho t^3 + 6974892\bar{h}\rho^3t^2 \\ & - 9514992\bar{h}\rho^2t^3 - 1826216\rho^3t^3 + 1055232\bar{h}^2\rho^2t \\ & + 3839436\bar{h}^2\rho t^2 + 7585920\rho^2t^3 - 123840\bar{h}^2\rho t \\ & - 4135320\bar{h}^2t^2 - 1179\bar{h}\rho^3 + 2743830\bar{h}\rho^2t \\ & + 11745742\bar{h}\rho t^2 - 10814940\bar{h}t^3 + 1666632\rho^3t \\ & - 8523066\rho^2t^2 - 779877\rho t^3 - 3948\bar{h}\rho^2 \\ & - 650169\bar{h}\rho t - 7227990\bar{h}t^2 - 4114\rho^3 \\ & + 2400546\rho^2t + 1512351\rho t^2 + 6166690t^3 \\ & - 1290\bar{h}\rho + 109530\bar{h}t - 13222\rho^2 \\ & - 728811\rho t - 6355470t^2 - 3663\rho \\ & + 188670t + 110\Big). \end{aligned} \tag{20}$$

To illustrate the absolute stability properties of the proposed method, two representative values, $\rho = -\frac{1}{2}$ and $\rho = \frac{1}{2}$, are selected from the interval $(-1, 1)$. The boundary locus method is employed to determine the regions of absolute stability. Specifically, the stability polynomial is evaluated along the unit circle by setting the amplification factor

$$t = e^{i\theta}, \qquad \theta \in [0, 2\pi],$$

thereby generating the boundary that separates the stable and unstable regions in the complex plane. The corresponding absolute stability regions, computed using Maple 18, are presented in Figures 2 and 3 for the selected values of ρ.

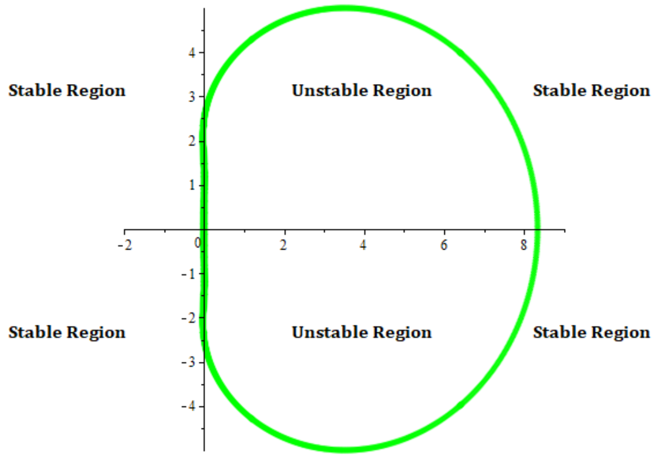

Figure 2. Stability region of 3SBEBDF when $\rho = -\frac{1}{2}$.

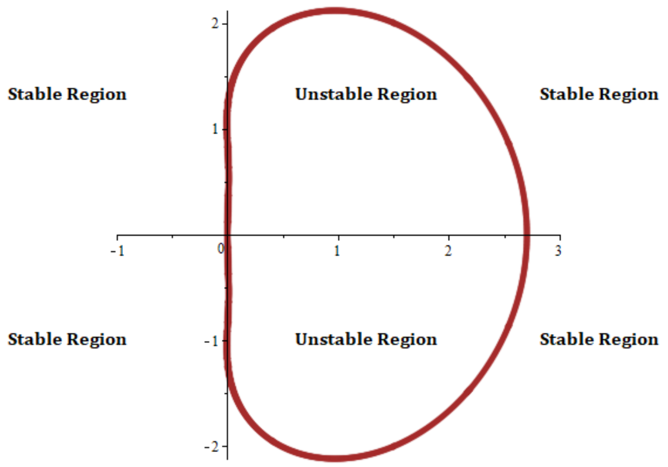

Figure 3. Stability region of 3SBEBDF when $\rho = \frac{1}{2}$.

The plots of the stability polynomial in Figures 2 and 3 reveal that the stability regions encompass the entire negative half-plane. This observation suggests that the method is A-stable and well-suited for solving stiff ODEs.

To show that the method is zero-stable, the first characteristics polynomial of the method is therefore given by

$$\det(A_0 t - A_1) = 0. \tag{21}$$

$$\left| \begin{pmatrix} 1 & -\dfrac{19\rho+25}{16(3\rho+1)} & -\dfrac{4\rho+3}{40(3\rho+1)} \\ -\dfrac{2(\rho-6)}{12\rho-1} & 1 & -\dfrac{167\rho+394}{50(12\rho-1)} \\ -\dfrac{20(628\rho+935)}{3(936\rho-4973)} & \dfrac{5(403\rho+1770)}{936\rho-4973} & 1 \end{pmatrix} t - \begin{pmatrix} -\dfrac{1}{80} & \dfrac{4\rho+1}{8(3\rho+1)} & \dfrac{5\rho-3}{4(3\rho+1)} \\ \dfrac{4\rho+3}{25(12\rho-1)} & -\dfrac{3\rho+2}{2(12\rho-1)} & \dfrac{4(2\rho+1)}{12\rho-1} \\ -\dfrac{167\rho+394}{3(936\rho-4973)} & \dfrac{5(88\rho+195)}{936\rho-4973} & -\dfrac{20(81\rho+160)}{936\rho-4973} \end{pmatrix} \right| = 0. \tag{22}$$

The first characteristic polynomial becomes:

$$\begin{aligned} &-\frac{1}{60(12\rho-1)(3\rho+1)(936\rho-4973)} \\ &\times\Big(-1826216\rho^3t^3 + 163698\rho^3t^2 + 6135742\rho^2t^3 \\ &+ 1666632\rho^3t - 8523066\rho^2t^2 - 779877\rho t^3 \\ &- 4114\rho^3 + 2400546\rho^2t + 1512351\rho t^2 \\ &+ 6166690t^3 - 13222\rho^2 - 728811\rho t \\ &- 6355470t^2 - 3663\rho + 188670t + 110\Big) = 0. \end{aligned} \tag{23}$$

In accordance with Definition 2, the 3SBEBDF method is zero-stable for all permissible values of the free parameter ρ within the interval -1,1. To further verify the suitability of ρ, three representative values were selected from this interval, and the corresponding roots of the first characteristic polynomial were computed, as presented in the Table 1.

Table 1. Roots for different values of ρ.

Free Parameter (ρ)	Roots ($\lvert t \rvert$)	Root Condition ($\lvert t \rvert \le 1$)	Zero-Stability
$\rho = -\dfrac{1}{2}$	$t_1 = 1,\ t_2 = 0.00090738252,\ t_3 = 0.1125739606$	Satisfied	Satisfied
$\rho = 0$	$t_1 = 1,\ t_2 = -0.00057200101,\ t_3 = 0.031184858$	Satisfied	Satisfied
$\rho = \dfrac{1}{2}$	$t_1 = 1,\ t_2 = 0.00995390406,\ t_3 = 0.07860193358$	Satisfied	Satisfied

Table 1 shows that, for each selected value of ρ, all the roots of the first characteristic polynomial lie within or on the unit circle, and every root on the unit circle is simple. Therefore, the root condition is satisfied for all the selected values of ρ. Consequently, the proposed 3SBEBDF method is zero-stable for these representative values of ρ. These numerical results provide evidence that the proposed method possesses satisfactory zero-stability over the admissible parameter interval $(-1, 1)$.

Definition 4 (Convergence). A linear multistep method (LMM) is said to be convergent if and only if it is both consistent and zero-stable [6].

Definition 5 (Consistency). A linear multistep method is said to be consistent if its order is at least one. Equivalently, it is consistent if the following conditions are satisfied [34]:

$$\text{(i)} \qquad \sum_{j=0}^{k} D_j = 0.$$

$$\text{(ii)} \qquad \sum_{j=0}^{k} jD_j = \sum_{j=0}^{k} G_j.$$

Theorem 1. The proposed 3SBEBDF scheme is convergent.

Proof. According to the Dahlquist equivalence theorem, a linear multistep method is convergent if and only if it is both consistent and zero-stable. Therefore, it suffices to verify these two properties for the proposed 3SBEBDF scheme.

The consistency of the proposed method has been established in the preceding section. In particular, the method satisfies the consistency conditions and has order six, which is greater than one. Hence, the proposed method is consistent.

Furthermore, the zero-stability of the proposed method has been verified in Section X by showing that the first characteristic polynomial satisfies the root condition. Therefore, the proposed method is zero-stable.

Since the proposed 3SBEBDF scheme is both consistent and zero-stable, it follows from the Dahlquist equivalence theorem that the method is convergent.

To verify the first consistency condition, let $D_0, D_1, \ldots, D_5$ and $G_0, G_1, \ldots, G_8$ be defined as in Section X. Then

$$\begin{aligned}
\sum_{j=0}^{5} D_j &= D_0 + D_1 + D_2 + D_3 + D_4 + D_5 \\
&= \begin{pmatrix} \frac{1}{80} \\ -\frac{4\rho+3}{25(12\rho-1)} \\ \frac{167\rho+394}{3(936\rho-4973)} \end{pmatrix} + \begin{pmatrix} -\frac{4\rho+1}{8(3\rho+1)} \\ \frac{3\rho+2}{2(12\rho-1)} \\ -\frac{5(88\rho+195)}{936\rho-4973} \end{pmatrix} \\
&\quad + \begin{pmatrix} -\frac{5\rho-3}{4(3\rho+1)} \\ -\frac{4(2\rho+1)}{12\rho-1} \\ \frac{20(81\rho+160)}{936\rho-4973} \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{2(\rho-6)}{12\rho-1} \\ -\frac{20(628\rho+935)}{3(936\rho-4973)} \end{pmatrix} \\
&\quad + \begin{pmatrix} -\frac{19\rho+25}{16(3\rho+1)} \\ 1 \\ \frac{5(403\rho+1770)}{936\rho-4973} \end{pmatrix} + \begin{pmatrix} -\frac{4\rho+3}{40(3\rho+1)} \\ -\frac{167\rho+394}{50(12\rho-1)} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{24}$$

Hence, the first condition is satisfied.

$$\begin{aligned}
\sum_{j=0}^{5} jD_j &= D_1 + 2D_2 + 3D_3 + 4D_4 + 5D_5 \\
&= \begin{pmatrix} -\frac{4\rho+1}{8(3\rho+1)} \\ \frac{3\rho+2}{2(12\rho-1)} \\ -\frac{5(88\rho+195)}{936\rho-4973} \end{pmatrix} + 2\begin{pmatrix} -\frac{5\rho-3}{4(3\rho+1)} \\ -\frac{4(2\rho+1)}{12\rho-1} \\ \frac{20(81\rho+160)}{936\rho-4973} \end{pmatrix} \\
&\quad + 3\begin{pmatrix} 1 \\ -\frac{2(\rho-6)}{12\rho-1} \\ -\frac{20(628\rho+935)}{3(936\rho-4973)} \end{pmatrix} + 4\begin{pmatrix} -\frac{19\rho+25}{16(3\rho+1)} \\ 1 \\ \frac{5(403\rho+1770)}{936\rho-4973} \end{pmatrix} \\
&\quad + 5\begin{pmatrix} -\frac{4\rho+3}{40(3\rho+1)} \\ -\frac{167\rho+394}{50(12\rho-1)} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3(\rho-3)}{4(3\rho+1)} \\ \frac{18(3\rho-4)}{5(12\rho-1)} \\ -\frac{20(145\rho+137)}{936\rho-4973} \end{pmatrix}.
\end{aligned} \tag{25}$$

$$
\begin{aligned}
\sum_{j=0}^{k+1} G_j &= G_0 + G_1 + G_2 + G_3 + G_4 + G_5 + G_6 \\
&= \begin{pmatrix} \frac{3\rho}{6\rho+2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{3}{6\rho+2} \\ \frac{12\rho}{12\rho-1} \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{3(\rho+1)}{4(3\rho+1)} \\ -\frac{12}{12\rho-1} \\ \frac{2940\rho}{936\rho-4973} \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ -\frac{6(\rho+2)}{5(12\rho-1)} \\ -\frac{2940}{936\rho-4973} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{40(\rho+5)}{936\rho-4973} \end{pmatrix} \\
&= \begin{pmatrix} \frac{3(\rho-3)}{4(3\rho+1)} \\ \frac{18(3\rho-4)}{5(12\rho-1)} \\ -\frac{20(145\rho+137)}{936\rho-4973} \end{pmatrix}.
\end{aligned} \tag{26}
$$

Hence,

$$
\sum_{j=0}^{5} jD_j = \sum_{j=0}^{k+1} G_j.
$$

Therefore, the consistency conditions are satisfied, and the proposed 3SBEBDF method is consistent.

Furthermore, the zero-stability of the proposed method has been established in the previous section. Since a linear multistep method is convergent if and only if it is both consistent and zero-stable according to the Dahlquist equivalence theorem [6], it follows that the proposed 3SBEBDF method is convergent. □

5 Implementation of the Method

Newton's method is employed to solve the nonlinear algebraic system generated by the proposed 3SBEBDF scheme. Specifically, the Newton iteration is given by

$$
y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \left(F_j'\left(y_{n+j}^{(i)}\right)\right)^{-1} F_j\left(y_{n+j}^{(i)}\right), \quad j = 1,2,3, \tag{27}
$$

where F_j denotes the nonlinear residual function associated with the jth equation, F_j' is its Jacobian (or derivative), and $y_{n+j}^{(i)}$ denotes the ith Newton iterate corresponding to the unknown solution value y_{n+j}.

Definition 6. The absolute error between two successive Newton iterates is defined by

$$
e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, \qquad j = 1,2,3. \tag{28}
$$

The maximum absolute error is defined as

$$
\mathrm{MAXE} = \max_{1\le t\le T} \left(\max_{1\le j\le N} |e_{n+j}^{(t)}| \right). \tag{29}
$$

where T denotes the total number of time steps and N is the number of unknown solution components.

The nonlinear functions F_1, F_2, and F_3 derived from the 3SBEBDF scheme in (11) are expressed as

$$
\begin{cases}
F_1 = y_{n+1} - \frac{19\rho+25}{16(3\rho+1)} y_{n+2} - \frac{4\rho+3}{40(3\rho+1)} y_{n+3} - \frac{3\rho}{6\rho+2} hf_n \\
\qquad + \frac{3}{6\rho+2} hf_{n+1} + \frac{3(\rho+1)}{4(3\rho+1)} hf_{n+2} - \varepsilon_1 = 0, \\
F_2 = -\frac{2(\rho-6)}{12\rho-1} y_{n+1} + y_{n+2} - \frac{167\rho+394}{50(12\rho-1)} y_{n+3} - \frac{12\rho}{12\rho-1} hf_{n+1} \\
\qquad + \frac{12}{12\rho-1} hf_{n+2} + \frac{6(\rho+2)}{5(12\rho-1)} hf_{n+3} - \varepsilon_2 = 0, \\
F_3 = -\frac{20(628\rho+935)}{3(936\rho-4973)} y_{n+1} + \frac{5(403\rho+1770)}{936\rho-4973} y_{n+2} + y_{n+3} \\
\qquad - \frac{2940\rho}{936\rho-4973} hf_{n+2} + \frac{2940}{936\rho-4973} hf_{n+3} \\
\qquad - \frac{40(\rho+5)}{936\rho-4973} hf_{n+4} - \varepsilon_3 = 0.
\end{cases} \tag{30}
$$

where $\varepsilon_i,\ i = 1,2,3$, are the back values defined by

$$
\begin{cases}
\varepsilon_1 = -\frac{1}{80} y_{n-2} + \frac{4\rho+1}{8(3\rho+1)} y_{n-1} + \frac{5\rho-3}{4(3\rho+1)} y_n, \\
\varepsilon_2 = \frac{4\rho+3}{25(12\rho-1)} y_{n-2} - \frac{3\rho+2}{2(12\rho-1)} y_{n-1} + \frac{4(2\rho+1)}{12\rho-1} y_n, \\
\varepsilon_3 = -\frac{167\rho+394}{3(936\rho-4973)} y_{n-2} + \frac{5(88\rho+195)}{936\rho-4973} y_{n-1} \\
\qquad - \frac{20(81\rho+160)}{936\rho-4973} y_n.
\end{cases} \tag{31}
$$

The Newton iteration in (27) can also be written in vector form as

$$\begin{pmatrix} y_{n+1}^{(i+1)} \\ y_{n+2}^{(i+1)} \\ y_{n+3}^{(i+1)} \end{pmatrix} = \begin{pmatrix} y_{n+1}^{(i)} \\ y_{n+2}^{(i)} \\ y_{n+3}^{(i)} \end{pmatrix} - \begin{pmatrix} \dfrac{\partial F_1^{(i)}}{\partial y_{n+1}^{(i)}} & \dfrac{\partial F_1^{(i)}}{\partial y_{n+2}^{(i)}} & \dfrac{\partial F_1^{(i)}}{\partial y_{n+3}^{(i)}} \\ \dfrac{\partial F_2^{(i)}}{\partial y_{n+1}^{(i)}} & \dfrac{\partial F_2^{(i)}}{\partial y_{n+2}^{(i)}} & \dfrac{\partial F_2^{(i)}}{\partial y_{n+3}^{(i)}} \\ \dfrac{\partial F_3^{(i)}}{\partial y_{n+1}^{(i)}} & \dfrac{\partial F_3^{(i)}}{\partial y_{n+2}^{(i)}} & \dfrac{\partial F_3^{(i)}}{\partial y_{n+3}^{(i)}} \end{pmatrix}^{-1} \begin{pmatrix} F_1^{(i)} \\ F_2^{(i)} \\ F_3^{(i)} \end{pmatrix}. \tag{32}$$

which leads to the linear system

$$\underbrace{\begin{pmatrix} \dfrac{\partial F_1^{(i)}}{\partial y_{n+1}^{(i)}} & \dfrac{\partial F_1^{(i)}}{\partial y_{n+2}^{(i)}} & \dfrac{\partial F_1^{(i)}}{\partial y_{n+3}^{(i)}} \\ \dfrac{\partial F_2^{(i)}}{\partial y_{n+1}^{(i)}} & \dfrac{\partial F_2^{(i)}}{\partial y_{n+2}^{(i)}} & \dfrac{\partial F_2^{(i)}}{\partial y_{n+3}^{(i)}} \\ \dfrac{\partial F_3^{(i)}}{\partial y_{n+1}^{(i)}} & \dfrac{\partial F_3^{(i)}}{\partial y_{n+2}^{(i)}} & \dfrac{\partial F_3^{(i)}}{\partial y_{n+3}^{(i)}} \end{pmatrix}}_{\text{Jacobian Matrix}} \begin{pmatrix} e_{n+1}^{(i+1)} \\ e_{n+2}^{(i+1)} \\ e_{n+3}^{(i+1)} \end{pmatrix} = - \begin{pmatrix} F_1^{(i)} \\ F_2^{(i)} \\ F_3^{(i)} \end{pmatrix}. \tag{33}$$

Let J denote the Jacobian matrix in (33). Then

$$J = \begin{pmatrix} 1 + \dfrac{3}{6\rho+2} h \dfrac{\partial F_{n+1}^{(i)}}{\partial y_{n+1}^{(i)}} & -\dfrac{19\rho+25}{16(3\rho+1)} + \dfrac{3(\rho+1)}{4(3\rho+1)} h \dfrac{\partial F_{n+1}^{(i)}}{\partial y_{n+2}^{(i)}} & -\dfrac{4\rho+3}{40(3\rho+1)} \dfrac{\partial F_{n+1}^{(i)}}{\partial y_{n+3}^{(i)}} \\ -\dfrac{2(\rho-6)}{12\rho-1} - \dfrac{12\rho}{12\rho-1} h \dfrac{\partial F_{n+2}^{(i)}}{\partial y_{n+1}^{(i)}} & 1 + \dfrac{12}{12\rho-1} h \dfrac{\partial F_{n+2}^{(i)}}{\partial y_{n+2}^{(i)}} & -\dfrac{167\rho+394}{50(12\rho-1)} + \dfrac{6(\rho+2)}{5(12\rho-1)} h \dfrac{\partial F_{n+2}^{(i)}}{\partial y_{n+3}^{(i)}} \\ -\dfrac{20(628\rho+935)}{3(936\rho-4973)} & \dfrac{5(403\rho+1770)}{936\rho-4973} - \dfrac{2940\rho}{936\rho-4973} h \dfrac{\partial F_{n+3}^{(i)}}{\partial y_{n+2}^{(i)}} & 1 + \dfrac{2940}{936\rho-4973} h \dfrac{\partial F_{n+3}^{(i)}}{\partial y_{n+3}^{(i)}} \end{pmatrix}.$$

Thus, the updated solution is obtained by

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} + e_{n+j}^{(i+1)}, \qquad j = 1, 2, 3. \tag{34}$$

A computer program written in the C programming language was developed to implement the proposed method. The implementation is carried out for the representative case $\rho = -\frac{1}{2}$. The same implementation procedure can be readily extended to any admissible value of ρ in the interval $(-1, 1)$.

The proposed 3SBEBDF method is implemented within a predictor–corrector framework. Specifically, the classical Euler method and the three-point block backward differentiation formula (3BBDF) [19] are employed as predictors, while the proposed 3SBEBDF scheme serves as the corrector.

6 Test Problems and Numerical Results

In this section, the numerical performance of the proposed 3SBEBDF method is compared with the fifth-order three-point diagonally implicit block backward differentiation formula (3DBBDF) and the sixth-order fully implicit three-point block extended backward differentiation formula (3BEBDF). To assess the efficiency and accuracy of these methods, the following stiff systems of ODEs are considered.

Problem 1. This problem is a highly stiff nonlinear system due to the presence of a large negative eigenvalue and the nonlinear coupling term y_2^2 [35, 36]:

$$\begin{aligned} y_1' &= -1002y_1 + 1000y_2^2, && y_1(0) = 1, \\ y_2' &= y_1 - y_2(1 + y_2), && y_2(0) = 1, \end{aligned} \qquad 0 \le x \le 20.$$

The exact solution is given by

$$\begin{aligned} y_1(x) &= e^{-2x}, \\ y_2(x) &= e^{-x}. \end{aligned}$$

The corresponding eigenvalue of the Jacobian matrix at the initial point is

$$\lambda = -1002.$$

Problem 2. Consider the following stiff initial-value problem taken from [6]:

$$\begin{aligned} y_1' &= -21y_1 + 19y_2 - 20y_3, && y_1(0) = 1, \\ y_2' &= 19y_1 - 21y_2 + 20y_3, && y_2(0) = 0, \quad 0 \le x \le 10. \\ y_3' &= 40y_1 - 40y_2 - 40y_3, && y_3(0) = -1, \end{aligned}$$

The exact solution of the system is given by

$$y_1(x) = \frac{1}{2}\left[e^{-2x} + e^{-40x}(\cos(40x) + \sin(40x))\right],$$
$$y_2(x) = \frac{1}{2}\left[e^{-2x} - e^{-40x}(\cos(40x) + \sin(40x))\right],$$
$$y_3(x) = 2e^{-40x}\left[-\frac{1}{2}\cos(40x) + \frac{1}{2}\sin(40x)\right].$$

The eigenvalues of the coefficient matrix associated with the system are

$$\lambda_1 = -2, \qquad \lambda_2 = -40 + 40i, \qquad \lambda_3 = -40 - 40i.$$

Problem 3. This problem is a moderately stiff linear system characterized by two negative eigenvalues with significantly different magnitudes. The coexistence of fast and slow decaying modes makes it a suitable benchmark for assessing the accuracy and stability of numerical methods for multiscale stiff problems [36]:

$$\begin{aligned} y_1' &= -y_1 + 95y_2, & y_1(0) &= 1, \\ y_2' &= -y_1 - 97y_2, & y_2(0) &= 1, \end{aligned} \qquad 0 \le x \le 10.$$

The exact solution is given by

$$y_1(x) = \frac{1}{47}\left(95e^{-2x} - 48e^{-96x}\right),$$
$$y_2(x) = \frac{1}{47}\left(48e^{-96x} - e^{-2x}\right).$$

The eigenvalues of the coefficient matrix are

$$\lambda_1 = -2, \qquad \lambda_2 = -96.$$

The following abbreviations are used in Tables 3–5, as defined in Table 2:

As shown in Table 3, the proposed 3SBEBDF method consistently produces the smallest maximum absolute error for all the tested step sizes. It also requires substantially less CPU time than the 3DBBDF and 3BEBDF methods. These results indicate that the proposed method provides improved accuracy and computational efficiency for Problem 1.

Table 4 shows that the proposed 3SBEBDF method yields the smallest MAXE for each tested step size. Moreover, the proposed method requires considerably less CPU time, particularly when the step size is reduced to 10^{-6}. Hence, the method demonstrates a favourable balance between accuracy and computational cost for Problem 2.

Table 2. Abbreviations used in the numerical comparisons.

Abbreviation	Description
h	Step size.
Method	Numerical method used.
TS	Total number of steps required to complete the integration.
MAXE	Maximum absolute error.
CPU time	Computation time in seconds.
3DBBDF	Three-point diagonally implicit block backward differentiation formula [20].
3BEBDF	Three-point block extended backward differentiation formula [28].
3SBEBDF	Proposed three-point superclass block extended backward differentiation formula.

Table 3. Comparison of the numerical performance of the proposed and existing methods for Problem 1.

Step size (h)	Method	MAXE	CPU time (s)
10^{-2}	3DBBDF	2.43175×10^{-2}	1.31600×10^{-2}
	3BEBDF	4.67864×10^{-3}	1.72100×10^{-3}
	3SBEBDF	$\mathbf{4.59639 \times 10^{-3}}$	$\mathbf{1.02100 \times 10^{-4}}$
10^{-4}	3DBBDF	1.10662×10^{-4}	3.18300×10^{-2}
	3BEBDF	7.01838×10^{-6}	2.27900×10^{-2}
	3SBEBDF	$\mathbf{4.28798 \times 10^{-6}}$	$\mathbf{2.21600 \times 10^{-3}}$
10^{-6}	3DBBDF	1.10755×10^{-6}	3.63200×10^{-1}
	3BEBDF	6.90874×10^{-8}	3.15800×10^{-1}
	3SBEBDF	$\mathbf{4.17943 \times 10^{-8}}$	$\mathbf{2.73100 \times 10^{-2}}$

Table 4. Comparison of the numerical performance of the proposed and existing methods for Problem 2.

Step size (h)	Method	MAXE	CPU time (s)
10^{-2}	3DBBDF	7.18748×10^{-2}	5.21500×10^{-2}
	3BEBDF	3.45336×10^{-2}	5.82100×10^{-3}
	3SBEBDF	$\mathbf{3.26853 \times 10^{-2}}$	$\mathbf{4.66400 \times 10^{-3}}$
10^{-4}	3DBBDF	1.10053×10^{-3}	5.73400×10^{-2}
	3BEBDF	9.19344×10^{-4}	5.26300×10^{-2}
	3SBEBDF	$\mathbf{8.50046 \times 10^{-4}}$	$\mathbf{2.11500 \times 10^{-2}}$
10^{-6}	3DBBDF	1.10361×10^{-5}	4.61500×10^{-1}
	3BEBDF	9.28143×10^{-6}	4.16800×10^{-1}
	3SBEBDF	$\mathbf{8.57965 \times 10^{-6}}$	$\mathbf{3.69400 \times 10^{-2}}$

Similar observations can be made from Table 5. The proposed 3SBEBDF method consistently achieves the smallest MAXE and the shortest CPU time for all the tested step sizes. The improvement is particularly

Table 5. Comparison of the numerical performance of the proposed and existing methods for Problem 3.

Step size (h)	Method	MAXE	CPU time (s)
10^{-2}	3DBBDF	3.41371×10^{-1}	5.72100×10^{-2}
	3BEBDF	3.06562×10^{-2}	5.24100×10^{-3}
	3SBEBDF	$\mathbf{2.88254 \times 10^{-2}}$	$\mathbf{3.52100 \times 10^{-3}}$
10^{-4}	3DBBDF	1.05259×10^{-2}	6.23100×10^{-2}
	3BEBDF	8.39129×10^{-3}	6.14700×10^{-2}
	3SBEBDF	$\mathbf{7.73325 \times 10^{-3}}$	$\mathbf{5.21600 \times 10^{-3}}$
10^{-6}	3DBBDF	1.08172×10^{-4}	4.62100×10^{-1}
	3BEBDF	9.09318×10^{-5}	4.21600×10^{-1}
	3SBEBDF	$\mathbf{8.40532 \times 10^{-5}}$	$\mathbf{3.31300 \times 10^{-2}}$

pronounced for smaller step sizes, indicating that the proposed scheme is effective in resolving both the rapidly and slowly decaying components of the moderately stiff system.

Overall, the numerical results in Tables 3–5 demonstrate that the proposed 3SBEBDF method compares favourably with the existing 3DBBDF and 3BEBDF methods. In all three problems, it produces smaller maximum absolute errors while requiring less computation time, thereby demonstrating superior numerical accuracy and computational efficiency.

To provide a visual comparison of the performance of the numerical methods, the efficiency diagram of $\log_{10}(\mathrm{MAXE})$ versus $\log_{10}(\mathrm{CPU\ time})$ is presented in Figures 4–6 for Problems 1, 2, and 3, respectively.

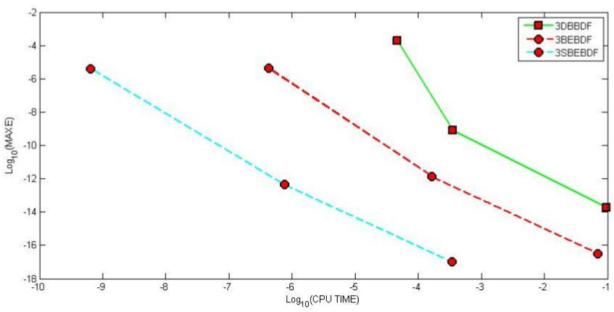

Figure 4. Efficiency Curve for Problem 1.

7 Discussion of Results

The numerical results presented in Tables 3–5 provide a comprehensive comparison of the performance of the 3DBBDF, 3BEBDF, and 3SBEBDF methods in terms of accuracy and computational efficiency. A consistent trend is observed across all cases as the step size h decreases from 10^{-2} to 10^{-6}, where the maximum error (MAXE) reduces significantly for all methods. This trend is consistent with numerical convergence as the step size decreases.

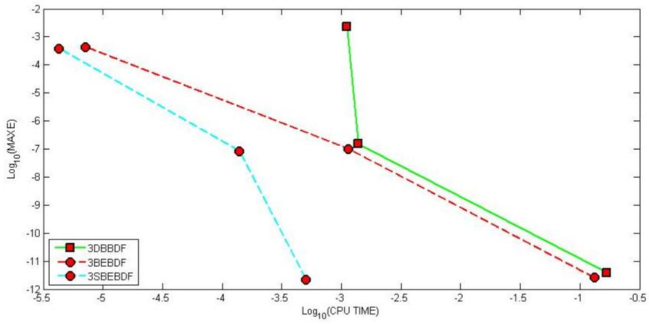

Figure 5. Efficiency Curve for Problem 2.

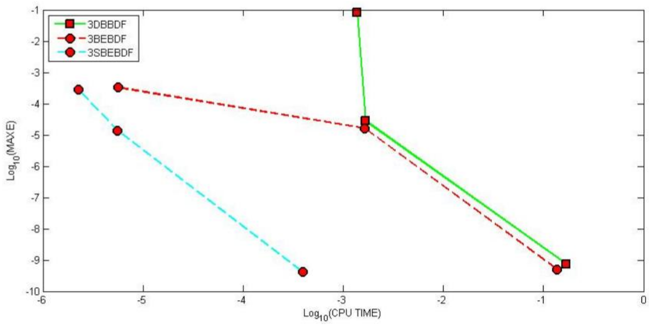

Figure 6. Efficiency Curve for Problem 3.

Despite the overall convergence of the methods, clear differences in performance are evident. The 3SBEBDF method consistently produces the smallest maximum error at all step sizes, indicating superior accuracy. The 3BEBDF method follows closely, while 3DBBDF records comparatively larger error values throughout. This pattern demonstrates the improved error control achieved by the 3SBEBDF scheme.

In terms of computational efficiency, the CPU time increases for all methods as the step size decreases, which is expected due to the higher computational workload associated with finer discretization. However, the 3SBEBDF method consistently requires significantly less CPU time compared to the other methods. The 3BEBDF method shows moderate computational performance, while 3DBBDF generally incurs the highest computational cost, particularly at smaller step sizes.

The combined results clearly indicate that the 3SBEBDF method achieves a better balance between accuracy and efficiency. Its ability to produce lower error values while maintaining reduced computational time suggests that the modifications incorporated into the scheme enhance both numerical stability and computational performance.

Overall, although all methods demonstrate satisfactory convergence properties, the 3SBEBDF scheme stands out as the most efficient and accurate approach for solving stiff initial value problems. Its consistent superiority across different step sizes highlights its robustness and suitability for practical scientific and engineering applications.

8 Conclusion

A three-point superclass block extended backward differentiation formula (3SBEBDF) with a stability control parameter ρ has been proposed for the numerical solution of stiff systems of ordinary differential equations. The method is constructed within a block multistep framework, allowing the simultaneous computation of multiple solution values while introducing a free parameter to improve the stability characteristics. The resulting nonlinear system is efficiently solved by Newton's iterative method.

Theoretical analysis has shown that the proposed method satisfies the consistency, zero-stability, and convergence requirements of linear multistep methods. Numerical experiments on several benchmark stiff problems demonstrate that all methods exhibit the expected convergence behaviour as the step size decreases. Compared with the existing 3DBBDF and 3BEBDF methods, the proposed 3SBEBDF scheme consistently achieves smaller maximum absolute errors while requiring less CPU time.

These results demonstrate that the introduction of the stability control parameter ρ significantly improves both the accuracy and computational efficiency of the proposed method. Therefore, the proposed 3SBEBDF scheme provides an accurate, efficient, and robust numerical technique for solving stiff systems of ordinary differential equations. Future work will focus on the optimal selection of the stability parameter ρ, the extension of the method to higher-order formulations, and its application to more general classes of stiff differential equations.

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Buhari Alhassan holds NCE (Mathematics/Chemistry) from Isah Kaita College of Education, Dutsinma, and BSc, MSc, and PhD degrees in Mathematics (Numerical Analysis) from Umaru Musa Yar'adua University, Katsina, Nigeria. He is a Lecturer in the Department of Mathematics and Statistics, Al-Qalam University, Katsina. His research interests include Numerical Analysis and Scientific Computing, with a focus on the development and implementation of numerical methods for ordinary and fractional differential equations. He has published several articles in peer-reviewed journals and is actively involved in teaching and research. (Email: buharialhassan@auk.edu.ng)

Hamisu Musa is a Professor of Numerical Analysis. He obtained his BSc and MSc degrees in Mathematics from Bayero University, Kano, Nigeria, and his PhD in Numerical Analysis from Universiti Putra Malaysia. His research interests include Numerical Analysis and Scientific Computing, with emphasis on numerical methods for differential equations. He has supervised numerous PhD and MSc students to successful completion and is actively engaged in teaching and research. (Email: hamisu.musa@umyu.edu.ng)

Abubakar Sadiq Saadu holds Diploma in Mathematics Education from Hassan Usman Katsina Polytechnic and BSc (Education) in Mathematics from Al-Qalam University Katsina. He is currently a postgraduate student in the MSc Mathematics programme at Al-Qalam University, Katsina, Nigeria. His research interests include Applied Mathematics and Numerical Analysis, with emphasis on numerical methods for ordinary differential equations. He is actively engaged in postgraduate research and the development of computational techniques. (Email: abubakarsadiqsaadu21@gmail.com)

Kabiru Abubakar Kantsi holds BSc in Mathematics from Kano State University of Science and Technology, Wudil, and MSc in Mathematics from Al-Qalam University, Katsina, Nigeria. His research interests include Applied Mathematics and Numerical Analysis, with emphasis on numerical methods for stiff ordinary differential equations. He is actively engaged in research and the development of computational techniques. (Email: kakantsi88@gmail.com)