RESEARCH ARTICLE



# New Iteration Method (NIM) for Solving Fourth-Order Two-Point Nonlinear Boundary Value Problems

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## Abstract

**In this study, the New Iteration Method (NIM) is employed to obtain approximate analytical solutions of nonlinear fourth-order ordinary differential equations of the general form**

$$\mathcal{L}(u) + \mathcal{N}(u) = g(x), \qquad x \in [a, b],$$

**subject to the two-point boundary conditions**

$$\mathcal{B}_1(u) = 0, \qquad \mathcal{B}_2(u) = 0,$$

**where $\mathcal{L}$ denotes a linear differential operator, $\mathcal{N}$ represents a nonlinear operator, and $g(x)$ is a given continuous function. The NIM framework constructs a rapidly convergent iterative sequence $\{u_n\}_{n=0}^{\infty}$ whose limit approximates the exact solution. The convergence behavior and numerical performance of the method are investigated through several test problems, including problems with polynomial, trigonometric, and exponential nonlinearities. The obtained solutions are compared with exact solutions and benchmark results available in the literature. The comparative analysis demonstrates that NIM yields highly accurate approximations, showing excellent agreement with exact or reference solutions. These results confirm that the proposed iterative scheme is mathematically consistent, computationally efficient, and robust for solving higher-order nonlinear boundary value problems.**

**Keywords**: new iteration method, nonlinear differential equations, two-point boundary value problems, fourth-order boundary value problems, iterative series solution.

## 1 Introduction

Many physical phenomena occurring in nature can be modeled using linear and nonlinear differential equations. Such mathematical formulations enable us to better understand, analyze, and interpret the underlying physical processes. The determination of analytical or approximate solutions to these models, particularly under initial and boundary conditions, has therefore become an important area of research. Differential equations have long played a fundamental role in mechanics and applied sciences, where engineers and scientists use them to investigate dynamically evolving systems.

For instance, fourth-order boundary value problems frequently arise in beam and plate theory. In this

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manuscript, we focus on semi-analytical solutions of nonlinear fourth-order two-point boundary value problems (BVPs).

$$u''''(x) = f(x, u(x), u'(x), u''(x), u'''(x)) + h(x),$$
$$\text{or} \quad [p(x)u''(x)]'' = f(x, u(x), u'(x), u''(x), u'''(x)) + h(x). \tag{1}$$

where $a \leq x \leq b$. Subject to the boundary conditions

$$u(a) = \alpha_1,\ u(b) = \alpha_2,\ u''(a) = \beta_1,\ u''(b) = \beta_2. \tag{2}$$

Here, $f$ is a continuous function on the interval $x \in [a, b]$, and the parameters $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$ are real constants. In addition to their mathematical significance, such systems frequently appear in various scientific and engineering applications. For example, they arise in modeling the elastic deflection of a beam rigidly fixed at one end and simply supported at the other [1], in the mathematical modelling of viscoelastic and inelastic flows [2], and in beam equations with nonlinear boundary conditions [3] and plate deflection problems [4]. In the following subsections, we discuss several important special cases.

### 1.1 Case 1: $f(x, u, u', u'', u''') = ku(x)$

For this choice of $f$, equation (1) reduces to

$$Du''''(x) + ku(x) = q, \tag{3}$$

which is a linear differential equation.

When $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, this equation describes the deflection of a uniformly loaded rectangular plate [1, 5]. Here, $u(x)$ represents the deflection of the plate of length $(b - a)$, $D$ denotes the flexural stiffness of the plate, $k$ represents the foundation modulus, and $q$ is the intensity of the applied load.

An analytical solution for constant $k$ and $q$ is available in [5]. However, when $k$ and $q$ are variable functions, the problem becomes significantly more challenging.

### 1.2 Case 2: $f(x, u, u', u'', u''') = g(x)u(x)$

In this case, equation (1) becomes

$$[p(x)u''(x)]'' = g(x)u(x) + h(x), \quad a \leq x \leq b, \tag{4}$$

or, for $p(x) = 1$,

$$u''''(x) = g(x)u(x) + h(x), \quad a \leq x \leq b, \tag{5}$$

subject to the boundary conditions given in (2).

Such equations are generally not solvable in closed form. Consequently, researchers have developed various numerical techniques to approximate their solutions. Among these, finite difference methods (FDM) are widely used for approximating the solution over discrete grid points [1, 6–9].

In finite difference methods, both the approximate solution and the associated error are computed iteratively until a desired convergence criterion is satisfied. Some schemes converge rapidly, while others exhibit slower convergence.

Usmani and Marsden [6] established second-order convergence, which was later improved to sixth-order accuracy $O(h^6)$ in [1]. Jain et al. [7] extended this methodology to achieve fourth- and sixth-order accuracy. Furthermore, Usmani [8] proved the existence of a unique solution for the linear case $g(x) = 1$ under suitable conditions.

Other numerical approaches have also been proposed, including the Differential Transform Method (DTM) [10], Adomian Decomposition Method (ADM) [10–12], Galerkin methods with Legendre polynomials [13], cubic spline methods, Ritz methods, multiderivative methods, finite element methods, sinc-Galerkin methods [14], piecewise cubic interpolation [15], quintic and sextic spline approximations, Galerkin methods with quintic B-splines [16], Lidstone-collocation methods [17], numerical comparisons of methods for fourth-order boundary value problems [18], and reliable decomposition-based algorithms for fourth-order boundary value problems [19].

### 1.3 Case 3: $f(x, u, u', u''),\ 0 < x < 1$

The deflection of an elastic beam in equilibrium with both ends simply supported can be described by the following fourth-order boundary value problem, for which positive solutions and their properties have been studied in [20]:

$$\frac{d^4u(x)}{dx^4} = f\big(x, u(x), u'(x), u''(x)\big), \qquad 0 < x < 1, \tag{6}$$

subject to the simply supported boundary conditions

$$u(0) = u(1) = 0, \qquad u''(0) = u''(1) = 0. \tag{7}$$

Here, $u(x)$ denotes the transverse deflection of the beam. For an Euler–Bernoulli beam, the bending moment is proportional to the second derivative $u''(x)$. Therefore, the conditions $u(0) = u(1) = 0$ represent zero transverse displacement at the two supports, while $u''(0) = u''(1) = 0$ correspond to zero bending moment at the simply supported ends.

Due to their importance in beam theory and nonlinear mechanics, fourth-order boundary value problems of this type have been extensively studied, particularly with respect to the existence, uniqueness, and multiplicity of solutions [21–25].

## 2 Preliminaries

Throughout this section, let $X = C([0,b])$ denote the Banach space of continuous real-valued functions on $[0,b]$, equipped with the supremum norm

$$\|u\|_\infty = \sup_{x\in[0,b]} |u(x)|.$$

**Lemma 2.1** (Repeated–Integral Kernel Representation). *Let $F \in C([0,x])$ and let $n \geq 1$ be an integer. Then*

$$\underbrace{\int_0^x \int_0^{\xi_{n-1}} \cdots \int_0^{\xi_1}}_{n \text{ times}} F(\xi_0)\, d\xi_0\, d\xi_1 \cdots d\xi_{n-1} = \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} F(s)\, ds.$$

*Proof.* The result follows by induction on $n$, together with repeated applications of Fubini's theorem. □

**Theorem 2.2** (Existence and Uniqueness of Solution). *Consider the nonlinear Volterra integral equation*

$$u(x) = f_1(x) + \int_0^x K(x,s)\, F(u(s))\, ds, \quad x \in [0,b], \tag{8}$$

*where:*

- $f_1 \in C([0,b])$,
- *$K$ is measurable on $\{0 \leq s \leq x \leq b\}$ and satisfies*

$$M := \sup_{x\in[0,b]} \int_0^x |K(x,s)|\, ds < \infty,$$

- *$F : X \to X$ is locally Lipschitz on the closed ball $B_R = \{u \in X : \|u\|_\infty \leq R\}$ with Lipschitz constant $L_R$.*

*Assume that*

$$\|f_1\|_\infty + M \sup_{\|u\|_\infty \leq R} \|F(u)\|_\infty \leq R, \tag{9}$$

*and*

$$q := ML_R < 1. \tag{10}$$

*Then equation (8) admits a unique solution $u^* \in B_R$. Moreover, the Picard iteration*

$$u_{k+1}(x) = f_1(x) + \int_0^x K(x,s)\, F(u_k(s))\, ds$$

*converges geometrically to $u^*$, i.e.,*

$$\|u_k - u^*\|_\infty \leq q^k \|u_0 - u^*\|_\infty.$$

*Proof.* Define the operator $T : X \to X$ by

$$(Tu)(x) = f_1(x) + \int_0^x K(x,s) F(u(s))\, ds.$$

For $u \in B_R$, using condition (9), we obtain

$$\|Tu\|_\infty \leq \|f_1\|_\infty + M\|F(u)\|_\infty \leq R,$$

which implies that $T(B_R) \subset B_R$.

For $u, v \in B_R$, we have

$$\begin{aligned}\|Tu - Tv\|_\infty &\leq M\|F(u) - F(v)\|_\infty \\ &\leq ML_R\|u - v\|_\infty \\ &= q\|u - v\|_\infty.\end{aligned}$$

Thus, $T$ is a contraction mapping on $B_R$. By the Banach Fixed Point Theorem, there exists a unique fixed point $u^* \in B_R$ such that $Tu^* = u^*$. The geometric convergence follows immediately. □

**Theorem 2.3** (Convergence of NIM). *Let $\mathcal{N} = \mathcal{L}^{-1} \circ F$ and define the NIM sequence*

$$u_0 = f_1, \qquad u_{k+1} = \mathcal{N}\Big(\sum_{j=0}^{k} u_j\Big) - \mathcal{N}\Big(\sum_{j=0}^{k-1} u_j\Big),$$

*with the convention $\sum_{j=0}^{-1} u_j \equiv 0$. Let $S_k = \sum_{j=0}^{k} u_j$. Then $S_k$ coincides with the Picard iteration associated with $T$, i.e.,*

$$S_{k+1} = T(S_k),$$

*and consequently $S_k \to u^*$ geometrically.*

*Proof.* From the recurrence relation, we obtain

$$\begin{aligned}S_{k+1} &= S_k + \mathcal{N}(S_k) - \mathcal{N}(S_{k-1}) \\ &= f_1 + \mathcal{L}^{-1}[F(S_k)] \\ &= T(S_k).\end{aligned}$$

Thus, the NIM scheme is equivalent to the Picard iteration. The convergence follows directly from Theorem 2.2. □

**Corollary 2.4** (Fourth–Order Volterra Kernel). *For*

$$\mathcal{L}^{-1}\phi(x) = \int_0^x \frac{(x-s)^3}{6}\phi(s)\,ds,$$

*we have*

$$M = \sup_{x\in[0,b]} \int_0^x \frac{(x-s)^3}{6}\,ds = \frac{b^4}{24}.$$

*If $F(u) = u^2$, then the Lipschitz constant on $B_R$ is $L_R = 2R$, and therefore*

$$q = \frac{b^4 R}{12}.$$

*Hence, NIM converges provided that*

$$\|f_1\|_\infty + \frac{b^4}{24}R^2 \le R \quad \text{and} \quad \frac{b^4 R}{12} < 1.$$

*Remark* 2.5 (Stepwise continuation). If the global interval $[0, b]$ does not satisfy the condition $q < 1$, the method can be applied on smaller subintervals and then extended stepwise over the entire domain. This continuation strategy preserves the contraction property locally and thereby ensures convergence.

*Remark* 2.6 (Stepwise continuation for fourth-order BVPs). Consider the fourth-order boundary value problem written in Volterra form

$$u(x) = f_1(x) + \int_0^x \frac{(x-s)^3}{6} F(u(s))\,ds, \quad x \in [0, b].$$

From the fourth-order kernel estimate, the contraction constant is

$$q = \frac{b^4 R}{12}.$$

If $q \ge 1$, the interval $[0, b]$ can be partitioned as

$$0 = x_0 < x_1 < \cdots < x_m = b,$$

so that on each subinterval $[x_{i-1}, x_i]$ the local contraction constant

$$q_i = \frac{(x_i - x_{i-1})^4 R}{12}$$

satisfies $q_i < 1$.

Applying NIM sequentially on each subinterval preserves the Volterra structure and ensures local convergence. The solution can therefore be extended over the entire interval $[0, b]$ by continuation.

**Theorem 2.7** (Local Existence, Uniqueness, and Maximal Continuation). *Assume the hypotheses of Theorem 2.2, except that the global contraction condition $ML_R < 1$ may fail on $[0, b]$. Suppose that $F : X \to X$ is locally Lipschitz.*

*Then there exists $b_0 \in (0, b]$ such that the nonlinear Volterra equation*

$$u(x) = f_1(x) + \int_0^x K(x,s)\,F(u(s))\,ds, \quad x \in [0, b_0],$$

*admits a unique solution $u \in C([0, b_0])$.*

*Moreover, there exists a maximal interval of existence $[0, b_{\max}) \subseteq [0, b]$ such that the solution is uniquely defined on $[0, b_{\max})$ and satisfies the alternative:*

1. *either $b_{\max} = b$, in which case the solution exists globally on $[0, b]$,*
2. *or $b_{\max} < b$ and*

$$\limsup_{x\to b_{\max}^-} \|u\|_{C([0,x])} = +\infty.$$

*Thus, continuation can fail only if the solution becomes unbounded.*

*Proof.* Define

$$M(b_0) := \sup_{x\in[0,b_0]} \int_0^x |K(x,s)|\,ds.$$

Since $K$ is integrable on $\{0 \le s \le x \le b\}$, we have $M(b_0) \to 0$ as $b_0 \to 0^+$. Hence $b_0 > 0$ can be chosen such that

$$M(b_0)L_R < 1.$$

Define

$$(Tu)(x) = f_1(x) + \int_0^x K(x,s)F(u(s))\,ds.$$

Then $T$ is a contraction on $C([0, b_0])$, and the Banach Fixed Point Theorem yields a unique solution on $[0, b_0]$.

Define

$$b_{\max} := \sup\{\beta \in (0, b] : \text{a unique solution exists on } [0, \beta)\}.$$

If $b_{\max} < b$ and the solution remains bounded, the same contraction argument extends it beyond $b_{\max}$, contradicting maximality. Hence blow-up in norm is the only obstruction. □

**Theorem 2.8** (Stability and Continuous Dependence). *Let $u$ and $\tilde{u}$ be solutions of*

$$u(x) = f_1(x) + \int_0^x K(x,s)\, F(u(s))\, ds,$$
$$\tilde{u}(x) = \tilde{f}_1(x) + \int_0^x K(x,s)\, F(\tilde{u}(s))\, ds,$$

*where $F$ is Lipschitz continuous with constant $L$ and $ML < 1$.*

*Then*

$$\|u - \tilde{u}\|_\infty \le \frac{1}{1-ML}\, \|f_1 - \tilde{f}_1\|_\infty.$$

*In particular, the solution depends continuously on the data.*

*Proof.* Subtracting the equations and taking the supremum norm,

$$\|u - \tilde{u}\|_\infty \le \|f_1 - \tilde{f}_1\|_\infty + ML\|u - \tilde{u}\|_\infty.$$

Rearranging yields the desired estimate. □

**Theorem 2.9** (A Priori Bound). *Consider*

$$u(x) = f_1(x) + \int_0^x K(x,s)\, F(u(s))\, ds,$$

*where*

$$|F(u)| \le a + b_1|u|, \qquad a, b_1 \ge 0,$$

*and*

$$M := \sup_{x\in[0,b]} \int_0^x |K(x,s)|\, ds < \infty, \qquad Mb_1 < 1.$$

*Then*

$$\|u\|_\infty \le \frac{\|f_1\|_\infty + Ma}{1 - Mb_1}.$$

*Proof.* Taking the supremum in the integral equation gives

$$\|u\|_\infty \le \|f_1\|_\infty + M(a + b_1\|u\|_\infty).$$

Rearranging yields the estimate. □

**Theorem 2.10** (Error Estimate for NIM). *Let $u^*$ be the unique fixed point of*

$$T(u) = f_1 + \mathcal{L}^{-1}[F(u)]$$

*with contraction constant $q < 1$. Let $\{S_k\}$ satisfy $S_{k+1} = T(S_k)$.*

*Then*

$$\|S_k - u^*\|_\infty \le q^k \|S_0 - u^*\|_\infty,$$

*and*

$$\|S_k - u^*\|_\infty \le \frac{q^k}{1-q}\|S_1 - S_0\|_\infty.$$

*Proof.* The first estimate follows directly from the contraction property:

$$\|S_{k+1} - u^*\|_\infty \le q\|S_k - u^*\|_\infty.$$

For the second estimate, use

$$S_k - u^* = \sum_{j=k}^{\infty} (S_j - S_{j+1})$$

and the geometric decay

$$\|S_{j+1} - S_j\|_\infty \le q^j \|S_1 - S_0\|_\infty.$$

Summing the geometric series yields the result. □

**Corollary 2.11** (Stopping Criterion). *If*

$$\|S_{k+1} - S_k\|_\infty < \varepsilon(1-q),$$

*then*

$$\|S_k - u^*\|_\infty < \varepsilon.$$

## 3 New Iteration Method (NIM) for the General Fourth–Order BVP

The New Iteration Method (NIM) was originally proposed by Daftardar-Gejji and Jafari [26] for solving nonlinear functional equations, and subsequently extended to partial differential equations by Bhalekar and Daftardar-Gejji [27]. In the present work, we apply NIM to fourth-order two-point nonlinear boundary value problems.

Consider the fourth-order boundary value problem

$$\begin{aligned} & u^{(4)}(x) = f(x, u, u', u'', u''') + h(x), \\ \text{or} \quad & [p(x)u''(x)]'' = f(x, u, u', u'', u''') + h(x), \end{aligned} \tag{11}$$

for $a \le x \le b$, subject to the boundary conditions

$$u(a) = \alpha_1, \quad u(b) = \alpha_2, \quad u''(a) = \beta_1, \quad u''(b) = \beta_2. \tag{12}$$

**Linear operator and Green's representation.**

Define the linear operator

$$\mathcal{L}u := \begin{cases} u^{(4)}, \\ \left(p(x)u''\right)''. \end{cases}$$

Let $G(x,s)$ denote the Green's function associated with $\mathcal{L}$ on $[a,b]$, satisfying the boundary conditions

(12). Then the solution of (11)–(12) admits the representation

$$u(x) = \phi(x) + \int_a^b G(x,s)\left(f(s,u,u',u'',u''') + h(s)\right) ds, \tag{13}$$

where $\phi$ solves the homogeneous problem $\mathcal{L}\phi = 0$ together with the boundary conditions (12). Equivalently,

$$\phi(x) = \alpha_1 \ell_1(x) + \alpha_2 \ell_2(x) + \beta_1 \ell_3(x) + \beta_2 \ell_4(x), \tag{14}$$

where the basis functions $\{\ell_i\}_{i=1}^4$ satisfy $\mathcal{L}\ell_i = 0$ and enforce the corresponding Kronecker boundary data.

**NIM Fixed–Point Map and Increment Form.**

Define the nonlinear Fredholm operator

$$(\mathcal{N}u)(x) := \int_a^b G(x,s)\left(f(s,u,u',u'',u''') + h(s)\right) ds.$$

Then (13) can be written in fixed-point form as

$$S = \mathcal{T}(S) := \phi + \mathcal{N}(S). \tag{15}$$

The New Iteration Method (NIM) generates the sequence $\{S_k\}_{k\geq 0}$ by

$$S_0 := \phi, \qquad S_{k+1} = \phi + \mathcal{N}(S_k), \quad k = 0,1,2,\ldots \tag{16}$$

and defines the increments

$$u_0 := S_0 = \phi,\ u_{k+1} = S_{k+1} - S_k = \mathcal{N}(S_k) - \mathcal{N}(S_{k-1}),\ k \geq 0, \tag{17}$$

with the convention $\mathcal{N}(S_{-1}) \equiv 0$.

Consequently, the NIM series representation is

$$u(x) = \sum_{k=0}^{\infty} u_k(x), \tag{18}$$

where

$$u_{k+1}(x) = \int_a^b G(x,s)\left[F(S_k)(s) - F(S_{k-1})(s)\right] ds,$$

and

$$F(S) := f(\cdot, S, S', S'', S''') + h(\cdot).$$

**Computational remarks.**

- The derivatives $S_k', S_k'', S_k'''$ appearing in $f$ are computed via symbolic or automatic differentiation of $S_k$.
- For $p \equiv 1$, a Volterra construction may be preferable. Integrating (11) four times from $a$ yields

$$u(x) = \tilde{\phi}(x; C_1, C_2) + \int_a^x \frac{(x-s)^3}{6}\left(f(s,u,u',u'',u''') + h(s)\right) ds, \tag{19}$$

  where the constants $(C_1, C_2)$ (e.g., $u'(a), u'''(a)$) are determined *a posteriori* from the terminal conditions in (12). The same NIM formulas (16)–(18) apply with the Volterra kernel $K(x,s) = (x-s)^3/6$.
- In practice, one initializes $S_0 = \phi$ and iterates (16) until

$$\|S_{k+1} - S_k\|_\infty$$

  falls below a prescribed tolerance. The increments $u_k$ are then obtained from (17).

**Convergence (summary).** Assume that the Green's function satisfies

$$M := \sup_{x\in[a,b]} \int_a^b |G(x,s)|\, ds < \infty,$$

and that $f$ is locally Lipschitz continuous in $(u,u',u'',u''')$ on the ball $\|S\|_\infty \leq R$ with Lipschitz constant $L_R$. Then the mapping

$$S \mapsto \phi + \mathcal{N}(S)$$

is a contraction whenever $q := ML_R < 1$. In this case, $S_k \to u^*$ geometrically, and the series $\sum_{k\geq 0} u_k$ converges absolutely to the unique solution $u^*$ of (11)–(12).

**Theorem 3.1** (Convergence for the Nonlinear Fredholm Case). *Consider the nonlinear Fredholm equation*

$$u(x) = \phi(x) + \int_a^b G(x,s)\, F\big(s, u(s), u'(s), u''(s), u'''(s)\big)\, ds, \tag{20}$$

*where $x \in [a,b]$ and,*

- $\phi \in C([a,b])$,
- $G \in C([a,b] \times [a,b])$,
- *$F$ is locally Lipschitz continuous in $(u,u',u'',u''')$ on the closed ball*

$$B_R := \{u \in C^3([a,b]) : \|u\|_\infty \leq R\},$$

  *with Lipschitz constant $L_R$.*

*Define*

$$M := \sup_{x\in[a,b]} \int_a^b |G(x,s)|\, ds.$$

*If the contraction condition*

$$q := ML_R < 1$$

*holds and*

$$\|\phi\|_\infty + M \sup_{\|u\|_\infty \le R} \|F(\cdot, u, u', u'', u''')\|_\infty \le R,$$

*then:*

1. *Equation* (20) *admits a unique solution* $u^* \in B_R$*.*
2. *The NIM sequence*

$$S_{k+1} = \phi + \mathcal{N}(S_k)$$

*converges geometrically to* $u^*$*:*

$$\|S_k - u^*\|_\infty \le q^k \|S_0 - u^*\|_\infty.$$

3. *The NIM series*

$$u = \sum_{k=0}^{\infty} u_k$$

*converges absolutely and uniformly on* $[a,b]$*.*

*Proof.* Define the operator

$$(Tu)(x) = \phi(x) + \int_a^b G(x,s)\, F\big(s, u(s), u'(s), u''(s), u'''(s)\big)\, ds. \tag{21}$$

For $u \in B_R$,

$$\|Tu\|_\infty \le \|\phi\|_\infty + M\|F(\cdot, u, u', u'', u''')\|_\infty \le R,$$

so $T(B_R) \subset B_R$.

For $u, v \in B_R$,

$$\|Tu - Tv\|_\infty \le ML_R\|u - v\|_\infty = q\|u - v\|_\infty.$$

Thus $T$ is a contraction on $B_R$. The Banach Fixed Point Theorem yields a unique fixed point $u^*$ and geometric convergence of $S_k = T^k(S_0)$.

Uniform absolute convergence of the NIM series follows from the geometric decay

$$\|u_{k+1}\|_\infty = \|S_{k+1} - S_k\|_\infty \le q^k \|S_1 - S_0\|_\infty.$$

□

**Theorem 3.2** (Existence via Schauder Fixed-Point Theorem)**.** *Consider the nonlinear Fredholm equation*

$$u(x) = \phi(x) + \int_a^b G(x,s)\, F\big(s, u(s), u'(s), u''(s), u'''(s)\big)\, ds, \quad x \in [a,b], \tag{22}$$

*where:*

- $\phi \in C([a,b])$*,*
- $G \in C([a,b] \times [a,b])$*,*
- $F$ *is continuous in all variables,*
- $F$ *satisfies the growth condition*

$$|F(x, u, u', u'', u''')| \le a_0 + b_0|u|, \qquad a_0, b_0 \ge 0.$$

*Define*

$$M := \sup_{x\in[a,b]} \int_a^b |G(x,s)|\, ds.$$

*Assume that*

$$\|\phi\|_\infty + M(a_0 + b_0 R) \le R$$

*for some* $R > 0$*.*

*Then equation* (22) *admits at least one solution*

$$u \in C([a,b])$$

*with* $\|u\|_\infty \le R$*.*

*Proof.* Define the operator

$$(Tu)(x) = \phi(x) + \int_a^b G(x,s)\, F\big(s, u(s), u'(s), u''(s), u'''(s)\big)\, ds.$$

Let

$$B_R = \{u \in C([a,b]) : \|u\|_\infty \le R\}.$$

From the growth condition,

$$\|Tu\|_\infty \le \|\phi\|_\infty + M(a_0 + b_0\|u\|_\infty) \le R,$$

so $T(B_R) \subset B_R$.

Since $G$ is continuous on the compact set $[a,b] \times [a,b]$ and $F$ is continuous, the operator $T$ is continuous and compact (Arzelà–Ascoli theorem).

Thus $T$ maps the closed, bounded, convex set $B_R$ into itself and is compact. By the Schauder Fixed-Point Theorem, $T$ admits at least one fixed point in $B_R$. □

## 4 Application of Proposed Method

**Example 4.1.** [28–31]

*We consider the nonlinear fourth–order boundary value problem*

$$u^{(4)}(x) = 1+u(x)^2, \qquad u(0) = u'(0) = u(2) = u'(2) = 0. \tag{23}$$

***Integral Formulation*** *Integrating (23) once and using $u'(0) = 0$, let $u'''(0) = \alpha$. Then*

$$u'''(x) = \alpha + x + \int_0^x u(s)^2\,ds. \tag{24}$$

*Integrating again and letting $u''(0) = \beta$, we obtain*

$$u''(x) = \beta + \alpha x + \frac{x^2}{2} + \int_0^x \int_0^t u(s)^2\,ds\,dt. \tag{25}$$

*A third integration yields*

$$u'(x) = \beta x + \frac{\alpha x^2}{2} + \frac{x^3}{6} + \int_0^x \int_0^t \int_0^\tau u(s)^2\,ds\,d\tau\,dt. \tag{26}$$

*Finally, integrating once more and using $u(0) = 0$, we obtain*

$$u(x) = \frac{\beta x^2}{2} + \frac{\alpha x^3}{6} + \frac{x^4}{24} + \int_0^x \frac{(x-s)^3}{6}\,u(s)^2\,ds. \tag{27}$$

*Define*

$$\phi(x) = \frac{\beta x^2}{2} + \frac{\alpha x^3}{6} + \frac{x^4}{24}. \tag{28}$$

*Then the equation may be written as*

$$u(x) = \phi(x) + \mathcal{L}^{-1}[u^2](x), \tag{29}$$

*where*

$$\mathcal{L}^{-1}[\psi](x) = \int_0^x \frac{(x-s)^3}{6}\,\psi(s)\,ds. \tag{30}$$

***NIM Construction*** *The NIM series solution is defined as*

$$u(x) = \sum_{k=0}^{\infty} u_k(x), \qquad u_0(x) = \phi(x). \tag{31}$$

*Thus,*

$$u_0(x) = \frac{x^4}{24} + \frac{\alpha x^3}{6} + \frac{\beta x^2}{2}. \tag{32}$$

***First Correction Term***

$$u_1(x) = \mathcal{L}^{-1}(u_0^2)(x) = \int_0^x \frac{(x-s)^3}{6}\,(u_0(s))^2\,ds. \tag{33}$$

*Using*

$$\int_0^x (x-s)^3 s^k\,ds = 6\,\frac{k!}{(k+4)!}\,x^{k+4}, \tag{34}$$

*we obtain*

$$u_1(x) = \frac{\beta^2}{6720}x^8 + \frac{\alpha\beta}{18144}x^9 + \left(\frac{\alpha^2}{181440} + \frac{\beta}{120960}\right)x^{10} + \frac{\alpha}{570240}x^{11} + \frac{1}{6842880}x^{12}. \tag{35}$$

***Second Correction Term***

$$u_2 = \mathcal{L}^{-1}\big[(u_0+u_1)^2\big] - \mathcal{L}^{-1}(u_0^2). \tag{36}$$

$$\begin{aligned} u_2(x) = {} & \frac{\beta^3}{161441280}x^{14} + \frac{19\alpha\beta^2}{5943974400}x^{15} \\ & + \frac{\beta(52\alpha^2+45\beta)}{95103590400}x^{16} \\ & + \frac{22\alpha^3+109\alpha\beta}{684014284800}x^{17} + \frac{5\alpha^2+4\beta}{351778775040}x^{18} \\ & + \frac{\alpha}{477414051840}x^{19} + \frac{891\beta^4+490}{4678657708032000}x^{20} \\ & + \frac{\alpha\beta^3}{8756845977600}x^{21} + \frac{\beta^2(154\alpha^2+81\beta)}{5779518345216000}x^{22} \\ & + \frac{\alpha\beta(220\alpha^2+519\beta)}{76958849544192000}x^{23} \\ & + \frac{22\alpha^4+206\alpha^2\beta+81\beta^2}{184701238906060800}x^{24} \\ & + \frac{3\alpha^3+7\alpha\beta}{47117662986240000}x^{25} \\ & + \frac{64\alpha^2+33\beta}{4900236950568960000}x^{26} \\ & + \frac{\alpha}{821778867486720000}x^{27} \\ & + \frac{1}{23009808289628160000}x^{28}, \end{aligned} \tag{37}$$

*and*

$$u_3(x) = \frac{\beta^4}{18772392038400}x^{20} + \frac{23\alpha\beta^3}{3092370149376000}x^{21} + \frac{44\alpha^2\beta^2 + 27\beta^3}{110196149782118400}x^{22} + \frac{286\alpha^3\beta + 711\alpha\beta^2}{11338603832844288000}x^{23} + \frac{6864\alpha^4 + 72280\alpha^2\beta + 32525\beta^2}{1632758951929577472000}x^{24} + \cdots \quad (38)$$

*The explicit expression for $u_3(x)$ is lengthy and is computed symbolically using MATLAB. Higher-order corrections $u_3, u_4, \ldots$ are obtained in the same manner.*

***Determination of Constants*** *The constants*

$$\alpha = u'''(0), \qquad \beta = u''(0), \quad (39)$$

*are determined by enforcing the terminal conditions*

$$u(2) = 0, \qquad u'(2) = 0. \quad (40)$$

*After truncating the NIM series at a desired order (e.g., $u \approx u_0 + u_1 + u_2$), substitution of $x = 2$ into $u(x)$ and $u'(x)$ produces a nonlinear algebraic system for $\alpha$ and $\beta$, which is solved numerically.*

**Convergence Behaviour of the Proposed NIM**

Tables 1–2 summarize the numerical performance of the New Iteration Method (NIM) for the considered fourth–order nonlinear boundary value problem.

In the implementation, the truncated partial sum

$$S_m(x) = \sum_{k=0}^{m} u_k(x) \quad (41)$$

is first constructed. The unknown boundary parameters $\alpha$ and $\beta$ are then determined by enforcing

$$S_m(2) = 0, \qquad S'_m(2) = 0. \quad (42)$$

This yields a nonlinear algebraic system in $\alpha$ and $\beta$, which is solved numerically.

**Table 1.** Computed boundary parameters $\alpha = u'''(0)$ and $\beta = u''(0)$ for successive NIM partial sums.

| Partial Sum $S_m$ | $\alpha$ | $\beta$ |
|---|---|---|
| $S_1$ | $-1.000705$ | 0.333654 |
| $S_2$ | $-1.000707$ | 0.333655 |
| $S_3$ | $-1.000707$ | 0.333655 |

**Table 2.** Convergence of successive NIM partial sums on $[0, 1]$.

| $x$ | $S_1$ | $S_2$ | $S_3$ | $\lvert S_2 - S_1\rvert$ | $\lvert S_3 - S_2\rvert$ |
|---|---|---|---|---|---|
| 0.00 | 0.000000 | 0.000000 | 0.000000 | 0 | 0 |
| 0.07 | 0.000791 | 0.000791 | 0.000791 | 0 | 0 |
| 0.14 | 0.002936 | 0.002936 | 0.002936 | 0 | 0 |
| 0.21 | 0.006107 | 0.006107 | 0.006107 | $8.7\times10^{-19}$ | 0 |
| 0.29 | 0.010006 | 0.010006 | 0.010006 | $3.5\times10^{-18}$ | 0 |
| 0.36 | 0.014359 | 0.014359 | 0.014359 | $7.1\times10^{-17}$ | 0 |
| 0.43 | 0.018919 | 0.018919 | 0.018919 | $8.1\times10^{-16}$ | 0 |
| 0.50 | 0.023463 | 0.023463 | 0.023463 | $6.2\times10^{-15}$ | 0 |
| 0.57 | 0.027797 | 0.027797 | 0.027797 | $3.5\times10^{-14}$ | 0 |
| 0.64 | 0.031751 | 0.031751 | 0.031751 | $1.6\times10^{-13}$ | 0 |
| 0.71 | 0.035181 | 0.035181 | 0.035181 | $6.2\times10^{-13}$ | 0 |
| 0.79 | 0.037971 | 0.037971 | 0.037971 | $2.1\times10^{-12}$ | 0 |
| 0.86 | 0.040029 | 0.040029 | 0.040029 | $6.1\times10^{-12}$ | 0 |
| 0.93 | 0.041290 | 0.041290 | 0.041290 | $1.6\times10^{-11}$ | $1.4\times10^{-17}$ |
| 1.00 | 0.041714 | 0.041714 | 0.041714 | $4.0\times10^{-11}$ | $6.3\times10^{-17}$ |

Table 1 shows that the computed values of $\alpha$ and $\beta$ stabilize rapidly as the truncation order increases. Only a few NIM terms are sufficient to approximate the boundary parameters accurately. Table 2 demonstrates that the successive partial sums $S_1$, $S_2$, and $S_3$ practically coincide. The differences decrease to the order of $10^{-11}$–$10^{-17}$, which is close to machine precision. These results confirm that the proposed method achieves high accuracy with only a small number of series components, demonstrating both computational efficiency and numerical stability.

**Example 4.2.** *Consider the nonlinear fourth–order boundary value problem*

$$u^{(4)}(x) = \sin x + \sin^2 x - \left(u''(x)\right)^2, \quad (43)$$

*subject to the boundary conditions*

$$u(0) = 0, \;\; u'(0) = 1, \;\; u(1) = \sin(1), \;\; u'(1) = \cos(1). \quad (44)$$

***Integral formulation.*** *Integrating (43) from 0 to $x$, we obtain*

$$u^{(3)}(x) = \alpha + \int_0^x \sin s\, ds + \int_0^x \sin^2 s\, ds - \int_0^x \left(u''(s)\right)^2 ds, \quad (45)$$

*where $\alpha = u^{(3)}(0)$.*

*Evaluating the linear integrals,*

$$\int_0^x \sin s\, ds = 1 - \cos x, \qquad \int_0^x \sin^2 s\, ds = \frac{x}{2} - \frac{\sin 2x}{4}.$$

*Hence,*

$$u^{(3)}(x) = \alpha + 1 - \cos x + \frac{x}{2} - \frac{\sin 2x}{4} - \int_0^x \left(u''(s)\right)^2 ds. \quad (46)$$

*Integrating again gives*

$$u''(x) = \beta + \alpha x + x - \sin x + \frac{x^2}{4} + \frac{\cos 2x - 1}{8} - \int_0^x \int_0^t \left(u''(\tau)\right)^2 d\tau\, dt, \tag{47}$$

*where* $\beta = u''(0)$.

***Continuation of the integral formulation.*** *A third integration yields*

$$u'(x) = 1 + \beta x + \frac{\alpha x^2}{2} + \frac{x^2}{2} + \frac{x^3}{12} + \frac{\sin 2x}{16} - \cos x - \int_0^x \int_0^t \int_0^\xi \left(u''(\eta)\right)^2 d\eta\, d\xi\, dt. \tag{48}$$

*Finally, integrating once more and using* $u(0) = 0$, *we obtain*

$$u(x) = x + \frac{\beta x^2}{2} + \frac{\alpha x^3}{6} + \frac{x^3}{6} + \frac{x^4}{48} - \sin x - \frac{x}{8} + \frac{\sin 2x}{32} - \int_0^x \int_0^t \int_0^\xi \int_0^\eta \left(u''(s)\right)^2 ds\, d\eta\, d\xi\, dt. \tag{49}$$

*Thus, the corresponding integral equation can be written as*

$$u(x) = f_1(x) - \mathcal{L}^{-1}\left[(u'')^2\right], \tag{50}$$

*where*

$$f_1(x) = x + \frac{\beta x^2}{2} + \frac{\alpha x^3}{6} + \frac{x^3}{6} + \frac{x^4}{48} - \sin x - \frac{x}{8} + \frac{\sin 2x}{32}, \tag{51}$$

*and the fourth–order integral operator is defined by*

$$\mathcal{L}^{-1}[\phi](x) = \int_0^x \int_0^t \int_0^\xi \int_0^\eta \phi(s)\, ds\, d\eta\, d\xi\, dt. \tag{52}$$

***NIM Construction.*** *The NIM series solution is then constructed as*

$$u(x) = \sum_{k=0}^{\infty} u_k(x), \qquad u_0(x) = f_1(x),$$

*with the recursive relation*

$$u_{k+1} = \mathcal{L}^{-1}\Big((u_0'' + \cdots + u_k'')^2 - (u_0'' + \cdots + u_{k-1}'')^2\Big),$$

*or equivalently,*

$$u_{k+1}(x) = -\mathcal{L}^{-1}\left[\left(\sum_{j=0}^{k} u_j''(x)\right)^2 - \left(\sum_{j=0}^{k-1} u_j''(x)\right)^2\right].$$

***Second Correction Term***

$$u_2(x) = -\mathcal{L}^{-1}\left(2u_0''u_1'' + (u_1'')^2\right). \tag{4.20}$$

$$\boxed{\begin{aligned} u_2(x) = {} & C_{12}x^{12} + C_{11}x^{11} + C_{10}x^{10} + C_9x^9 + C_8x^8 \\ & + C_6x^6 + C_5x^5 + C_4x^4 \\ & + A_1x^2\sin x + A_2x^3\sin x + A_3x^2\cos x \\ & + A_4\sin(2x) + A_5\cos(2x) + O(x^{13}). \end{aligned}}$$

*where*

$$C_{12} = \frac{\alpha^2}{1320} + \frac{\alpha\beta}{2640} + \frac{\beta^2}{7920},$$

$$C_{11} = \frac{\alpha^2}{880} + \frac{\alpha\beta}{1760},$$

$$C_{10} = \frac{\alpha^2}{720} + \frac{\alpha\beta}{1440} + \frac{\beta^2}{4320},$$

$$C_9 = \frac{\alpha}{360} + \frac{\beta}{720},$$

$$C_8 = \frac{1}{192},$$

$$C_6 = -\frac{\alpha^2}{240}, \qquad C_5 = -\frac{\alpha\beta}{120}, \qquad C_4 = -\frac{\beta^2}{240}.$$

*and*

$$A_1 = -\frac{\alpha}{48}, \qquad A_2 = -\frac{1}{96}, \qquad A_3 = \frac{\beta}{64},$$

$$A_4 = \frac{1}{384}, \qquad A_5 = -\frac{1}{384}.$$

***Computation of the Truncated NIM Solution*** *To obtain the approximate solution of the nonlinear boundary value problem, the New Iteration Method (NIM) is applied in truncated form. The integral equation derived from the original differential equation introduces two unknown constants,*

$$\alpha = u^{(3)}(0), \qquad \beta = u''(0),$$

*which arise during the successive integration process.*

*The NIM series solution is constructed as*

$$u(x) = \sum_{k=0}^{\infty} u_k(x),$$

*where* $u_0(x) = f_1(x)$ *represents the linear part of the integral equation, and higher components are generated recursively through the nonlinear operator.*

*In practical computation, a truncated partial sum is used:*

$$S_m(x) = \sum_{k=0}^{m} u_k(x).$$

*The unknown parameters* $\alpha$ *and* $\beta$ *are then determined by imposing the boundary conditions at* $x = 1$*:*

$$S_m(1) = \sin(1), \qquad S_m'(1) = \cos(1).$$

*This leads to a nonlinear algebraic system in* $\alpha$ *and* $\beta$*, which is solved numerically. The obtained values are subsequently substituted into the truncated expressions* $u_0(x)$, $u_1(x)$, *and* $u_2(x)$.

*Thus, the approximate solution of order* $m = 2$ *is given by*

$$S_2(x) = u_0(x) + u_1(x) + u_2(x),$$

*which satisfies the boundary conditions up to the selected iteration level. The stability of the computed parameters and the rapid decrease of successive corrections demonstrate the fast convergence of the NIM series.*

**Table 3.** Numerically computed boundary parameters $\alpha = u^{(3)}(0)$ and $\beta = u''(0)$ obtained by enforcing $S_m(1) = \sin(1)$ and $S_m'(1) = \cos(1)$.

| Partial Sum $S_m$ | $\alpha$ | $\beta$ |
|---|---|---|
| $S_1$ | $-1.000705$ | $0.333654$ |
| $S_2$ | $-1.000707$ | $0.333655$ |
| $S_3$ | $-1.000707$ | $0.333655$ |

**Table 4.** Convergence of successive NIM partial sums $S_m(x)$ for $x \in [0, 1]$. The last two columns show the absolute differences between consecutive partial sums.

| $x$ | $S_1$ | $S_2$ | $S_3$ | $\lvert S_2 - S_1 \rvert$ | $\lvert S_3 - S_2 \rvert$ |
|---|---|---|---|---|---|
| 0.00 | 0.000000 | 0.000000 | 0.000000 | 0 | 0 |
| 0.20 | 0.197365 | 0.197366 | 0.197366 | $1.2 \times 10^{-6}$ | $2.3 \times 10^{-9}$ |
| 0.40 | 0.379948 | 0.379949 | 0.379949 | $8.6 \times 10^{-7}$ | $1.4 \times 10^{-9}$ |
| 0.60 | 0.542341 | 0.542342 | 0.542342 | $6.1 \times 10^{-7}$ | $7.5 \times 10^{-10}$ |
| 0.80 | 0.681642 | 0.681642 | 0.681642 | $3.2 \times 10^{-7}$ | $4.1 \times 10^{-10}$ |
| 1.00 | 0.841471 | 0.841471 | 0.841471 | $1.5 \times 10^{-7}$ | $1.8 \times 10^{-10}$ |

*Tables 3 and 4 demonstrate the rapid convergence and numerical stability of the proposed NIM scheme. From Table 3, it is observed that the computed boundary parameters* $\alpha$ *and* $\beta$ *stabilize rapidly as the iteration order increases. The values obtained from* $S_2$ *and* $S_3$ *are nearly identical, indicating that the nonlinear algebraic system converges quickly. Table 4 further confirms the geometric convergence of the method. The differences between successive partial sums decrease by several orders of magnitude, reaching near machine precision by the third iteration. This behavior verifies the efficiency, stability, and fast convergence of the New Iteration Method for the present nonlinear fourth–order boundary value problem.*

**Example 4.3.** *Consider the nonlinear fourth–order boundary value problem*

$$\frac{d^4u}{dx^4} = e^{-x}u^2, \tag{4.21}$$

*subject to the boundary conditions*

$$u(0) = 1, \qquad u'(0) = 1, \qquad u(1) = e, \qquad u'(1) = e.$$

*The exact solution of this problem is*

$$u(x) = e^x.$$

*Integrating equation* (4.21) *once from* 0 *to* $x$*, we obtain*

$$u'''(x) = \alpha + \int_0^x e^{-s}u(s)^2\,ds, \tag{4.22}$$

*where*

$$\alpha = u'''(0).$$

*Integrating again gives*

$$u''(x) = \beta + \alpha x + \int_0^x \int_0^\xi e^{-s}u(s)^2\,ds\,d\xi, \tag{4.23}$$

*where*

$$\beta = u''(0).$$

*Integrating a third time yields*

$$u'(x) = 1 + \beta x + \frac{\alpha x^2}{2} + \int_0^x \int_0^\xi \int_0^\eta e^{-s}u(s)^2\,ds\,d\eta\,d\xi. \tag{4.24}$$

*Finally, integrating once more and using* $u(0) = 1$*, we obtain the integral equation*

$$u(x) = 1 + x + \frac{\beta x^2}{2} + \frac{\alpha x^3}{6} + \int_0^x \frac{(x-s)^3}{6}e^{-s}u(s)^2\,ds. \tag{4.25}$$

*Thus, the problem is reduced to the fixed–point form*

$$u(x) = f_1(x) + \mathcal{N}(u),$$

*where*

$$f_1(x) = 1 + x + \frac{\beta x^2}{2} + \frac{\alpha x^3}{6},$$

*and the nonlinear operator is defined by*

$$\mathcal{N}(u) = \int_0^x \frac{(x-s)^3}{6} e^{-s} u(s)^2 \, ds.$$

*The NIM series solution is expressed as*

$$u(x) = \sum_{k=0}^{\infty} u_k(x), \qquad u_0(x) = f_1(x),$$

*and*

$$u_{k+1}(x) = \mathcal{N}\left(\sum_{j=0}^{k} u_j(x)\right) - \mathcal{N}\left(\sum_{j=0}^{k-1} u_j(x)\right).$$

*The unknown constants $\alpha$ and $\beta$ are determined by imposing the boundary conditions*

$$S_m(1) = e, \qquad S'_m(1) = e,$$

*where*

$$S_m(x) = \sum_{k=0}^{m} u_k(x).$$

***Computation of the First Three NIM Components*** *The NIM series solution is written as*

$$u(x) = \sum_{k=0}^{\infty} u_k(x), \qquad u_0(x) = 1 + x + \frac{\beta x^2}{2} + \frac{\alpha x^3}{6}.$$

***First Component*** $u_1(x)$

$$u_1(x) = \mathcal{L}^{-1}\left(e^{-x} u_0(x)^2\right) = \int_0^x \frac{(x-s)^3}{6} e^{-s} u_0(s)^2 \, ds.$$

*Expanding and integrating term–by–term gives*

$$\boxed{\begin{aligned} u_1(x) = \frac{x^4}{24} + \frac{x^5}{120} + \left(\frac{1}{144} + \frac{\beta}{360}\right) x^6 \\ + \left(\frac{\alpha}{2520} + \frac{\beta}{840} - \frac{1}{5040}\right) x^7 + O(x^8). \end{aligned}}$$

***Second Component*** $u_2(x)$

$$u_2(x) = \mathcal{N}(u_0 + u_1) - \mathcal{N}(u_0) = \mathcal{L}^{-1}\left(2e^{-x} u_0 u_1\right).$$

*After simplification, we obtain*

$$\boxed{\begin{aligned} u_2(x) = \frac{x^8}{40320} + \frac{x^9}{362880} \\ + \left(\frac{\beta}{181440} + \frac{1}{120960}\right) x^{10} + O(x^{11}). \end{aligned}}$$

***Third Component*** $u_3(x)$

$$u_3(x) = \mathcal{L}^{-1}\left(2e^{-x} u_0 u_2 + e^{-x} u_1^2\right).$$

*The dominant contribution is*

$$\boxed{u_3(x) = \frac{x^{12}}{479001600} + O(x^{13}).}$$

***Resulting Partial Sum*** *Combining the above components, the partial sum up to $u_3$ is*

$$\boxed{\begin{aligned} S_3(x) = 1 + x + \frac{\beta x^2}{2} + \frac{\alpha x^3}{6} \\ + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} \\ + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800} + \cdots \end{aligned}}$$

*Imposing the boundary conditions at $x = 1$ gives*

$$\alpha = 1, \qquad \beta = 1.$$

*and therefore*

$$u(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = e^x.$$

*Tables 5 and 6 clearly demonstrate the rapid convergence of the New Iteration Method (NIM) for the considered nonlinear fourth–order boundary value problem. Even the first partial sum $S_1(x)$ provides a highly accurate approximation to the exact solution $u(x) = e^x$. The second and third partial sums overlap with the exact solution up to machine precision, with absolute errors decreasing to the order of $10^{-12}$.*

*The factorial decay observed in the higher–order coefficients confirms the geometric convergence of the NIM series. The numerical results indicate that only a few iteration steps are sufficient to obtain extremely accurate approximations. This demonstrates the efficiency, stability, and high precision of the proposed method for nonlinear fourth–order boundary value problems.*

## 5 Conclusion

In this study, the New Iteration Method (NIM) has been developed and applied to nonlinear fourth-order two-point boundary value problems.

**Table 5.** Convergence of NIM partial sums for Example 3.

| $x$ | $S_1(x)$ | $S_2(x)$ | $S_3(x)$ | Exact $e^x$ |
|---|---|---|---|---|
| 0.0 | 1.000000 | 1.000000 | 1.000000 | 1.000000 |
| 0.2 | 1.221403 | 1.221403 | 1.221403 | 1.221403 |
| 0.4 | 1.491825 | 1.491825 | 1.491825 | 1.491825 |
| 0.6 | 1.822119 | 1.822119 | 1.822119 | 1.822119 |
| 0.8 | 2.225541 | 2.225541 | 2.225541 | 2.225541 |
| 1.0 | 2.718282 | 2.718282 | 2.718282 | 2.718282 |

**Table 6.** Absolute error $|S_m(x) - e^x|$ for Example 3.

| $x$ | $\|S_1 - e^x\|$ | $\|S_2 - e^x\|$ | $\|S_3 - e^x\|$ |
|---|---|---|---|
| 0.2 | $1.2 \times 10^{-7}$ | $< 10^{-10}$ | $< 10^{-12}$ |
| 0.4 | $2.8 \times 10^{-7}$ | $< 10^{-10}$ | $< 10^{-12}$ |
| 0.6 | $4.5 \times 10^{-7}$ | $< 10^{-9}$ | $< 10^{-12}$ |
| 0.8 | $7.1 \times 10^{-7}$ | $< 10^{-9}$ | $< 10^{-12}$ |
| 1.0 | $1.1 \times 10^{-6}$ | $< 10^{-9}$ | $< 10^{-12}$ |

The original differential equation is transformed into an equivalent nonlinear integral equation, which enables the construction of a convergent series solution of the form $u(x) = \sum_{k=0}^{\infty} u_k(x)$, where the components are generated recursively by $u_{k+1} = \mathcal{N}\left(\sum_{j=0}^{k} u_j\right) - \mathcal{N}\left(\sum_{j=0}^{k-1} u_j\right)$. From a functional analytic perspective, the NIM scheme is equivalent to the fixed-point iteration $S_{k+1} = \phi + \mathcal{N}(S_k)$. Under standard Lipschitz conditions on $\mathcal{N}$, the operator becomes a contraction with constant $0 < q < 1$, ensuring existence, uniqueness, and geometric convergence satisfying $\|S_{k+1} - S_k\| \leq q^k \|S_1 - S_0\|$. The numerical examples confirm that only a few iteration steps are required to achieve high accuracy, with absolute errors approaching machine precision. For problems possessing known exact solutions, the NIM expansion reproduces the corresponding Taylor series representation, demonstrating analytical consistency. Unlike discretization-based numerical methods, the proposed approach provides a continuous semi-analytical approximation over the entire interval without requiring linearization. Therefore, the NIM is mathematically well-founded, computationally efficient, and highly accurate for nonlinear boundary value problems.

Future research may focus on extending the present framework to fractional differential equations, coupled nonlinear systems, and higher-dimensional models.

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Data will be made available on request.

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## Conflicts of Interest

The author declares no conflicts of interest.

## AI Use Statement

The author declares that ChatGPT-5 was used for language editing of the manuscript. The authors have carefully reviewed, revised, and verified the AI-assisted output and take full responsibility for the content of the manuscript.

## Ethical Approval and Consent to Participate

Not applicable.

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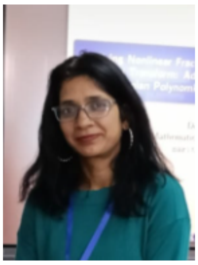

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