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RESEARCH ARTICLE

The Integration of Stiff Systems of Ordinary Differential Equations Using a Time-Stepping Algorithm with Off-Step Points

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Abstract

Stiff systems of ordinary differential equations (ODEs) arise frequently in science, engineering, and applied mathematics, and their numerical solutions require methods with strong stability properties to avoid severe step-size restrictions. This motivates the development of efficient and stable numerical schemes tailored for such problems. In this paper, a fully implicit hybrid block method, termed the Two-Point Superclass Block Backward Differentiation Formula with Off-step Points (2SBBDFO), is proposed for solving stiff ODE systems. The method extends the classical two-point block BDF by incorporating a non-zero coefficient β_{k-1}, enabling the simultaneous computation of two grid-point and two off-step solution values within each block. Theoretical analysis shows that the method is consistent, zero-stable, and convergent, with order-five accuracy. Stability analysis using the boundary-locus technique confirms that the method is nearly A-stable (A(α)-stable). The resulting nonlinear system is solved using Newton's iteration for efficient convergence. Numerical experiments on selected stiff initial value problems (IVPs) demonstrate that the method achieves higher accuracy and competitive computational efficiency compared to existing block BDF methods, particularly for smaller step sizes. These results indicate that the 2SBBDFO method is an effective and reliable tool for solving stiff ODEs.

Keywords: Stiff ODEs, Block Methods, Backward Differentiation Formula, A-stability, Newton Iteration.

1 Introduction

Stiff ordinary differential equations (ODEs) constitute a class of problems that pose significant challenges in numerical computation due to their inherent multi-scale behavior. Such equations frequently arise in the mathematical modeling of real-world phenomena in science and engineering, including reaction kinetics, population dynamics, and electrical circuit analysis. A defining feature of stiff systems is the presence of widely varying time scales, where

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rapidly changing transient components coexist with slowly varying dynamics [1, 2].

This disparity in time scales gives rise to stiffness, a condition under which numerical solutions become highly sensitive to step size, initial conditions, and parameter variations. Consequently, conventional explicit numerical methods often fail to provide stable and accurate solutions, leading to numerical instability, spurious oscillations, or lack of convergence [3]. These challenges necessitate the development of specialized numerical techniques with strong stability properties.

In this study, we consider the numerical approximation of the first-order stiff initial value problem of the form

$$y' = f(x, y), \quad y(a) = y_0, \quad a \leq x \leq b. \tag{1}$$

Over the years, several implicit numerical methods have been proposed to address stiffness. Foundational BDF-based approaches include the work of Gear [4] and Cash [5, 6], who developed and extended backward differentiation formulae for stiff ODEs and DAEs. Block methods were subsequently introduced and refined by Ibrahim et al. [7], Abasi et al. [8], Zawawi et al. [9], and Musa and Alhassan [10], with comparative evaluations provided by Yatim et al. [11]. Related stability theory for linear multistep methods was established by Widlund [12] and Enright [13]. Standard theoretical references include Butcher [14] and Iserles [15]. These methods have undergone continuous refinement to enhance their stability, accuracy, and computational efficiency.

Motivated by these developments, this paper presents a novel two-point fully implicit block hybrid method for solving first-order stiff ODEs. The proposed method is formulated with a free stability control parameter $\rho \in (-1, 1)$, which provides flexibility in improving both stability and convergence properties. By appropriately selecting values of ρ, the method achieves reliable and accurate numerical approximations for stiff problems.

Furthermore, the hybrid nature of the scheme incorporates off-step points within the integration interval, particularly when the step size is refined, thereby enhancing accuracy while preserving stability. This combination of block structure and hybrid formulation offers an efficient and robust computational framework for the numerical solution of stiff ODEs.

2 Derivation of the Method

Consider the conventional 2-point block backward differentiation formula with off-step points (2BBDFO) for stiff ODEs as developed in [8], given by

$$\begin{aligned} &\sum_{j=0}^{1} \alpha_{j,i}\, y_{n+j-1} + \sum_{j=0}^{3} \alpha_{j+2,i}\, y_{n+\frac{j+1}{2}} \\ &\quad = h\beta_{k,i} f_{n+k}, \\ &\quad k = i = \tfrac{1}{2}, 1, \tfrac{3}{2}, 2. \end{aligned} \tag{2}$$

where $\beta_{0,i} = \beta_{1,i} = \beta_{2,i} = \ldots = \beta_{k-1,i} = 0$ and $\beta_{k,i} \neq 0$. The values $k = i = \frac{1}{2}, 1, \frac{3}{2}, 2$ correspond respectively to the first off-step, the first grid point, the second off-step, and the second grid point of the method.

In this study, the above formulation is extended by relaxing the restriction imposed on the β-coefficients and introducing additional flexibility through two non-zero trailing coefficients. Specifically, we consider the case

$$\begin{aligned} \beta_{0,i} = \beta_{1,i} = \beta_{2,i} = \cdots = \beta_{k-2,i} = 0, \\ \beta_{k-1,i} \neq 0, \qquad \beta_{k,i} \neq 0. \end{aligned}$$

which leads to the definition of a superclass of the method in (2). This generalized formulation extends the conventional 2BBDFO by incorporating an additional degree of freedom, thereby facilitating the construction of more stable, accurate, and computationally efficient hybrid block implicit methods for stiff ODEs. It is noteworthy that the standard 2BBDFO in (2) is recovered as a special case when $\beta_{k-1,i} = 0$. Furthermore, this generalization allows for the generation of a family of methods parameterized by

$$\rho = \frac{\beta_{k-1,i}}{\beta_{k,i}},$$

which enhances the stability characteristics of the scheme. In particular, improved stability properties, including enlarged regions of absolute stability, are achieved when the parameter $\rho \in (-1, 1)$, as reported in the literature [9].

In this section, we present the formulation of the proposed two-point fully implicit block hybrid method for the numerical integration of stiff initial value problems (IVPs). The method simultaneously computes two grid-point solution values and two off-step solution values within each block. The general form of the method is given in Definition 2.1.

Table 1. The coefficients of the first off-step point formula.

$\alpha_{0,\frac{1}{2}}$	$\alpha_{1,\frac{1}{2}}$	$\alpha_{2,\frac{1}{2}}$	$\alpha_{3,\frac{1}{2}}$	$\alpha_{4,\frac{1}{2}}$	$\alpha_{5,\frac{1}{2}}$	$\beta_{\frac{1}{2},\frac{1}{2}}$
$-\frac{1}{20}\frac{4\rho-1}{16\rho-3}$	$-\frac{1}{4}\frac{38\rho+9}{16\rho-3}$	1	$-\frac{9}{4}\frac{4\rho-3}{16\rho-3}$	$\frac{1}{5}\frac{16\rho-9}{16\rho-3}$	$-\frac{1}{4}\frac{2\rho-1}{16\rho-3}$	$\frac{3}{16\rho-3}$

Definition 2.1. The proposed fully implicit 2-point superclass of block backward differentiation formula with off-step points (2SBBDFO) is defined by

$$\sum_{j=0}^{1}\alpha_{j,i}\,y_{n+j-1}+\sum_{j=0}^{3}\alpha_{j+2,i}\,y_{n+\frac{j+1}{2}} = h\beta_{k,i}\Big(f_{n+k}+\rho f_{n+k-\frac{1}{2}}\Big),\quad k=i=\tfrac{1}{2},1,\tfrac{3}{2},2. \tag{3}$$

The method (3) is derived through Taylor series expansion about any point x_n.

Definition 2.2. The linear operator L_i associated with 2SBBDFO is defined by

$$\begin{aligned} L_i[y(x_n),h] &:= \alpha_{0,i}y_{n-1}+\alpha_{1,i}y_n+\alpha_{2,i}y_{n+\frac{1}{2}} \\ &\quad+\alpha_{3,i}y_{n+1}+\alpha_{4,i}y_{n+\frac{3}{2}}+\alpha_{5,i}y_{n+2} \\ &\quad-h\beta_{k,i}\Big(f_{n+k}+\rho f_{n+k-\frac{1}{2}}\Big)=0, \\ &\quad k=i=\tfrac{1}{2},1,\tfrac{3}{2},2. \end{aligned} \tag{4}$$

First Off-step Point: $k=\frac{1}{2}$**.** To obtain the first off-step point $y_{n+\frac{1}{2}}$, we introduce the following linear operator:

$$\begin{aligned} L_{\frac{1}{2}}[y(x_n),h] &:= \alpha_{0,\frac{1}{2}}y_{n-1}+\alpha_{1,\frac{1}{2}}y_n+\alpha_{2,\frac{1}{2}}y_{n+\frac{1}{2}} \\ &\quad+\alpha_{3,\frac{1}{2}}y_{n+1}+\alpha_{4,\frac{1}{2}}y_{n+\frac{3}{2}}+\alpha_{5,\frac{1}{2}}y_{n+2} \\ &\quad-h\beta_{\frac{1}{2},\frac{1}{2}}\left(f_{n+\frac{1}{2}}+\rho f_n\right)=0. \end{aligned} \tag{5}$$

The associated approximate relationship for (3) can be expressed as

$$\begin{aligned} &\alpha_{0,\frac{1}{2}}y(x_n-h)+\alpha_{1,\frac{1}{2}}y(x_n)+\alpha_{2,\frac{1}{2}}y\big(x_n+\tfrac{1}{2}h\big) \\ &+\alpha_{3,\frac{1}{2}}y(x_n+h)+\alpha_{4,\frac{1}{2}}y\big(x_n+\tfrac{3}{2}h\big)+\alpha_{5,\frac{1}{2}}y(x_n+2h) \\ &-h\beta_{\frac{1}{2},\frac{1}{2}}\left[f\big(x_n+\tfrac{1}{2}h\big)+\rho f(x_n)\right]=0. \end{aligned} \tag{6}$$

Expanding equation (6) as a Taylor series about x_n, then equating and collecting like terms yields

$$\begin{aligned} &C_{0,\frac{1}{2}}y(x_n)+C_{1,\frac{1}{2}}hy'(x_n)+C_{2,\frac{1}{2}}h^2y''(x_n) \\ &+C_{3,\frac{1}{2}}h^3y'''(x_n)+C_{4,\frac{1}{2}}h^4y^{(4)}(x_n) \\ &+C_{5,\frac{1}{2}}h^5y^{(5)}(x_n)+\cdots=0. \end{aligned} \tag{7}$$

where

$$\begin{aligned} C_{0,\frac{1}{2}} &= \alpha_{0,\frac{1}{2}}+\alpha_{1,\frac{1}{2}}+\alpha_{2,\frac{1}{2}}+\alpha_{3,\frac{1}{2}} \\ &\quad+\alpha_{4,\frac{1}{2}}+\alpha_{5,\frac{1}{2}}=0, \\ C_{1,\frac{1}{2}} &= -\alpha_{0,\frac{1}{2}}+\tfrac{1}{2}\alpha_{2,\frac{1}{2}}+\alpha_{3,\frac{1}{2}}+\tfrac{3}{2}\alpha_{4,\frac{1}{2}} \\ &\quad+2\alpha_{5,\frac{1}{2}}-\beta_{\frac{1}{2},\frac{1}{2}}(\rho+1)=0, \\ C_{2,\frac{1}{2}} &= \tfrac{1}{2}\alpha_{0,\frac{1}{2}}+\tfrac{1}{8}\alpha_{2,\frac{1}{2}}+\tfrac{1}{2}\alpha_{3,\frac{1}{2}}+\tfrac{9}{8}\alpha_{4,\frac{1}{2}} \\ &\quad+2\alpha_{5,\frac{1}{2}}-\tfrac{1}{2}\beta_{\frac{1}{2},\frac{1}{2}}=0, \\ C_{3,\frac{1}{2}} &= -\tfrac{1}{6}\alpha_{0,\frac{1}{2}}+\tfrac{1}{48}\alpha_{2,\frac{1}{2}}+\tfrac{1}{6}\alpha_{3,\frac{1}{2}}+\tfrac{9}{16}\alpha_{4,\frac{1}{2}} \\ &\quad+\tfrac{4}{3}\alpha_{5,\frac{1}{2}}-\tfrac{1}{8}\beta_{\frac{1}{2},\frac{1}{2}}=0, \\ C_{4,\frac{1}{2}} &= \tfrac{1}{24}\alpha_{0,\frac{1}{2}}+\tfrac{1}{384}\alpha_{2,\frac{1}{2}}+\tfrac{1}{24}\alpha_{3,\frac{1}{2}}+\tfrac{27}{128}\alpha_{4,\frac{1}{2}} \\ &\quad+\tfrac{2}{3}\alpha_{5,\frac{1}{2}}-\tfrac{1}{48}\beta_{\frac{1}{2},\frac{1}{2}}=0, \\ C_{5,\frac{1}{2}} &= -\tfrac{1}{120}\alpha_{0,\frac{1}{2}}+\tfrac{1}{3840}\alpha_{2,\frac{1}{2}}+\tfrac{1}{120}\alpha_{3,\frac{1}{2}} \\ &\quad+\tfrac{81}{1280}\alpha_{4,\frac{1}{2}}+\tfrac{4}{15}\alpha_{5,\frac{1}{2}}-\tfrac{1}{384}\beta_{\frac{1}{2},\frac{1}{2}}=0. \end{aligned} \tag{8}$$

In deriving the first off-step point $y_{n+\frac{1}{2}}$, the coefficient $\alpha_{2,\frac{1}{2}}$ is normalized to unity. Solving the system of equations in (8) simultaneously for the coefficients $\alpha_{j,i}$ and $\beta_{j,i}$ yields the values reported in Table 1.

Substituting these values in equation (5), we obtain

$$\begin{aligned} y_{n+\frac{1}{2}} &= \tfrac{1}{20}\tfrac{4\rho-1}{16\rho-3}y_{n-1}+\tfrac{1}{4}\tfrac{38\rho+9}{16\rho-3}y_n \\ &\quad+\tfrac{9}{4}\tfrac{4\rho-3}{16\rho-3}y_{n+1}-\tfrac{1}{5}\tfrac{16\rho-9}{16\rho-3}y_{n+\frac{3}{2}} \\ &\quad+\tfrac{1}{4}\tfrac{2\rho-1}{16\rho-3}y_{n+2}+\tfrac{3}{16\rho-3}hf_{n+\frac{1}{2}} \\ &\quad+\tfrac{3\rho}{16\rho-3}hf_n. \end{aligned} \tag{9}$$

A similar procedure to that used in deriving the first off-step point is applied to obtain the first, second off-step, and second points, respectively. By combining the resulting expressions, the derived

2SBBDFO scheme is given as follows:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} =& \tfrac{1}{20}\tfrac{4\rho-1}{16\rho-3}y_{n-1} + \tfrac{1}{4}\tfrac{38\rho+9}{16\rho-3}y_n \\ &+ \tfrac{9}{4}\tfrac{4\rho-3}{16\rho-3}y_{n+1} - \tfrac{1}{5}\tfrac{16\rho-9}{16\rho-3}y_{n+\frac{3}{2}} \\ &+ \tfrac{1}{4}\tfrac{2\rho-1}{16\rho-3}y_{n+2} + \tfrac{3}{16\rho-3}hf_{n+\frac{1}{2}} \\ &+ \tfrac{3\rho}{16\rho-3}hf_n. \\ y_{n+1} =& -\tfrac{1}{45}\tfrac{3\rho-2}{9\rho+2}y_{n-1} + \tfrac{1}{3}\tfrac{9\rho-4}{9\rho+2}y_n \\ &+ \tfrac{4}{9}\tfrac{9\rho+16}{9\rho+2}y_{n+\frac{1}{2}} + \tfrac{4}{15}\tfrac{9\rho-16}{9\rho+2}y_{n+\frac{3}{2}} \\ &- \tfrac{1}{9}\tfrac{3\rho-4}{9\rho+2}y_{n+2} + \tfrac{4}{9(9\rho+2)}hf_{n+1} \\ &+ \tfrac{4\rho}{9(9\rho+2)}hf_{n+\frac{1}{2}}, \\ y_{n+\frac{3}{2}} =& \tfrac{1}{12}\tfrac{2\rho-3}{16\rho+31}y_{n-1} - \tfrac{5}{4}\tfrac{4\rho-5}{16\rho+31}y_n \\ &+ \tfrac{5}{3}\tfrac{16\rho-15}{16\rho+31}y_{n+\frac{1}{2}} - \tfrac{15}{4}\tfrac{2\rho-15}{16\rho+31}y_{n+1} \\ &+ \tfrac{5}{12}\tfrac{4\rho-15}{16\rho+31}y_{n+2} + \tfrac{15}{16\rho+31}hf_{n+\frac{3}{2}} \\ &+ \tfrac{15\rho}{16\rho+31}hf_{n+1}, \\ y_{n+2} =& -\tfrac{1}{5}\tfrac{\rho-4}{5\rho+54}y_{n-1} + \tfrac{5\rho-18}{5\rho+54}y_n \\ &- \tfrac{4(5\rho-16)}{5\rho+54}y_{n+\frac{1}{2}} + \tfrac{9(5\rho-12)}{5\rho+54}y_{n+1} \\ &- \tfrac{4}{5}\tfrac{31\rho-144}{5\rho+54}y_{n+\frac{3}{2}} + \tfrac{12}{5\rho+54}hf_{n+2} \\ &+ \tfrac{12\rho}{5\rho+54}hf_{n+\frac{3}{2}}. \end{aligned}\right\} \quad (10)$$

For the purpose of maintaining zero and absolute stability of the method (10), the free parameter ρ is restricted to the interval $-1 < \rho < 1$ as indicated in [1, 16–18]. It is important to note that the fully implicit 2SBBDFO (10) reduces to the fully implicit 2-point block backward differentiation formula with off-step points (2BBDFO), as developed by [8], when $\rho = 0$. Accordingly, the formulae corresponding to the case $\rho = 0$ are expressed as follows:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} =& \tfrac{1}{60}y_{n-1} - \tfrac{3}{4}y_n + \tfrac{9}{4}y_{n+1} \\ &- \tfrac{3}{5}y_{n+\frac{3}{2}} + \tfrac{1}{12}y_{n+2} - hf_{n+\frac{1}{2}}, \\ y_{n+1} =& \tfrac{1}{45}y_{n-1} - \tfrac{2}{3}y_n + \tfrac{32}{9}y_{n+\frac{1}{2}} \\ &- \tfrac{32}{15}y_{n+\frac{3}{2}} + \tfrac{2}{9}y_{n+2} + \tfrac{2}{9}hf_{n+1}, \\ y_{n+\frac{3}{2}} =& -\tfrac{1}{124}y_{n-1} + \tfrac{25}{124}y_n - \tfrac{25}{31}y_{n+\frac{1}{2}} \\ &+ \tfrac{225}{124}y_{n+1} - \tfrac{25}{124}y_{n+2} + \tfrac{15}{31}hf_{n+\frac{3}{2}}, \\ y_{n+2} =& \tfrac{2}{135}y_{n-1} - \tfrac{1}{3}y_n + \tfrac{32}{27}y_{n+\frac{1}{2}} \\ &- 2y_{n+1} + \tfrac{32}{15}y_{n+\frac{3}{2}} + \tfrac{2}{9}hf_{n+2}. \end{aligned}\right\} \quad (11)$$

The 2BBDFO scheme has been formulated to achieve fifth-order accuracy, with its error constant given by

$$C_6 = \begin{bmatrix} -\frac{1}{20} \\ -\frac{4}{45} \\ \frac{5}{124} \\ -\frac{4}{45} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

3 Derivation of Order, Error Constant and Consistency of the 2SBBDFO Method

In this section, the detailed derivation of the order and the error constant of the 2SBBDFO scheme, corresponding to the equations in (10), is presented. To determine the order of the method, the formulae in (10) can equivalently be expressed as

$$\left.\begin{aligned} &-\tfrac{1}{20}\tfrac{4\rho-1}{16\rho-3}y_{n-1} - \tfrac{1}{4}\tfrac{38\rho+9}{16\rho-3}y_n + y_{n+\frac{1}{2}} \\ &\quad - \tfrac{9}{4}\tfrac{4\rho-3}{16\rho-3}y_{n+1} + \tfrac{1}{5}\tfrac{16\rho-9}{16\rho-3}y_{n+\frac{3}{2}} - \tfrac{1}{4}\tfrac{2\rho-1}{16\rho-3}y_{n+2} \\ &\quad = \tfrac{3}{16\rho-3}hf_{n+\frac{1}{2}} + \tfrac{3\rho}{16\rho-3}hf_n, \\ &\tfrac{1}{45}\tfrac{3\rho-2}{9\rho+2}y_{n-1} - \tfrac{1}{3}\tfrac{9\rho-4}{9\rho+2}y_n - \tfrac{4}{9}\tfrac{9\rho+16}{9\rho+2}y_{n+\frac{1}{2}} \\ &\quad + y_{n+1} - \tfrac{4}{15}\tfrac{9\rho-16}{9\rho+2}y_{n+\frac{3}{2}} + \tfrac{1}{9}\tfrac{3\rho-4}{9\rho+2}y_{n+2} \\ &\quad = \tfrac{4}{9(9\rho+2)}hf_{n+1} + \tfrac{4\rho}{9(9\rho+2)}hf_{n+\frac{1}{2}}, \\ &-\tfrac{1}{12}\tfrac{2\rho-3}{16\rho+31}y_{n-1} + \tfrac{5}{4}\tfrac{4\rho-5}{16\rho+31}y_n - \tfrac{5}{3}\tfrac{16\rho-15}{16\rho+31}y_{n+\frac{1}{2}} \\ &\quad + \tfrac{15}{4}\tfrac{2\rho-15}{16\rho+31}y_{n+1} + y_{n+\frac{3}{2}} - \tfrac{5}{12}\tfrac{4\rho-15}{16\rho+31}y_{n+2} \\ &\quad = \tfrac{15}{16\rho+31}hf_{n+\frac{3}{2}} + \tfrac{15\rho}{16\rho+31}hf_{n+1}, \\ &\tfrac{1}{5}\tfrac{\rho-4}{5\rho+54}y_{n-1} - \tfrac{5\rho-18}{5\rho+54}y_n + \tfrac{4(5\rho-16)}{5\rho+54}y_{n+\frac{1}{2}} \\ &\quad - \tfrac{9(5\rho-12)}{5\rho+54}y_{n+1} + \tfrac{4}{5}\tfrac{31\rho-144}{5\rho+54}y_{n+\frac{3}{2}} + y_{n+2} \\ &\quad = \tfrac{12}{5\rho+54}hf_{n+2} + \tfrac{12\rho}{5\rho+54}hf_{n+\frac{3}{2}}. \end{aligned}\right\} \quad (12)$$

The general matrix representation of (12) is expressed as

$$\sum_{j=0}^{1} \sigma_j^* Y_{m+j-1} = h \sum_{j=0}^{1} \delta_j^* F_{m+j-1}, \qquad (13)$$

where σ_0^*, σ_1^*, δ_0^*, and δ_1^* are 4×4 square matrices defined as

$$\sigma_0^* = \begin{bmatrix} 0 & -\frac{1}{20}\frac{4\rho-1}{16\rho-3} & 0 & -\frac{1}{4}\frac{38\rho+9}{16\rho-3} \\ 0 & \frac{1}{45}\frac{3\rho-2}{9\rho+2} & 0 & -\frac{1}{3}\frac{9\rho-4}{9\rho+2} \\ 0 & -\frac{1}{12}\frac{2\rho-3}{16\rho+31} & 0 & \frac{5}{4}\frac{4\rho-5}{16\rho+31} \\ 0 & \frac{1}{5}\frac{\rho-4}{5\rho+54} & 0 & -\frac{5\rho-18}{5\rho+54} \end{bmatrix},$$

$$\sigma_1^* = \begin{bmatrix} 1 & -\frac{9}{4}\frac{4\rho-3}{16\rho-3} & \frac{1}{5}\frac{16\rho-9}{16\rho-3} & -\frac{1}{4}\frac{2\rho-1}{16\rho-3} \\ -\frac{4}{9}\frac{9\rho+16}{9\rho+2} & 1 & -\frac{4}{15}\frac{9\rho-16}{9\rho+2} & \frac{1}{9}\frac{3\rho-4}{9\rho+2} \\ -\frac{5}{3}\frac{16\rho-15}{16\rho+31} & \frac{15}{4}\frac{2\rho-15}{16\rho+31} & 1 & -\frac{5}{12}\frac{4\rho-15}{16\rho+31} \\ \frac{4(5\rho-16)}{5\rho+54} & -\frac{9(5\rho-12)}{5\rho+54} & \frac{4}{5}\frac{31\rho-144}{5\rho+54} & 1 \end{bmatrix},$$

$$\delta_0^* = \begin{bmatrix} 0 & 0 & 0 & \frac{3\rho}{16\rho-3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\delta_1^* = \begin{bmatrix} \frac{3}{16\rho-3} & 0 & 0 & 0 \\ \frac{4\rho}{9(9\rho+2)} & \frac{4}{9(9\rho+2)} & 0 & 0 \\ 0 & \frac{15\rho}{16\rho+31} & \frac{15}{16\rho+31} & 0 \\ 0 & 0 & \frac{12\rho}{5\rho+54} & \frac{12}{5\rho+54} \end{bmatrix}.$$

and Y_{m-1}, Y_m, F_{m-1}, F_m are the column vectors

$$Y_{m-1} = \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix}, \qquad Y_m = \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix},$$

$$F_{m-1} = \begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix}, \qquad F_m = \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}.$$

The square matrices can equivalently be written column-wise as

$$\sigma_0^* = (\sigma_0\ \sigma_1\ \sigma_2\ \sigma_3), \quad \sigma_1^* = (\sigma_4\ \sigma_5\ \sigma_6\ \sigma_7),$$
$$\delta_0^* = (\delta_0\ \delta_1\ \delta_2\ \delta_3), \quad \delta_1^* = (\delta_4\ \delta_5\ \delta_6\ \delta_7),$$

where

$$\sigma_1 = \begin{bmatrix} -\frac{1}{20}\frac{4\rho-1}{16\rho-3} \\ \frac{1}{45}\frac{3\rho-2}{9\rho+2} \\ -\frac{1}{12}\frac{2\rho-3}{16\rho+31} \\ \frac{1}{5}\frac{\rho-4}{5\rho+54} \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} -\frac{1}{4}\frac{38\rho+9}{16\rho-3} \\ -\frac{1}{3}\frac{9\rho-4}{9\rho+2} \\ \frac{5}{4}\frac{4\rho-5}{16\rho+31} \\ -\frac{5\rho-18}{5\rho+54} \end{bmatrix},$$

$$\sigma_4 = \begin{bmatrix} 1 \\ -\frac{4}{9}\frac{9\rho+16}{9\rho+2} \\ -\frac{5}{3}\frac{16\rho-15}{16\rho+31} \\ \frac{4(5\rho-16)}{5\rho+54} \end{bmatrix}, \qquad \sigma_5 = \begin{bmatrix} -\frac{9}{4}\frac{4\rho-3}{16\rho-3} \\ 1 \\ \frac{15}{4}\frac{2\rho-15}{16\rho+31} \\ -\frac{9(5\rho-12)}{5\rho+54} \end{bmatrix}.$$

$$\sigma_6 = \begin{bmatrix} \frac{1}{5}\frac{16\rho-9}{16\rho-3} \\ -\frac{4}{15}\frac{9\rho-16}{9\rho+2} \\ 1 \\ \frac{4}{5}\frac{31\rho-144}{5\rho+54} \end{bmatrix}, \qquad \sigma_7 = \begin{bmatrix} -\frac{1}{4}\frac{2\rho-1}{16\rho-3} \\ \frac{1}{9}\frac{3\rho-4}{9\rho+2} \\ -\frac{5}{12}\frac{4\rho-15}{16\rho+31} \\ 1 \end{bmatrix},$$

$$\sigma_0 = \sigma_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\delta_3 = \begin{bmatrix} \frac{3\rho}{16\rho-3} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \delta_4 = \begin{bmatrix} \frac{3}{16\rho-3} \\ \frac{4\rho}{9(9\rho+2)} \\ 0 \\ 0 \end{bmatrix},$$

$$\delta_5 = \begin{bmatrix} 0 \\ \frac{4}{9(9\rho+2)} \\ \frac{15\rho}{16\rho+31} \\ 0 \end{bmatrix}, \qquad \delta_6 = \begin{bmatrix} 0 \\ 0 \\ \frac{15}{16\rho+31} \\ \frac{12\rho}{5\rho+54} \end{bmatrix},$$

$$\delta_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{12}{5\rho+54} \end{bmatrix}, \qquad \delta_0 = \delta_1 = \delta_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let the linear operator L be defined by

$$L\{y(x), h\} = \sum_{j=0}^{k} \left[\sigma_j y\left(x + j\tfrac{h}{2}\right) - h\delta_j y'\left(x + j\tfrac{h}{2}\right)\right]. \tag{14}$$

where σ_j and δ_j are constants. Expanding $y\left(x + j\frac{h}{2}\right)$ and $y'\left(x + j\frac{h}{2}\right)$ in Taylor series about the point x_n, we obtain

$$\begin{aligned} y\left(x + j\tfrac{h}{2}\right) &= y(x) + \tfrac{jh}{2}y'(x) + \tfrac{(jh)^2}{2!\,2^2}y''(x) \\ &\quad + \tfrac{(jh)^3}{3!\,2^3}y'''(x) + \cdots, \\ y'\left(x + j\tfrac{h}{2}\right) &= y'(x) + \tfrac{jh}{2}y''(x) + \tfrac{(jh)^2}{2!\,2^2}y'''(x) \\ &\quad + \tfrac{(jh)^3}{3!\,2^3}y^{(4)}(x) + \cdots. \end{aligned}$$

Substituting these expansions into equation (14), we obtain

$$L\{y(x),h\} = \sum_{j=0}^{k}\Bigg[\sigma_j\Big(y(x)+\tfrac{jh}{2}y'(x) + \tfrac{(jh)^2}{2!\,2^2}y''(x)+\cdots\Big) - h\delta_j\Big(y'(x)+\tfrac{jh}{2}y''(x) + \tfrac{(jh)^2}{2!\,2^2}y'''(x)+\cdots\Big)\Bigg]. \tag{15}$$

Collecting like terms in powers of h, we obtain

$$L\{y(x),h\} = A_0y(x_n) + A_1hy'(x_n) + A_2h^2y''(x_n) + A_3h^3y'''(x_n) + \cdots . \tag{16}$$

where the coefficients A_m are given by

$$A_0 = \sum_{j=0}^{k}\sigma_j,$$
$$A_1 = \tfrac{1}{2}\sum_{j=0}^{k}j\sigma_j - \sum_{j=0}^{k}\delta_j,$$
$$A_m = \tfrac{1}{2^m m!}\sum_{j=0}^{k}j^m\sigma_j - \tfrac{1}{2^{m-1}(m-1)!}\sum_{j=0}^{k}j^{m-1}\delta_j, \quad m\geq 2.$$

Definition 3.1 (Order)**.** A linear multistep method (LMM) is defined to be of order p if the order conditions $A_0 = A_1 = A_2 = \cdots = A_p = 0$ and $A_{p+1}\neq 0$ are satisfied [19].

Using the previously obtained coefficients, direct computation of the sums $A_0, A_1, \dots, A_5$ gives, for the representative case,

$$A_0 = \sum_{j=0}^{7}\sigma_j = \sigma_1+\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_7 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \tag{17}$$

and, proceeding identically,

$$A_1 = A_2 = A_3 = A_4 = A_5 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}. \tag{18}$$

The first non-vanishing coefficient is

$$A_6 = \tfrac{1}{2^6 6!}\sum_{j=0}^{7}j^6\sigma_j - \tfrac{1}{2^5 5!}\sum_{j=0}^{7}j^5\delta_j = \begin{bmatrix}-\frac{1}{20}\frac{8\rho-3}{16\rho-3}\\ \frac{1}{45}\frac{9\rho-8}{9\rho+2}\\ -\frac{1}{12}\frac{8\rho-15}{16\rho+31}\\ \frac{1}{5}\frac{5\rho-24}{5\rho+54}\end{bmatrix} \neq \mathbf{0}. \tag{19}$$

Since $A_0 = A_1 = A_2 = A_3 = A_4 = A_5 = 0$ and $A_6\neq 0$, the method is of order $p = 5$. The error constant is given by the coefficient of the first non-vanishing term, A_6. Thus, the local truncation error (LTE) is

$$\text{LTE} = \pm A_6h^6y^{(6)}(x_n) + \dots \tag{20}$$

Remark. Setting $\rho = 0$ in A_6 recovers $C_6 = [-\frac{1}{20}, -\frac{4}{45}, \frac{5}{124}, -\frac{4}{45}]^\top$, confirming consistency with the 2BBDFO error constant.

Definition 3.2 (Consistency)**.** A linear multistep method is said to be consistent if it has order at least one. Equivalently, an LMM is consistent if the following conditions are satisfied [20]:

(i) $\displaystyle\sum_{j=0}^{7}\sigma_j = 0,$

(ii) $\displaystyle\sum_{j=0}^{7}j\sigma_j = 2\sum_{j=0}^{7}\delta_j.$

Proposition 3.1. The fully implicit 2-point superclass of block backward differentiation formula with off-step points (2SBBDFO) is consistent.

Proof. The consistency of the 2SBBDFO method is established by verifying that it satisfies the conditions stated in Definition 3.2. From the results obtained above, the method is of order five, which is greater than one; hence it is consistent. Moreover, using the previously defined vectors,

$$\sum_{j=0}^{7}\sigma_j = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \tag{21}$$

so condition (i) holds, and

$$\sum_{j=0}^{7}j\sigma_j = \begin{bmatrix}\frac{6(\rho+1)}{16\rho-3}\\ \frac{8(\rho+1)}{9(9\rho+2)}\\ \frac{30(\rho+1)}{16\rho+31}\\ \frac{24(\rho+1)}{5\rho+54}\end{bmatrix} = 2\sum_{j=0}^{7}\delta_j, \tag{22}$$

$$A = \begin{bmatrix} 1 - \dfrac{3\lambda h}{16\rho - 3} & -\dfrac{9}{4}\dfrac{4\rho - 3}{16\rho - 3} & \dfrac{1}{5}\dfrac{16\rho - 9}{16\rho - 3} & -\dfrac{1}{4}\dfrac{2\rho - 1}{16\rho - 3} \\ -\dfrac{4\lambda h\rho}{9(9\rho + 2)} - \dfrac{4}{9}\dfrac{9\rho + 16}{9\rho + 2} & 1 - \dfrac{4\lambda h}{9(9\rho + 2)} & -\dfrac{4}{15}\dfrac{9\rho - 16}{9\rho + 2} & \dfrac{1}{9}\dfrac{3\rho - 4}{9\rho + 2} \\ -\dfrac{5}{3}\dfrac{16\rho - 15}{16\rho + 31} & -\dfrac{15\lambda h\rho}{16\rho + 31} + \dfrac{15}{4}\dfrac{2\rho - 15}{16\rho + 31} & 1 - \dfrac{15\lambda h}{16\rho + 31} & -\dfrac{5}{12}\dfrac{4\rho - 15}{16\rho + 31} \\ \dfrac{4(5\rho - 16)}{5\rho + 54} & -\dfrac{9(5\rho - 12)}{5\rho + 54} & \dfrac{4}{5}\dfrac{31\rho - 144}{5\rho + 54} - \dfrac{12\lambda h\rho}{5\rho + 54} & 1 - \dfrac{12\lambda h}{5\rho + 54} \end{bmatrix}$$

so condition (ii) also holds. Therefore the 2SBBDFO method is consistent. □

4 Stability and Convergence Analysis of the Method

Stability analysis is crucial for evaluating the reliability of any LMM, especially for stiff IVPs. Numerical solutions can become unstable if errors are not controlled, making stability essential [3]. A method is stable if stepwise errors remain bounded and the solution stays close to the exact solution. For stiff ODEs, zero-stability and A-stability are key criteria, ensuring bounded error growth and stability for eigenvalues in the left-half complex plane [21].

Definition 4.1 (Zero-Stability)**.** An LMM is said to be zero-stable if all the roots of its first characteristic polynomial have moduli less than or equal to unity, and any root lying on the unit circle is simple [22].

Definition 4.2 (A-Stability)**.** A LMM is said to be A-stable if its absolute stability region encompasses the entire left half of the complex plane [19].

Definition 4.3 ($A(\alpha)$-Stability)**.** A LMM is said to be $A(\alpha)$-stable (nearly A-stable) for some $\alpha \in [0, \frac{\pi}{2})$ if the wedge $S_\alpha = \{z : |\mathrm{Arg}(-z)| < \alpha,\ z \neq 0\}$ is contained in its region of absolute stability [12].

The stability characteristics of the method are analyzed by applying the linear test equation $y' = \lambda y$, where λ is a complex constant with negative real part, $\mathrm{Re}(\lambda) < 0$. Substituting the linear test equation into (10) yields the system

$$AY_m = BY_{m-1}, \tag{23}$$

where

$$B = \begin{bmatrix} 0 & \frac{1}{20}\frac{4\rho-1}{16\rho-3} & 0 & \frac{1}{4}\frac{38\rho+9}{16\rho-3} + \frac{3\lambda h\rho}{16\rho-3} \\ 0 & -\frac{1}{45}\frac{3\rho-2}{9\rho+2} & 0 & \frac{1}{3}\frac{9\rho-4}{9\rho+2} \\ 0 & \frac{1}{12}\frac{2\rho-3}{16\rho+31} & 0 & -\frac{5}{4}\frac{4\rho-5}{16\rho+31} \\ 0 & -\frac{1}{5}\frac{\rho-4}{5\rho+54} & 0 & \frac{5\rho-18}{5\rho+54} \end{bmatrix}.$$

Here, if m is the number of blocks and r the number of points per block, then $n = mr$ with $r = 2$ and $n = 2m$; the block indexing follows [23].

The stability polynomial $\gamma(t, \bar{h})$ of the method, where $\bar{h} = \lambda h$ and the free parameter is chosen as $\rho = -\frac{3}{4}$, is obtained by evaluating

$$\gamma(t, \bar{h}) = \det(At - B). \tag{24}$$

This implies that

$$\begin{aligned} \gamma(t, \bar{h}) = & -\tfrac{32}{1088415}t^2 \Big(-1080\bar{h}^4t^2 + \tfrac{10935}{32}\bar{h}^4t \\ & + \tfrac{232029}{64}\bar{h}^3t^2 - \tfrac{8091}{8}\bar{h}^3t - \tfrac{4617}{128}\bar{h}^3 \\ & - \tfrac{1021425}{128}\bar{h}^2t^2 + \tfrac{936117}{256}\bar{h}^2t - \tfrac{64347}{256}\bar{h}^2 \\ & + \tfrac{90193}{8}\bar{h}t^2 - \tfrac{3645}{8}\bar{h}t - \tfrac{25747}{32}\bar{h} \\ & - \tfrac{62189}{8}t^2 + \tfrac{565083}{64}t - \tfrac{67571}{64} \Big) = 0. \end{aligned} \tag{25}$$

The first characteristic polynomial $\pi(t)$ is obtained by setting $\bar{h} = 0$ in (25) and removing the nonzero scalar factor, yielding

$$\pi(t) = t^2 \left(-\tfrac{67571}{64} - \tfrac{62189}{8}t^2 + \tfrac{565083}{64}t\right) = 0. \tag{26}$$

Solving (26) for t gives the roots

$$t = 0, \quad t = 0, \quad t = 1, \quad t = 0.1358178295.$$

These roots confirm that the 2SBBDFO method is zero-stable, since all the roots of the first characteristic polynomial lie within or on the unit circle, and the root with modulus one is simple (non-repeated).

The boundary of the region of absolute stability for the 2SBBDFO method, corresponding to $\rho = -\frac{3}{4}$, is determined by the boundary-locus technique, which substitutes $t = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, into the stability polynomial (25). The resulting stability region is plotted using Maple 18 and is shown in Figure 1.

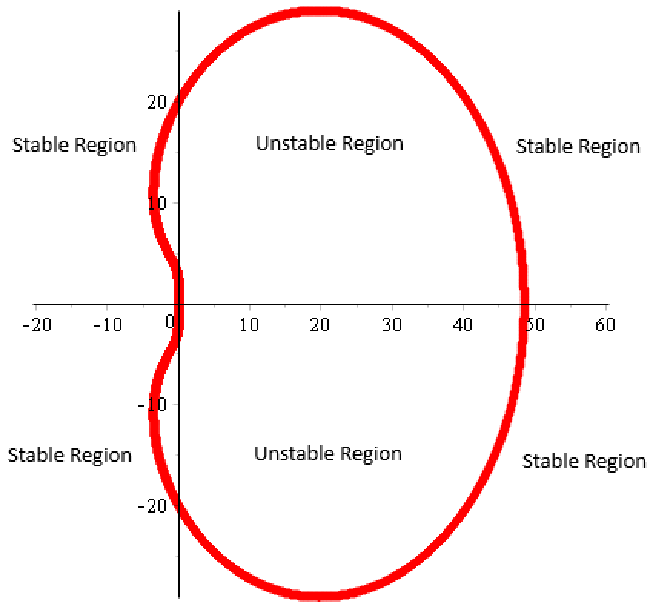

Figure 1. Stability region of 2SBBDFO.

The stability region of the method extends beyond the circular boundary, indicating that the 2SBBDFO method is nearly A-stable, as it covers almost the entire left half of the complex plane. Having satisfied both the zero-stability condition and exhibiting near A-stability, the 2SBBDFO method is well-suited for the numerical solution of first-order stiff ODEs.

Theorem 4.1 (Dahlquist Equivalence Theorem)**.** *A block numerical method is convergent if and only if it is both consistent and zero-stable* [24].

Theorem 4.2. *The 2SBBDFO method is convergent.*

Proof. By Theorem 4.1, a linear multistep method is convergent if and only if it is both consistent and zero-stable. The 2SBBDFO method satisfies the consistency conditions and is of order five, implying that the local truncation error tends to zero as $h \to 0$. Moreover, all roots of its first characteristic polynomial satisfy $|t| \leq 1$, with simple roots on the unit circle, confirming zero-stability. Hence the method is convergent, and the global error approaches zero as $h \to 0$. □

5 Implementation of the Method

Newton's iteration is applied for the implementation of the method. We first define the error.

Definition 5.1 (Absolute Error)**.** Let y_i and $y(x_i)$ be the approximate and exact solutions of (1), respectively. Then the absolute error is

$$(\text{error}_i)_t = |(y_i)_t - (y(x_i))_t| \,. \tag{27}$$

The maximum error is defined by

$$\text{MAXE} = \max_{1\le i\le T}\left(\max_{1\le t\le N}(\text{error}_i)_t\right), \tag{28}$$

where T is the total number of steps and N is the number of equations.

By substituting $\rho = -\frac{3}{4}$ into equation (10), the function evaluations at both off-step and grid points are obtained and denoted by $F_{\frac{1}{2}}$, F_1, $F_{\frac{3}{2}}$, and F_2, where

$$\left.\begin{aligned}
F_{\frac{1}{2}} &= y_{n+\frac{1}{2}} - \tfrac{9}{10}y_{n+1} + \tfrac{7}{25}y_{n+\frac{3}{2}} \\
&\quad - \tfrac{1}{24}y_{n+2} + \tfrac{1}{5}hf_{n+\frac{1}{2}} - \tfrac{3}{20}hf_n + \xi_{\frac{1}{2}}, \\
F_1 &= y_{n+1} + \tfrac{148}{171}y_{n+\frac{1}{2}} - \tfrac{364}{285}y_{n+\frac{3}{2}} \\
&\quad + \tfrac{25}{171}y_{n+2} + \tfrac{16}{171}hf_{n+1} - \tfrac{12}{171}hf_{n+\frac{1}{2}} + \xi_1, \\
F_{\frac{3}{2}} &= y_{n+\frac{3}{2}} + \tfrac{45}{19}y_{n+\frac{1}{2}} - \tfrac{495}{152}y_{n+1} \\
&\quad + \tfrac{15}{38}y_{n+2} - \tfrac{15}{19}hf_{n+\frac{3}{2}} + \tfrac{45}{76}hf_{n+1} + \xi_{\frac{3}{2}}, \\
F_2 &= y_{n+2} - \tfrac{316}{201}y_{n+\frac{1}{2}} + \tfrac{189}{67}y_{n+1} \\
&\quad - \tfrac{892}{335}y_{n+\frac{3}{2}} - \tfrac{16}{67}hf_{n+2} + \tfrac{12}{67}hf_{n+\frac{3}{2}} + \xi_2.
\end{aligned}\right\} \tag{29}$$

and $\xi_{\frac{1}{2}}$, ξ_1, $\xi_{\frac{3}{2}}$, ξ_2 are the backward values

$$\left.\begin{aligned}
\xi_{\frac{1}{2}} &= -\tfrac{1}{75}y_{n-1} - \tfrac{13}{40}y_n \\
\xi_1 &= \tfrac{17}{855}y_{n-1} - \tfrac{43}{57}y_n \\
\xi_{\frac{3}{2}} &= \tfrac{3}{152}y_{n-1} - \tfrac{10}{19}y_n \\
\xi_2 &= -\tfrac{19}{1005}y_{n-1} + \tfrac{29}{67}y_n
\end{aligned}\right\}. \tag{30}$$

Let $y_{n+j}^{(i+1)}$, $j = \frac{1}{2}, 1, \frac{3}{2}, 2$, denote the $(i+1)$-th iterative values of y_{n+j} and define

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, \quad j = \tfrac{1}{2}, 1, \tfrac{3}{2}, 2. \tag{31}$$

Newton's iteration for the 2SBBDFO takes the form

$$\begin{bmatrix} y_{n+\frac{1}{2}}^{(i+1)} \\ y_{n+1}^{(i+1)} \\ y_{n+\frac{3}{2}}^{(i+1)} \\ y_{n+2}^{(i+1)} \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{2}}^{(i)} \\ y_{n+1}^{(i)} \\ y_{n+\frac{3}{2}}^{(i)} \\ y_{n+2}^{(i)} \end{bmatrix} - J^{-1}\begin{bmatrix} F_{\frac{1}{2}}^{(i)} \\ F_1^{(i)} \\ F_{\frac{3}{2}}^{(i)} \\ F_2^{(i)} \end{bmatrix}, \tag{32}$$

which can be written as $J\,e^{(i+1)} = -F^{(i)}$, where J denotes the Jacobian matrix

$$J = \begin{bmatrix} 1+\frac{1}{5}h\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}} & -\frac{9}{10} & \frac{7}{25} & -\frac{1}{24} \\ \frac{148}{171}-\frac{12}{171}h\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}} & 1+\frac{16}{171}h\frac{\partial f_{n+1}}{\partial y_{n+1}} & -\frac{364}{285} & \frac{25}{171} \\ \frac{45}{19} & -\frac{495}{152}+\frac{45}{76}h\frac{\partial f_{n+1}}{\partial y_{n+1}} & 1-\frac{15}{19}h\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}} & \frac{15}{38} \\ -\frac{316}{201} & \frac{189}{67} & -\frac{892}{335}+\frac{12}{67}h\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}} & 1-\frac{16}{67}h\frac{\partial f_{n+2}}{\partial y_{n+2}} \end{bmatrix}. \tag{33}$$

A computer code in the C programming language was written for the implementation of the method.

6 Test Problems and Numerical Results

The following stiff IVPs are selected to evaluate and compare the performance of the proposed method. These problems consist of linear first-order stiff systems obtained from the existing literature and are widely recognized as standard benchmark tests for assessing the accuracy, stability, and computational efficiency of numerical methods designed for stiff differential equations. The notations and abbreviations used in reporting the numerical results are summarized in Table 2.

Table 2. Notations used in the numerical result tables.

Notation	Meaning
h	Step size
METHOD	The methods used
MAXE	Maximum absolute error
CPU TIME	Computation time in seconds
2BBDF(5)	Fifth-order 2-point block backward differentiation formula [25]
2BBDFO	2-point block backward differentiation formula with off-step points [8]
2SBBDFO	Fully implicit 2-point superclass of block backward differentiation formula with off-step points (proposed)

Problem 1. A linear stiff system [26]:

$$\begin{aligned} y_1' &= -20y_1 - 19y_2, \quad y_1(0) = 2, \quad 0 \le x \le 20, \\ y_2' &= -19y_1 - 20y_2, \quad y_2(0) = 0. \end{aligned}$$

The exact solutions are $y_1(x) = e^{-39x} + e^{-x}$ and $y_2(x) = e^{-39x} - e^{-x}$. The system matrix has negative real eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -39$, which lead to rapidly decaying transient components, thereby exhibiting stiff behavior.

Problem 2. A highly stiff problem taken from [7]:

$$\begin{aligned} y_1' &= 198y_1 + 199y_2, \quad y_1(0) = 1, \quad 0 \le x \le 10, \\ y_2' &= -398y_1 - 399y_2, \quad y_2(0) = -1. \end{aligned}$$

The exact analytical solutions are $y_1(x) = e^{-x}$ and $y_2(x) = -e^{-x}$. The system has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -200$, both real and negative, indicating strong stiffness due to widely separated decay rates.

Problem 3. A severely stiff problem taken from [27]:

$$\begin{aligned} y_1' &= 32y_1 + 66y_2 + \tfrac{2}{3}x + \tfrac{2}{3}, \quad y_1(0) = \tfrac{1}{3}, \quad 0 \le x \le 1, \\ y_2' &= -66y_1 - 133y_2 - \tfrac{1}{3}x - \tfrac{1}{3}, \quad y_2(0) = \tfrac{1}{3}, \end{aligned}$$

whose exact solutions are $y_1(x) = \frac{2}{3}x + \frac{2}{3}e^{-x} - \frac{1}{3}e^{-100x}$ and $y_2(x) = -\frac{1}{3}x - \frac{1}{3}e^{-x} + \frac{2}{3}e^{-100x}$. The system has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -100$, which are real and negative, leading to purely exponential decay with multiple time scales. The coexistence of rapidly decaying and slowly varying components induces strong stiffness, although no oscillatory behavior is present in the exact solution.

The problems above are solved using the fifth-order 2-point block backward differentiation formula (2BBDF(5)), the 2-point block backward differentiation formula with off-step points (2BBDFO), and the proposed fully implicit 2-point superclass of block backward differentiation formula with off-step points (2SBBDFO). To compare accuracy and efficiency, the maximum absolute errors and computation times are reported for different step sizes h in Tables 3–5.

For a clearer visual comparison, the relationship between accuracy and step size is illustrated by plotting $\log_{10}(\mathrm{MAXE})$ versus h for the three methods, as shown in Figures 2–4.

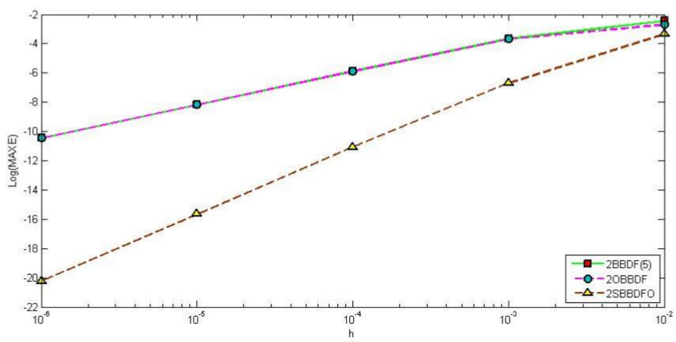

Figure 2. Accuracy graph for Problem 1.

7 Discussion

The numerical results reported in Tables 3–5 demonstrate that the proposed method outperforms

Table 3. Numerical results for Problem 1.

h	METHOD	MAXE	CPU TIME
10^{-2}	2BBDF(5)	8.81087E−02	1.96250E−03
	2BBDFO	7.00088E−02	3.07107E−03
	2SBBDFO	3.59295E−02	2.29500E−01
10^{-3}	2BBDF(5)	2.61263E−02	2.26088E−02
	2BBDFO	2.58857E−02	3.00581E−02
	2SBBDFO	1.25911E−03	4.12800E−01
10^{-4}	2BBDF(5)	2.84765E−03	1.94097E−01
	2BBDFO	2.84492E−03	2.92691E−01
	2SBBDFO	1.56717E−05	2.70500E+00
10^{-5}	2BBDF(5)	2.87178E−04	2.80007E+00
	2BBDFO	2.87150E−04	4.18854E+00
	2SBBDFO	1.62279E−07	2.54400E+01
10^{-6}	2BBDF(5)	2.87419E−05	3.34886E+01
	2BBDFO	2.87417E−05	5.30707E+01
	2SBBDFO	1.63066E−09	2.54000E+02

Table 4. Numerical results for Problem 2.

h	METHOD	MAXE	CPU TIME
10^{-2}	2BBDF(5)	7.13926E−03	1.22776E−03
	2BBDFO	7.17251E−03	3.79395E−03
	2SBBDFO	9.81843E−05	1.76900E−01
10^{-3}	2BBDF(5)	7.33994E−04	1.66640E−02
	2BBDFO	7.33813E−04	2.66598E−02
	2SBBDFO	1.05866E−06	3.71700E−01
10^{-4}	2BBDF(5)	7.35582E−05	9.89231E−02
	2BBDFO	7.35564E−05	1.38798E−01
	2SBBDFO	1.07041E−08	1.49300E+00
10^{-5}	2BBDF(5)	7.35741E−06	1.06522E+00
	2BBDFO	7.35740E−06	2.46586E+00
	2SBBDFO	1.07172E−10	1.28800E+01
10^{-6}	2BBDF(5)	7.35744E−07	1.83490E+01
	2BBDFO	7.35775E−07	2.63500E+01
	2SBBDFO	7.34015E−10	1.27800E+02

Table 5. Numerical results for Problem 3.

h	METHOD	MAXE	CPU TIME
10^{-2}	2BBDF(5)	1.21580E−02	3.32570E−04
	2BBDFO	1.20347E−02	5.87940E−04
	2SBBDFO	1.59223E−02	1.82500E−01
10^{-3}	2BBDF(5)	3.79477E−02	3.76255E−03
	2BBDFO	3.72766E−02	5.66602E−03
	2SBBDFO	4.20242E−03	1.62600E−01
10^{-4}	2BBDF(5)	4.78743E−03	1.32177E−02
	2BBDFO	4.77571E−03	2.15800E−02
	2SBBDFO	6.54527E−05	2.97000E−01
10^{-5}	2BBDF(5)	4.89264E−04	8.46313E−02
	2BBDFO	4.89142E−04	1.44442E−01
	2SBBDFO	7.05740E−07	1.50900E+00
10^{-6}	2BBDF(5)	4.90322E−05	9.19200E−01
	2BBDFO	4.90310E−05	2.37217E+00
	2SBBDFO	7.13570E−09	1.42300E+01

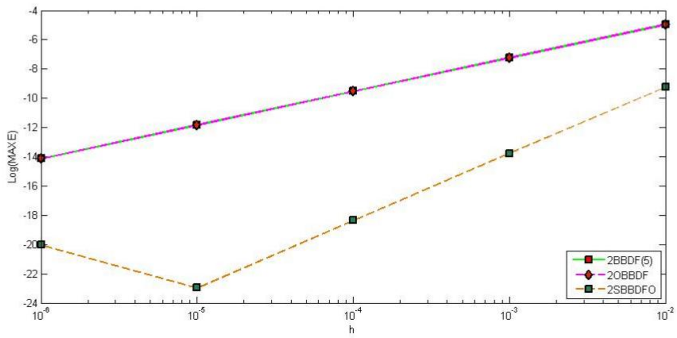

Figure 3. Accuracy graph for Problem 2.

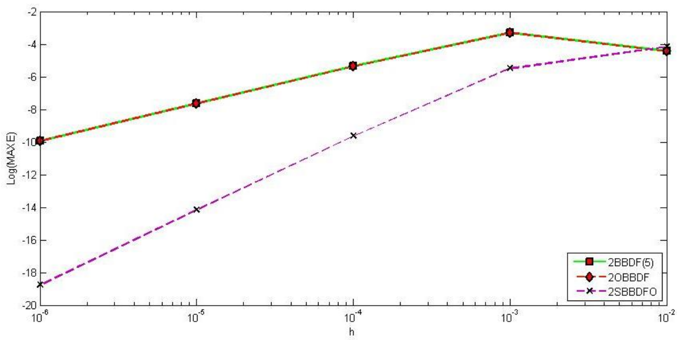

Figure 4. Accuracy graph for Problem 3.

the existing 2BBDF(5) and 2BBDFO schemes in accuracy for the great majority of the step sizes considered. In particular, for Problems 1 and 2, 2SBBDFO yields the lowest maximum absolute error (MAXE) at every step size tested, indicating a higher degree of precision. The only exception occurs for Problem 3 at the largest step size $h = 10^{-2}$, where 2SBBDFO (MAXE $= 1.59223\text{E}{-}02$) is slightly less accurate than 2BBDF(5) ($1.21580\text{E}{-}02$) and 2BBDFO ($1.20347\text{E}{-}02$); as the step size is refined, however, 2SBBDFO again attains the smallest errors by a wide margin. A general improvement in accuracy is observed as the step size is reduced, reflecting the strong convergence of the proposed formulation, except at the smallest step size of Problem 2 ($h = 10^{-6}$), where round-off error begins to dominate.

On the other hand, the improved accuracy of 2SBBDFO comes with an increase in computational effort. The method requires more execution time than both 2BBDF(5) and 2BBDFO, and this computational cost grows as the step size becomes smaller. This trade-off suggests that, while the method is highly accurate, it is also more computationally intensive, particularly for finer discretizations.

Overall, the findings indicate that 2SBBDFO is a robust and highly accurate method for solving stiff initial value problems. It provides superior error control and improved numerical performance compared to the competing methods, making it particularly suitable for applications where accuracy is prioritized over computational efficiency.

To further illustrate these observations, graphical comparisons were carried out by plotting $\log_{10}(\mathrm{MAXE})$ against the step size h for all three methods (Figures 2–4). The graphs reveal a consistent trend in which the error increases with the step size for all methods considered. Across the range of step sizes, 2SBBDFO generally maintains lower error levels than both 2BBDF(5) and 2BBDFO, providing further visual evidence of the accuracy and reliability of the proposed method.

8 Conclusion

This study introduces a modified fully implicit hybrid block method, 2SBBDFO, for solving stiff ordinary differential equations. The key contribution of this work lies in extending the classical two-point block backward differentiation formula through the incorporation of two off-step points and a free parameter ρ, which enhances the flexibility of the scheme and enables efficient simultaneous computation within each block.

Theoretical analysis establishes that the method is consistent, zero-stable, and convergent, while attaining fifth-order accuracy. In addition, the scheme is shown to be nearly A-stable ($\mathrm{A}(\alpha)$-stable), ensuring reliable performance for stiff systems. Numerical results further demonstrate that the proposed method achieves significantly lower maximum absolute errors and improved stability characteristics compared to existing block methods, with only a modest increase in computational cost.

Overall, the 2SBBDFO method provides a highly accurate, stable, and robust framework for the numerical integration of stiff IVPs and offers a strong basis for the development of more advanced block methods.

Data Availability Statement

The data used to support the findings of this study are available from the corresponding author upon request.

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Conflicts of Interest

The authors declare no conflicts of interest.

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Ethical Approval and Consent to Participate

Not applicable.

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