RESEARCH ARTICLE

# A New Natural Boundary Element Method for the 2D Viscoelastic Wave Equation

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## Abstract

**The natural boundary element (NBE) method is herein mainly adopted to compute the numerical solutions of the viscoelastic wave equation in a two-dimensional (2D) unbounded domain. To this end, a new time semi-discrete (TSD) scheme is constructed for the viscoelastic wave equation, and the existence, stability, and error estimates of the time semi-discretized solutions are analyzed. Subsequently, based on the natural boundary naturalization principle, a fully discrete NBE scheme is established. The existence and stability of the fully discrete NBE solutions are examined, and the errors between the analytical solution and the fully discrete NBE solutions are estimated. Finally, several numerical experiments are conducted to verify the effectiveness of the NBE method in solving the viscoelastic wave equation in the 2D unbounded domain.**

**Keywords**: natural boundary element, viscoelastic wave equation, existence and stability, error estimate.

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## 1 Introduction

Let $\Omega \subset R^2$ be a connected and bounded domain with the boundary $\Sigma := \partial\Omega$, $\Omega^c := R^2 \backslash \bar{\Omega}$ be the outer region, $\boldsymbol{z} = (x, y)$, and $|\boldsymbol{z}| = \sqrt{x^2 + y^2}$. For the known time upper bound $T$, the following 2D viscoelastic wave equation is considered.

**Problem 1.** *Seek $u$ from the following equation*

$$\begin{cases} u_{tt} - \alpha\Delta u_t - \beta\Delta u \\ = f_1(\boldsymbol{z}, t), \quad (\boldsymbol{z}, t) \in \Omega^c \times [0, T], \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_1(\boldsymbol{z}, t), \quad (\boldsymbol{z}, t) \in \Sigma \times [0, T], \\ u(\boldsymbol{z}, 0) = u_0(\boldsymbol{z}),\ u_t(\boldsymbol{z}, 0) = u_1(\boldsymbol{z}), \boldsymbol{z} \in \Omega^c, \end{cases} \tag{1}$$

*in which $u_{tt} = \partial^2 u/\partial t^2$, $u_t = \partial u/\partial t$, $\alpha$ and $\beta$ are two positive constants, $f(\boldsymbol{z}, t)$ and $g(\boldsymbol{z}, t)$ are separately the source term and the boundary value function, $u_0(\boldsymbol{z})$ and $u_1(\boldsymbol{z})$ are two given initial value functions, $\partial/\partial\boldsymbol{n}$ is the external normal derivative, and $\boldsymbol{n}$ is a unit normal vector on the boundary $\Sigma$ from the outer domain $\Omega^c$ toward the inner region $\Omega$.*

Problem 1 is an important topic in both mathematics and physics, with wide-ranging applications. It is often used to describe the propagation of seismic waves underground and radiation phenomena in nature [1]. However, when the computational domain is irregular, the initial and boundary conditions are complex, or the source function is complicated, obtaining an

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analytical solution becomes infeasible. Therefore, seeking numerical solutions presents a practical and effective alternative.

For the solution of the viscoelastic wave equation, there are already many numerical methods such as the finite element (FE) method [2], the finite volume element (FVE) method [3], the finite difference (FD) scheme [4]. However, the above-mentioned methods can only calculate the inner problems with the bounded computational domain, and the same is true of the traditional boundary element (TBE) method. While on the contrary, the NBE method initially proposed by Feng and Yu [5–7] in the late 1970s can be accustomed to calculate the outer problem with unbounded infinity domain. Therefore, the NBE method is more useful than the TBE method and has more special advantages than the TBE method. For instance, maintains the invariance of the energy functional form, has a sole form for thr boundary integral equation, possesses excellent numerical stability, needs not to calculate a large number of singular integrations in practice, and uses the same variation principle as the FE method. It can also couple with FE method to solve more complex partial differential equations (PDEs). It has been used to solve the second-order elliptic equation [5–7], the standard parabolic equation [8], the Sobolev equation [9], the uniform high-voltage transmission line problem [10], and hyperbolic equation (see [11]).

Although a reduced-order NBE method for the viscoelastic wave equation has been studied in [12], unfortunately, the classical NBE method of the viscoelastic wave equation has been studies, and the existence, stability, and error estimations for the NBE solutions has yet not been discussed. Therefore, this paper is to build the classical NBE method of the viscoelastic wave equation and provide the existence, stability, and error estimations for the NBE solutions, which is a very necessary supplement to the research on the viscoelastic wave equation. Especially, the viscoelastic wave equation not only includes the second-order derivative terms about time and spatial variables, but also contains some mixed derivative terms about the first-order derivative of time and the second-order derivatives of spatial variables. Hence, either the construction for the NBE method or the theory analysis of the existence, stability, and errors for the NBE solutions require more techniques and faces more difficulties than the second-order elliptic equation in [5–7], the standard parabolic equation in [8], the Sobolev equation in [9], the uniform high-voltage transmission line problem in [10], and the hyperbolic equation in [11], but as mentioned above, it has very important applications. Hence, it is worth to research the NBE method for the viscoelastic wave equation in the 2D unbounded domain, which can fill in the study gap of the classical NBE method for the viscoelastic wave equation.

To this end, Section 2 presents the construction of a time semi-discrete (TSD) method with second-order accuracy in time for the viscoelastic wave equation in the 2D unbounded region. The existence, stability, and error estimates of the TSD solutions are analyzed. In Section 3, a new fully discrete natural boundary element (NBE) method is developed based on the natural boundary naturalization theory. The existence and stability of the fully discrete NBE solutions are discussed, and the errors between the analytical solution and the fully discrete NBE solutions are estimated. Section 4 provides numerical simulations to verify the consistency between the numerical results and the theoretical analysis. Finally, the main conclusions are summarized in Section 5.

## 2 The TSD method and the existence, stability, and error estimates for the TSD solutions

The Sobolev spaces and their norms used subsequently are classical (see [13]). The semi-discrete formulation in time may be stated as follows.

**Problem 2.** *$\forall t \in (0,T)$, seek $u(t) \in H^1(\Omega^c)$ from the following system:*

$$\begin{aligned}&(u_{tt},\nu)+\alpha(\nabla u_t,\nabla\nu)+\beta(\nabla u,\nabla\nu)\\&=(f_1,\nu)+\beta\left\langle g_1,\nu\right\rangle,\ \ \forall\nu\in H^1(\Omega^c),\end{aligned}\qquad(2)$$

*herein $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ indicate the inner product in $L^2(\Omega^c)$ and $L^2(\Sigma)$ separately.*

The uniqueness and existence of the solution for Problem 2 have been provided in [4].

Supposed that $N$ is a positive integer, $\tau = T/N$ is the time-step, $u_{tt}^k$ is approximated by $(u^{k+1}-2u^k+u^{k-1})/\tau^2$, $u_t^k$ is approximated by $(u^{k+1}-u^{k-1})/2\tau$, and $u^k$ is approximated by $(u^{k+1}+u^{k-1})/2$. Thus, a new TSD scheme can be constructed as follows.

**Problem 3.** *Calculate $u^{k+1}\in H^1(\Omega^c)$ from the following system*

$$\begin{aligned}&(u^{k+1}-2u^k+u^{k-1},\nu)+\frac{\tau\alpha}{2}(\nabla(u^{k+1}-u^{k-1}),\nabla\nu)\\&+\frac{\tau^2\beta}{2}(\nabla(u^{k+1}+u^{k-1}),\nabla\nu)\\&=\tau^2(f_1^k,\nu)+\tau^2\beta\langle g_1^k,\nu\rangle,\end{aligned}$$

$$\forall v \in H^1(\Omega^c),\ k = 1, 2, \cdots, N, \quad (3)$$
$$u^0 = u_0(\boldsymbol{z}),\ u^1 = u_0(\boldsymbol{z}) + \tau u_1(\boldsymbol{z}),\ \boldsymbol{z} \in \Omega^c, \quad (4)$$

*where* $f_1^k = f_1(\boldsymbol{z}, t_k)$ *and* $g_1^k = (\boldsymbol{z}, t_k)$.

For Problem 3, the existence, stability, and error results of the TSD solutions are established as follows.

**Theorem 1.** *If* $f_1 \in L^2(0,T; L^2(\Omega^c))$, $g_1 \in L^2(0,T; L^2(\Sigma))$, *and* $u_0$, $u_1 \in H^1(\Omega^c)$, *then a unique solution set* $\{u^k\}_{k=1}^N \subset H^1(\Omega^c)$ *for Problem 3 to satisfy the following stability:*

$$\|\nabla u^k\|_0 \leqslant \Big(\beta^{-1}\|u_1\|_0^2 + \|\nabla u_0\|_0^2 + \|\nabla u_1\|_0^2 + T[2\tau\beta^{-1}\|f_1^k\|_0^2 + C_1\beta\alpha^{-1}\|g_1^k\|_{0,\Sigma}^2]\Big)^{\frac{1}{2}}, \quad (5)$$

*where* $k = 1, 2, \cdots, N$, $C_1$ *is the nonnegative constant in the trace theorem. Moreover, when* $u \in H^4(0,T; L^2(\Omega)) \cap H^3(0,T; H^1(\Omega))$, *then the following error estimates are obtained:*

$$\|\nabla(u(t_k) - u^k)\|_0 = \|\nabla e^k\|_0 \leqslant C\tau^2, \quad (6)$$

*where* $k = 1, 2, \cdots, N$, $C^2 = T\beta^{-1}(\tau/72\|u^{(4)}(\xi_1^k)\|_0^2 + (16\alpha)^{-1}\|\nabla u^{(3)}(\xi_2^k)\|_0^2 + (8\alpha)^{-1}\|\nabla u_{tt}(\xi_3^2)\|_0^2)$, $t_{k-1} \leqslant \xi_1^k, \xi_2^k, \xi_3^k \leqslant t_{k+1}$, *and* $u(t_k)$ *are the cases of* $u$ *at* $t = t_k$.

*Proof.* Since Problem 3 is a linear system on unknown function $u$, to demonstrate the existence and uniqueness of solutions for Problem 3, it is only necessary to demonstrate that it has only a zero solution when $f_1 = g_1 = u_0 = u_1 = 0$.

Taking $\nu = u^{k+1} - u^{k-1}$ in Problem 3, and using the Cauchy-Schwarz inequality and the trace theorem in [13], the following estimate is obtained:

$$\begin{aligned}&\|u^{k+1} - u^k\|_0^2 - \|u^k - u^{k-1}\|_0^2 \\ &+\tau^2\beta[\|\nabla u^{k+1}\|_0^2 - \|u^{k-1}\|_0^2] \\ &\leqslant 2\tau^3[\tau\|f_1^k\|_0^2 + C_1\beta^2\alpha^{-1}\|g_1^k\|_{0,\Sigma}^2],\end{aligned} \quad (7)$$

where $C_1$ is the nonnegative constant in the trace theorem in [13].

Summing (7) from 1 until $k$, the following result is obtained:

$$\begin{aligned}&\|u^{k+1} - u^k\|_0^2 + \tau^2\beta[\|\nabla u^{k+1}\|_0^2 + \|\nabla u^k\|_0^2] \\ &\leqslant \tau^2[\|u_1\|_0^2 + \beta(\|\nabla u_0\|_0^2 + \|\nabla u_1\|_0^2) + 2T\tau\|f_1^k\|_0^2 \\ &+C_1T\beta^2\alpha^{-1}\|g_1^k\|_{0,\Sigma}^2],\ 1 \leqslant k \leqslant N.\end{aligned} \quad (8)$$

Thus, when $f_1 = g_1 = u_0 = u_1 = 0$, from (8) can gain $\|\nabla u^k\|_0 = 0$. It follows that $u^k = 0$. Hence, Problem 3 has a sole solution set.

It can be obtained from (8) that

$$\begin{aligned}&\|\nabla u^k\|_0^2 \leqslant \|u_1\|_0^2\beta^{-1} + \|\nabla u_0\|_0^2 + \|\nabla u_1\|_0^2 \\ &+T[2\tau\beta^{-1}\|f_1^k\|_0^2 + C_1\beta\alpha^{-1}\|g_1^k\|_{0,\Sigma}^2].\end{aligned} \quad (9)$$

Using Taylor's expansion formula, one can obtain

$$u_{tt}(t_k) = \frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2} - \frac{\tau^2}{12}u^{(4)}(\xi_1^k), \quad (10)$$

$$u_t(t_k) = \frac{u(t_{k+1}) - u(t_{k-1})}{2\tau} - \frac{\tau^2}{6}u^{(3)}(\xi_2^k), \quad (11)$$
$$u(t_k) = \frac{u(t_{k+1}) + u(t_{k-1})}{2} - \frac{\tau^2}{4}u_{tt}(\xi_3^k), \quad (12)$$

where $t_{k-1} \leqslant \xi_1^k, \xi_2^k, \xi_3^k \leqslant t_{k+1}$.

From Problem 2 one can obtain

$$\begin{aligned}&(u(t_{k+1}) - u(t_k), \nu) - (u(t_k) - u(t_{k-1}), \nu) \\ &+\frac{\alpha\tau}{2}(\nabla(u(t_{k+1}) - u(t_{k-1})), \nabla\nu) \\ &+\frac{\beta\tau^2}{2}(\nabla(u(t_{k+1}) + u(t_{k-1})), \nabla\nu) \\ &= (\tau^2 f_1(t_k), \nu) + \tau^2\beta\langle g_1(t_k), \nu\rangle \\ &+\frac{\tau^4}{12}(u^{(4)}(\xi_1^k), \nu) + \frac{\tau^4}{6}(\nabla u^{(3)}(\xi_2^k), \nabla\nu) \\ &+\frac{\tau^4}{4}(\nabla u_{tt}(\xi_3^k), \nabla\nu), \forall\nu \in H^1(\Omega^c).\end{aligned} \quad (13)$$

Let $e^{k+1} = u(t_{k+1}) - u^{k+1}$. Subtracting (3) from (13) and taking $t = t_{k+1}$, one can obtain

$$\begin{aligned}&(e^{k+1} - e^k, \nu) + (e^k - e^{k-1}, \nu) + \frac{\alpha\tau}{2}(\nabla e^{k+1}, \nabla\nu) \\ &-\frac{\alpha\tau}{2}(\nabla e^{k-1}, \nabla\nu) + \frac{\beta\tau^2}{2}(\nabla(e^{k+1} + e^{k-1}), \nabla\nu) \\ &= \frac{\tau^4}{12}(u^{(4)}(\xi_1^k), \nu) + \frac{\tau^4}{6}(\nabla u^{(3)}(\xi_2^k), \nabla\nu) \\ &+\frac{\tau^4}{4}(\nabla u_{tt}(\xi_3^k), \nabla\nu).\end{aligned} \quad (14)$$

Taking $\nu = e^{k+1} - e^{k-1}$ in (14) and using the Cauchy-Schwarz inequality yield that

$$\begin{aligned}&\|e^{k+1} - e^k\|_0^2 - \|e^k - e^{k-1}\|_0^2 \\ &+\frac{\alpha\tau}{2}\|\nabla(e^{k+1} - e^{k-1})\|_0^2 \\ &+\frac{\beta\tau^2}{2}(\|\nabla e^{k+1}\|_0^2 - \|\nabla e^{k-1}\|_0^2) \\ &\leqslant \frac{\tau^4}{12}\|u^{(4)}(\xi_1^k)\|_0(\|e^{k+1} - e^k\|_0 + \|e^k - e^{k-1}\|_0) \\ &+\frac{\tau^4}{6}\|\nabla u^{(3)}(\xi_2^k)\|_0\|\nabla(e^{k+1} - e^{k-1})\|_0 \\ &+\frac{\tau^4}{4}\|\nabla u_{tt}(\xi_3^k)\|_0\|\nabla(e^{k+1} - e^{k-1})\|_0.\end{aligned} \quad (15)$$

Thus, it follows that

$$\begin{aligned}&\|e^{k+1}-e^k\|_0^2-\|e^k-e^{k-1}\|_0^2\\&+\beta\tau^2(\|\nabla e^{k+1}\|_0^2-\beta\tau^2\|\nabla e^{k-1}\|_0^2)\\&\leqslant\frac{\tau^8}{72}\|u^{(4)}(\xi_1^k)\|_0^2+\frac{\tau^7}{16\alpha}\|\nabla u^{(3)}(\xi_2^k)\|_0^2\\&+\frac{\tau^7}{8\alpha}\|\nabla u_{tt}(\xi_3^k)\|_0^2.\end{aligned}\tag{16}$$

Summating (16) from 1 until $k$ yields

$$\begin{aligned}&\|e^{k+1}-e^k\|_0^2+\beta\tau^2(\|\nabla e^{k+1}\|_0^2+\|\nabla e^k\|_0^2)\\&\leqslant T[\frac{\tau^7}{72}\|u^{(4)}(\xi_1^k)\|_0^2+\frac{\tau^6}{16\alpha}\|\nabla u^{(3)}(\xi_2^k)\|_0^2\\&+\frac{\tau^6}{8\alpha}\|\nabla u_{tt}(\xi_3^k)\|_0^2].\end{aligned}\tag{17}$$

It follows from (17) that

$$\|\nabla e^k\|_0=\|\nabla(u(t_k)-u^k)\|_0\leqslant C\tau^2,\tag{18}$$

where $C^2=T\beta^{-1}[\frac{\tau}{72}\|(u^{(4)}(\xi_1^k)\|_0^2+\frac{1}{16\alpha}\|\nabla u^{(3)}(\xi_2^k)\|_0^2+\frac{1}{8\alpha}\|\nabla u_{tt}(\xi_3^k)\|_0^2]$. The proof for Theorem 1 is completed. □

## 3 The NBE method

### 3.1 Naturalization of the natural boundary

When discretizing the time of the governing equation for Problem 1, it is also necessary to discretize its boundary conditions at the same time. If we set that $\mu:=(\sqrt{\tau(\alpha+\tau\beta)/2})^{-1}$ and $\tilde f_1^k:=-(2u^k-u^{k-1}-\tau\alpha\Delta u^{k-1}/2+\tau^2\beta\Delta u^{k-1}/2+\tau^2 f_1^k)$, when one obtains

$$\begin{cases}\Delta u^k-\mu^2u^k=\mu^2\tilde f_1^k, & \boldsymbol{z}\in\Omega^c,\\ \dfrac{\partial u^k}{\partial n}=g_1^k, & \boldsymbol{z}\in\Sigma,\\ |u^k|<+\infty, & |\boldsymbol{z}|\to+\infty.\end{cases}\tag{19}$$

From (19), It is easy to known that next main task is to solve the elliptic boundary value problems per time step.

Let $I_n(x)$ and $K_n(x)$ $(n=0,1,2,\cdots)$ be separately the first and second type of modified Bessel functions in [14, 15], and $\Re_\mu$ and $\Im_\mu$ be separately the natural and Poisson integral operators in [7, 8]. By using the NBE method in [8, 16, 17], we can assert that Dirichlet boundary $\hat u_0^k$ and Neumann boundary $\frac{\partial u^k}{\partial \boldsymbol{n}}$ satisfy the following equation

$$\frac{\partial u^k}{\partial \boldsymbol{n}}+\mathcal{N}(\mu,r;\tilde f_1^{\,k},\theta)=\Re_\mu\hat u_0^k,\tag{20}$$

and the relationship between the solution $u^k$ of Problem 3 and its Dirichlet boundary value $\hat u_0^k$ is as follows:

$$u^k=\Im_\mu\hat u_0^k+\mathcal{F}(\mu,r;\tilde f_1^{\,k},R,\theta),\tag{21}$$

where

$$\begin{aligned}&\mathcal{N}(\mu,r;\tilde f_1^{\,k},\theta)\\&=\frac{\mu^2}{2}\sum_{n=0}^{+\infty}\xi_n\int_r^{+\infty}\bar G_n(\mu,r;\sigma)\cdot\Big[\tilde f_{1n}^{k+1,c}(\sigma)\cos n\theta\\&+\tilde f_{1n}^{k+1,s}(\sigma)\sin n\theta\Big]\mathrm{d}\sigma,\\&\xi_0=1;\ \ \xi_n=2,n=1,2,\cdots,\\&\bar G_n(\mu,r;\sigma)=-\frac{K_n(\mu\sigma)}{K_n(\mu r)}\cdot\frac{\sigma}{r},\ n=0,1,2,\cdots,\\&\mathcal{F}(\mu,r;\tilde f_1^{\,k},R,\theta)\\&=\frac{\mu^2}{2}\sum_{n=0}^{+\infty}\xi_n\int_r^{+\infty}\sigma^2G_n(R,\sigma)\Big[\tilde f_{1n}^{k+1,c}(\sigma)\cos n\theta\\&+\tilde f_{1n}^{k+1,s}(\sigma)\sin n\theta\Big]\mathrm{d}\sigma,\\&\sigma^2G_n(R,\sigma)\\&=\begin{cases}\dfrac{\phi_n(\sigma)\psi_n(R)}{E_n(\sigma)},R\leqslant\sigma,\\ \dfrac{\psi_n(\sigma)\phi_n(R)}{E_n(\sigma)},R\geqslant\sigma,\end{cases}\quad n=0,1,2,\cdots,\\&\phi_n(\sigma)=K_n(\mu\sigma),\\&\psi_n(\sigma)=I_n(\mu\sigma)K_n(\mu r)-K_n(\mu\sigma)I_n(\mu r),\\&n=0,1,2,\cdots,\\&E_n(\sigma)=\psi_n(\sigma)\phi_n'(\sigma)\\&-\phi_n(\sigma)\psi_n'(\sigma),n=0,1,2,\cdots,\\&\tilde f_{1n}^{k+1,c}(\sigma)\\&=\frac{1}{\pi}\int_0^{2\pi}\tilde f_1^{\,k}(\sigma,\theta)\cos n\theta\mathrm{d}\theta,n=0,1,2,\cdots,\\&\tilde f_{1n}^{k+1,s}(\sigma)\\&=\frac{1}{\pi}\int_0^{2\pi}\tilde f_1^{\,k}(\sigma,\theta)\sin n\theta\mathrm{d}\theta,\ \ n=1,2,\cdots.\end{aligned}$$

The above equations (20) and (21) are separately known as the natural and the Poisson integral equations. Further, (20) is equivalent to the following variation form.

**Problem 4.** *Seek $u_0^k\in H^{\frac12}(\Sigma)$ $(1\leqslant k\leqslant N)$ from the below system*

$$\begin{aligned}&\hat D(u_0^k,v^k)\\&=\Big\langle g_1^k(r,\theta)+\mathcal{N}(\mu,r;\tilde f_1^{\,k},\theta),v^k\Big\rangle,\\&\forall v^k\in H^{\frac12}(\Sigma),\ 1\leqslant k\leqslant N-1,\end{aligned}\tag{22}$$

*where* $\langle u, \ell\rangle = \int_\Sigma u\ell \mathrm{d}s$ *and* $\hat{D}(u_0^k, v^k) = \left\langle \Re_\mu \hat{u}_0^k, v^k \right\rangle = \int_\Sigma (\Re_\mu \hat{u}_0^k) v^k \mathrm{d}s$.

### 3.2 The expressions of naturalization problem in polar coordinates

For the convenience of discussion, assume that the region $\Omega$ is a circle with radius $r$ and center at origin and the solutions $u^k$ to Problem 3 are appropriate smooth. Under the polar coordinates, the external normal derivative operator on $\Sigma$ satisfies $\partial/\partial \boldsymbol{n} = -\partial/\partial \boldsymbol{R}$, $\Sigma = \{(R,\theta) : R = r,\ \theta \in [0, 2\pi]\}$, and $\Omega^c = \{(R,\theta) : R = |\boldsymbol{z}| > r, \theta \in [0, 2\pi]\}$. Thus, in the polar coordinates, the solutions to the equations (19) can be denoted by

$$U^k(R,\theta) = a_0(R)/2 + \sum_{n=1}^{+\infty} \left[a_n(R)\cos n\theta + b_n(R)\sin n\theta\right], \tag{23}$$

where

$$a_n(R) = \frac{1}{\pi}\int_0^{2\pi} u^k(R,\theta)\cos n\theta \mathrm{d}\theta, n = 1, 2, \cdots;$$
$$b_n(R) = \frac{1}{\pi}\int_0^{2\pi} u^k(R,\theta)\sin n\theta \mathrm{d}\theta,\ n = 1, 2, \cdots.$$

Further, the solution $u^k(R,\theta)$ of the equations (19)can be denoted by

$$u^k(R,\theta) = \frac{1}{2}\pi \sum_{n=0}^{+\infty} \xi_n \int_0^{2\pi} \frac{K_n(\mu R)}{K_n(\mu r)} \cos n(\theta-\theta') u^k(r,\theta')\mathrm{d}\theta' + \mathcal{F}(\mu, r; \tilde{f_1}^k, R, \theta),\quad R > r, \tag{24}$$

$$\frac{\partial u^k(r,\theta)}{\partial \boldsymbol{n}} + \mathcal{N}(\mu, r; \tilde{f_1}^k, \theta) = \frac{\mu}{2\pi}\int_0^{2\pi} \bar{K}_n(\mu, r; \theta-\theta') \cdot u^k(r,\theta')\mathrm{d}\theta', \tag{25}$$

where $\bar{K}_n(\mu, r; \theta-\theta')$=-$\sum_{n=0}^{+\infty} \xi_n \cdot \frac{K_n'(\mu r)}{K_n(\mu r)} \cdot \cos n(\theta-\theta')$.

### 3.3 The NBE method and the existence, stability, and error estimations of the NBE solutions

For simplicity, the circumference $\Sigma$ is partitioned into some uniform regular FEs (see [18]). Let $\mathcal{S}_h(\Sigma) \subset H^{1/2}(\Sigma)$ is an FE subspace spanned by proper basis functions on the uniform regular FE subdivision. Thus, the NBE method for Problem 2 is stated in the following.

**Problem 5.** *Seek* $u_{0h}^k \in \mathcal{S}_h(\Sigma)$ $(1 \leqslant k \leqslant N)$ *from the following system*

$$\hat{D}(u_{0h}^k, v^k) = \left\langle g_1^k(r,\theta) + \mathcal{N}(\mu, r; \tilde{f_1}^k, \theta), v^k \right\rangle,\quad \forall v^k \in \mathcal{S}_h(\Sigma),\ 1 \leqslant k \leqslant N, \tag{26}$$

*and*

$$u_h^k(R,\theta) = \frac{1}{2\pi}\sum_{n=0}^{+\infty} \xi_n \int_0^{2\pi} \frac{K_n(\mu R)}{K_n(\mu r)} \cos n(\theta-\theta') u_{0h}^k(r,\theta')\mathrm{d}\theta' + \mathcal{F}(\mu, r; \tilde{f_1}^k, R, \theta),\quad R > r. \tag{27}$$

It is easy to see that the formula (27) is approximated for the Poisson integral formula (24).

To analyze the errors of NBE solutions of Problem 5 needs to use the following NBE-projection.

**Definition 1.** *An operator* $P_h : H^{\frac{1}{2}}(\Sigma) \to \mathcal{S}_h(\Sigma)$ *is known as the NBE projection, if for any* $v \in L^2(\Sigma)$, *there has a unique* $P_h v \in \mathcal{S}_h(\Sigma)$ *satisfying*

$$\hat{D}(v - P_h v, v_h) = 0,\quad \forall v_h \in \mathcal{S}_h(\Sigma),$$

*where* $\mathcal{S}_h(\Sigma) \subset H^{\frac{1}{2}}(\Sigma)$ *is an FE space.*

It follows from References [5–7] that the NBE projection $P_h$ has the following property.

**Lemma 1.** *If* $\mathcal{S}_h(\Sigma)$ *is a subspace spanned by piecewise linear polynomials and* $v \in H^2(\Sigma)$, *then the NBE projection* $P_h$ *satisfies*

$$\|v - P_h v\|_s \leqslant Ch^{2-s}\|v\|_{2,\Sigma},\quad s = -1, 0, 1,$$

*where* $C > 0$ *is a constant independent of* $h$ *and* $\tau$.

For Problem 5, the following results are obtained.

**Theorem 2.** *If* $u_0^k \in H^2(\Sigma)$ *and* $u_{0h}^k$ *are separately the solutions to* (22) *together with* (26), *and* $\mathcal{S}_h(\Sigma)$ *be the subspace formed by piecewise linear polynomials, then they have the below error estimations:*

$$\| u_0^k - u_{0h}^k \|_{0,\Sigma} \leqslant Ch^2,\ k = 1, 2, \cdots, N. \tag{28}$$

*Proof.* Subtracting (26) from (22) and taking $v^k = v_h^k$ yield that

$$\hat{D}(u_0^k - u_{0h}^k, v_h^k) = \left\langle \frac{\mu^2}{2\pi}\sum_{n=0}^{+\infty} \xi_n \int_r^{+\infty} \bar{G}_n(\mu, r; \sigma) \left[\int_0^{2\pi} [2(u_0^k - u_{0h}^k) - (u_0^{k-1} - u_{0h}^{k-1}) - \frac{\tau\alpha}{2}\Delta(u_0^{k-1} - u_{0h}^{k-1}) + \frac{\tau^2\beta}{2}\Delta(u_0^{k-1} - u_{0h}^{k-1})] \cos n\hat{\theta}\mathrm{d}\hat{\theta} \cos n\theta\right.$$

$$+\int_0^{2\pi}[2(u_0^k-u_{0h}^k)-(u_0^{k-1}-u_{0h}^{k-1})$$
$$-\frac{\tau\alpha}{2}\Delta(u_0^{k-1}-u_{0h}^{k-1})+$$
$$\left.\left.\frac{\tau^2\beta}{2}\Delta(u_0^{k-1}-u_{0h}^{k-1})]\sin n\hat{\theta}\mathrm{d}\hat{\theta}\sin n\theta\right]\mathrm{d}\sigma,v_h^k\right\rangle,$$
$$\forall v_h^k\in\mathcal{S}_h(\Sigma). \tag{29}$$

Noting that $\hat{D}(\cdot,\cdot)$ is positive definite on $H^{\frac{1}{2}}(\Sigma)\times H^{\frac{1}{2}}(\Sigma)$ (see [8]), using the Hölder inequality and the NBE projection, one can obtained

$$M\|P_hu_0^k-u_{0h}^k\|_{0,\Sigma}^2$$
$$\leqslant|\hat{D}(P_hu_0^k-u_{0h}^k,P_hu_0^k-u_{0h}^k)|$$
$$=|\hat{D}(u_0^k-u_{0h}^k,P_hu_0^k-u_{0h}^k)|$$
$$=\left|\left\langle\frac{\mu^2}{2\pi}\sum_{n=0}^{+\infty}\xi_n\int_r^{+\infty}\bar{G}_n(\mu,r;\sigma)\right.\right.$$
$$\left[\int_0^{2\pi}[2(u_0^k-u_{0h}^k)-(u_0^{k-1}-u_{0h}^{k-1})\right.$$
$$-\frac{\tau\alpha}{2}\Delta(u_0^{k-1}-u_{0h}^{k-1})$$
$$+\frac{\tau^2\beta}{2}\Delta(u_0^{k-1}-u_{0h}^{k-1})]\cos n\hat{\theta}\mathrm{d}\hat{\theta}\cos n\theta$$
$$+\int_0^{2\pi}\left[2(u_0^k-u_{0h}^k)-(u_0^{k-1}-u_{0h}^{k-1})\right.$$
$$-\frac{\tau\alpha}{2}\Delta(u_0^{k-1}-u_{0h}^{k-1})$$
$$\left.\left.+\frac{\tau^2\beta}{2}\Delta(u_0^{k-1}-u_{0h}^{k-1})\right]\sin n\hat{\theta}\mathrm{d}\hat{\theta}\sin n\theta\right]\mathrm{d}\sigma,$$
$$\left.\left.P_hu_0^k-u_{0h}^k\right\rangle\right|. \tag{30}$$

By [8, 15], it follows that $\bar{G}_n(\mu,r;\sigma)=-\frac{K_n(\mu\sigma)\sigma}{[K_n(\mu r)r]}\to 0$ $(n\to\infty)$. Hance, we set $\frac{\mu^2}{2}\sum\limits_{n=0}^{+\infty}\varepsilon_n\int_r^{+\infty}\bar{G}_n(\mu,r;\sigma)\mathrm{d}\sigma$ $\leqslant C\tau$. Thus, from (30), the following estimate is obtained:

$$M\|P_hu_0^k-u_{0h}^k\|_{0,\Sigma}^2\leqslant$$
$$C\tau(3+\frac{\tau\alpha}{2}+\frac{\beta\tau^2}{2})\|u_0^k-u_{0h}^k\|_{0,\Sigma}\|P_hu_0^k-u_{0h}^k\|_{0,\Sigma}.$$

Thereupon, the following is obtained

$$\|P_hu_0^k-u_{0h}^k\|_{0,\Sigma}$$
$$\leqslant C\tau(3+\frac{\tau\alpha}{2}+\frac{\beta\tau^2}{2})\|u_0^k-u_{0h}^k\|_{0,\Sigma}. \tag{31}$$

Further, by the Cauchy inequality and Lemma 1, the following is obtained

$$\|P_hu_0^k-u_{0h}^k\|_{0,\Sigma}\leqslant C\tau h^2$$
$$+C\tau(3+\frac{\tau\alpha}{2}+\frac{\beta\tau^2}{2})\|P_hu_0^k-u_{0h}^k\|_{0,\Sigma}. \tag{32}$$

Summing (32) from 1 to $k$ and using Gronwall lemma (see [14, 18]), the following is obtained

$$\|P_hu_0^k-u_{0h}^k\|_{0,\Sigma}\leqslant Ch^2. \tag{33}$$

Combining Lemma 1 with (33) yields (28). This finishes the proof of Theorem 2. □

The solutions obtained by the approximate expression (27) have the below error estimations.

**Theorem 3.** *If $h=O(\tau)$, and $u^k$ and $u_h^k$ are separately the solutions of (24) and (27), then they meet the below error estimations:*

$$\|u^k-u_h^k\|_{0,\infty,\Omega^c}\leqslant Ch^2,\quad 1\leqslant k\leqslant N. \tag{34}$$

*Proof.* By Reference [15], one can immediately derive

$$K_n(z)=\sqrt{\frac{\pi}{2n}}\left(\frac{2n}{ez}\right)^n[1+o(n^{-1})],\quad n\to+\infty.$$

When $r<R$, $\frac{K_n(\mu R)}{K_n(\mu r)}\to 0$ and $\sigma^2G_n(R,\sigma)\to 0$ $(n\to\infty)$. Thus, assume that $\frac{1}{2\pi}\sum_{n=0}^{+\infty}\xi_n\int_0^{2\pi}\frac{K_n(\mu R)}{K_n(\mu r)}\mathrm{d}\theta'\leqslant C\tau$ and $\frac{\mu^2}{2}\sum_{n=0}^{+\infty}\int_r^{+\infty}\sigma^2G_n(R,\sigma)\mathrm{d}\sigma\leqslant C\tau$ in the following discussions. Therefore, the following is obtained

$$\|u^k-u_h^k\|_0$$
$$\leqslant\frac{1}{2\pi}\left|\sum_{n=0}^{+\infty}\xi_n\int_0^{2\pi}\frac{K_n(\mu R)}{K_n(\mu r)}\right.$$
$$\left.\cdot\cos n(\theta-\theta')(u_0^k-u_{0h}^k)\mathrm{d}\theta'\right|$$
$$+\frac{\mu^2}{2\pi}\sum_{n=0}^{+\infty}\xi_n\int_r^{+\infty}\sigma^2G_n(R,\sigma)\left\{\int_0^{2\pi}\left|2(u^k-u_h^k)\right.\right.$$
$$-(u^{k-1}-u_h^{k-1})-\frac{\tau\alpha}{2}\Delta(u^{k-1}-u_h^{k-1})$$
$$\left.+\frac{\tau^2\beta}{2}\Delta(u^{k-1}-u_h^{k-1})\right|\cos n\hat{\theta}\mathrm{d}\hat{\theta}\cos n\theta$$
$$+\int_0^{2\pi}\left|2(u^k-u_h^k)-(u^{k-1}-u_h^{k-1})\right.$$
$$-\frac{\tau\alpha}{2}\Delta(u^{k-1}-u_h^{k-1})$$
$$\left.\left.+\frac{\tau^2\beta}{2}\Delta(u^{k-1}-u_h^{k-1})\right|\sin n\hat{\theta}\mathrm{d}\hat{\theta}\sin n\theta\right\}\mathrm{d}\sigma$$
$$\leqslant C\tau\|u_0^k-u_{0h}^k\|_{0,\Sigma}+C\tau(3$$
$$+\frac{\alpha\tau}{2}+\frac{\beta\tau^2}{2})\|u^k-u_h^k\|_0,\quad 1\leqslant k\leqslant N. \tag{35}$$

Summing (35) from 1 until $k$ and using Theorem 2, the following is obtained

$$\|u^k - u_h^k\|_0 \leqslant C\tau \sum_{i=0}^{k-1} \|u^{i+1} - u_h^{i+1}\|_0 + Ch^2, \ \ 1 \leqslant k \leqslant N. \tag{36}$$

By applying Gronwall lemma to (36), one can immediately get

$$\|u^k - u_h^k\|_0 \leqslant Ch^2, \ \ 1 \leqslant k \leqslant N. \tag{37}$$

This finishes the proof of Theorem 3. □

For the fully discretized NBE solutions, the following results on existence, stability, and error estimations are obtained.

**Theorem 4.** *If $f_1^k \in L^2(\Omega^c)$, $g_1 \in L^0(\Sigma)$, and $u_0, u_1 \in H^1(\Omega^c)$, Problem 5 has a sole set of solutions $\{u_{0h}^k\}_{k=1}^N \subset \mathcal{S}_h(\Sigma)$ satisfying the below stability*

$$\|u_{0h}^k\|_0 \leqslant \frac{\tau}{M} \sum_{i=0}^{k} (\|g_1^{i+1}\|_{0,\Sigma} + C\tau^3 \|f_1^i\|_0) \cdot \exp[CM^{-1}k\tau], \ \ 1 \leqslant k \leqslant N, \tag{38}$$

*and the below error estimations:*

$$\| u(t_k) - u_h^k \|_{0,\Omega^c} \leqslant C(\tau^2 + h^2), \ \ 1 \leqslant k \leqslant N. \tag{39}$$

*Proof.* Noting that the bilinear functional $\hat{D}(\cdot,\cdot)$ on $H^{\frac{1}{2}}(\Sigma) \times H^{\frac{1}{2}}(\Sigma)$ is symmetrical, continuous, and positive definite (see [7, 8]), by Lax-Milgram theorem (see [7, 8, 18]), the Problem 5 has a sole set of solutions.

Taking $v_h^k = u_{0h}^k$ in (26) and using the Hölder inequality, the following is obtained

$$\begin{aligned}
&M\|u_{0h}^k\|_{0,\Sigma}^2 \leqslant |\hat{D}(u_{0h}^k, u_{0h}^k)| \\
&= |\langle g_1^k(r,\theta) + \mathcal{N}(\mu, r; \tilde{f_1}^k, \theta), u_{0h}^k \rangle| \\
&\leqslant \Bigg[ \|g_1^k\|_{0,\Sigma} + \| \frac{\mu^2}{2\pi} \sum_{n=0}^{+\infty} \xi_n \int_r^{+\infty} \bar{G}_n(\mu, r; \sigma) \\
&\Bigg\{ \int_0^{2\pi} \Big[ 2(u_0^k - u_{0h}^k) - (u_0^{k-1} - u_{0h}^{k-1}) \\
&-\frac{\tau\alpha}{2}\Delta(u_0^{k-1} - u_{0h}^{k-1}) + \frac{\tau^2\beta}{2}\Delta(u_0^{k-1} - u_{0h}^{k-1}) \\
&+\tau^2 f_1^k \Big] \cdot \cos n\hat{\theta} \mathrm{d}\hat{\theta} \cos n\theta \\
&+ \int_0^{2\pi} \Big[ 2(u_0^k - u_{0h}^k) - (u_0^{k-1} - u_{0h}^{k-1}) \\
&+\frac{\tau^2\beta}{2}\Delta(u_0^{k-1} - u_{0h}^{k-1}) - \frac{\tau\alpha}{2}\Delta(u_0^{k-1} - u_{0h}^{k-1}) \\
&+\tau^2 f_1^k \Big] \sin n\hat{\theta} \mathrm{d}\hat{\theta} \sin n\theta \Bigg\} \mathrm{d}\sigma \|_{0,\Sigma} \Bigg] \|u_{0h}^k\|_{0,\Sigma} \\
&\leqslant [C\tau(3 + \frac{\alpha\tau}{2} + \frac{\beta\tau^2}{2})\|u_{0h}^k\|_{0,\Sigma} \\
&+C\tau^3 \|f_1^k\|_{0,\Omega^c} + \|g_1^k\|_{0,\Sigma}] \|u_{0h}^k\|_{0,\Sigma}.
\end{aligned}$$

It follows that

$$\|u_{0h}^k\|_0 \leqslant \frac{1}{M}[C\tau\|u_{0h}^k\|_{0,\Sigma} + C\tau^3 \|f_1^k\|_{0,\Omega^c} + \|g_1^k\|_{0,\Sigma}], \ \ 1 \leqslant k \leqslant N \tag{40}$$

Summing (40) from 1 until $k$ and using Gronwall lemma (see [14, 18]) yield (38).

Using the triangle inequality,

$$\| u(t_k) - u_h^k \|_{0,\Omega^c} \leqslant \|u(t_k) - u^k\|_{0,\Omega^c} + \|u^k - u_h^k\|_{0,\infty,\Omega^c}, \ \ 1 \leqslant k \leqslant N, \tag{41}$$

and associating Theorems 1 and 3 with (41) yield (39). This finishes the proof of Theorem 4. □

## 4 Numerical simulations

Here, numerical simulations are conducted to verify that the numerical errors align with the theoretical error estimates and to demonstrate the effectiveness of the NBE method in solving the viscoelastic wave equation in a 2D unbounded region. For ease of comparison, simulations are performed using cases where the viscoelastic wave equation admits an analytical solution. However, in general, analytical solutions are unavailable when the boundary conditions, initial data, and source terms are complex.

Suppsoe that there occurs earthquake in the circle $\Omega$ and need to simulate the earthquake wave cases in the external region $\Omega^c$. Thus, the earthquake source is assumed to be as follows

$$f_1(\boldsymbol{z}, t) = \big[3\pi(1-\pi)\cos(\pi R/2)/2R^3 + (5\pi^2 - 4\pi^3 + 4\pi - 4)\sin(\pi R/2)/R^2\big] e^{-\pi t},$$

where $R = |\boldsymbol{z}| = \sqrt{x^2+y^2} \geqslant 2$ and take $\alpha = \beta = 1$. The boundary and initial functions are separately chosen as $g_1(\boldsymbol{z}, t) = 0$, $u_0(\boldsymbol{z}) = \frac{1}{R^2}\sin(\pi R)$, and $u_1(\boldsymbol{z}) = -\pi\sin(\pi R)/R^2$. In this case, $\Sigma_r = \{(x,y) : (x,y) \in R^2; R = 2\}$. Thus, this problem has an analytical solution $u = \frac{1}{R^2}\sin(\pi R)e^{-\pi t}$. We approximately replace $\sum_{n=1}^{+\infty}$ with $\sum_{n=1}^{M}$ and adopt the numerical integral to calculate $\mathcal{N}(\mu, r; \tilde{f_1}^k, \theta)$ and $\mathcal{F}(\mu, r; \tilde{f_1}^k, R, \theta)$ in the numerical simulations.

The circumference $\Gamma$ is divided into 64 arc paragraphes with same side length $\Delta\theta = \pi/32$, which satisfy the uniform regular conditions. The parameter $M = 120$ is chosen, and the time step size is set as $\tau =$

0.0125. The analytical solution $u(t)$ and the numerical solutions $u_h^k$ at time $t = 0.2, 0.4, 0.6, 0.8, 1.0$, as shown in the graphs (a) and (b) in Figures. 1 to 5 separately. Each pair of graphs in Figures. 1 to 5 show that the analytical solutions are basically the same as the numerical solutions. Whereas the errors between the analytical solution $u(t)$ and numerical solutions $u_h^k$ at $t = 0.2, 0.4, 0.6, 0.8, 1.0$ are shown graphically in Figure 6, which also shows that the numerical simulating errors are consistent with the theory errors since both theory errors and numerical simulating errors are $O(10^{-3})$. These sufficiently indicate that the NBE method is effective and feasible for solving the viscoelastic wave equation in 2D unbounded region.

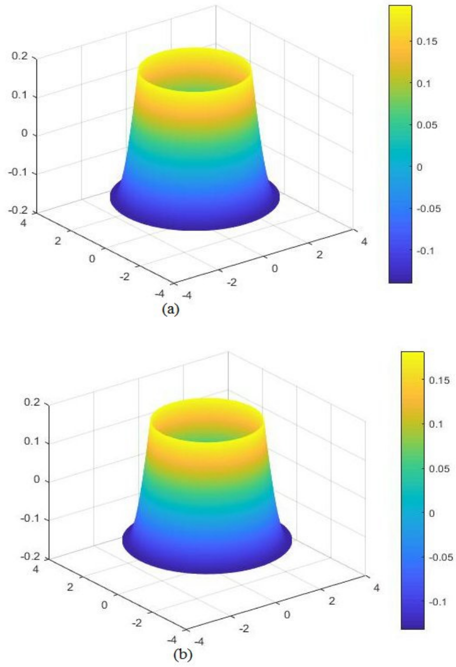


**Figure 1.** (a) The analytical solution $u(t)$ when $t = 0.2$. (b) The NBE solution $u_h^k$ when $t = 0.2$.

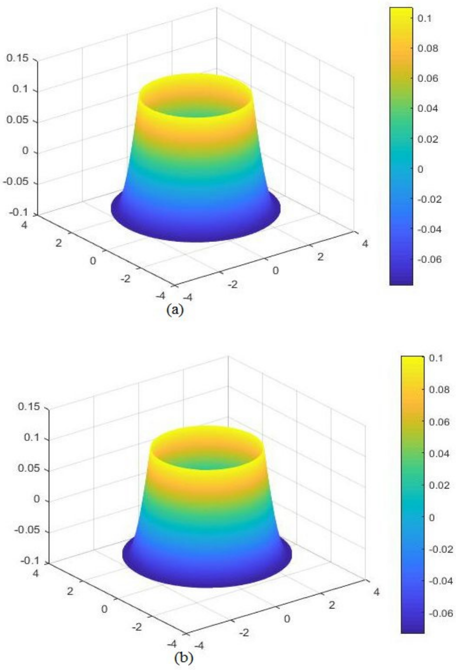


**Figure 2.** (a) The analytical solution $u(t)$ when $t = 0.4$. (b) The NBE solution $u_h^k$ when $t = 0.4$.

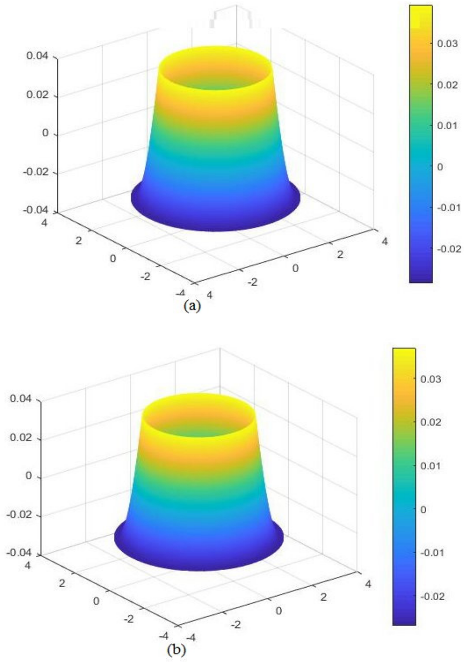


**Figure 3.** (a) The analytical solution $u(t)$ when $t = 0.6$. (b) The NBE solution $u_h^k$ when $t = 0.6$.

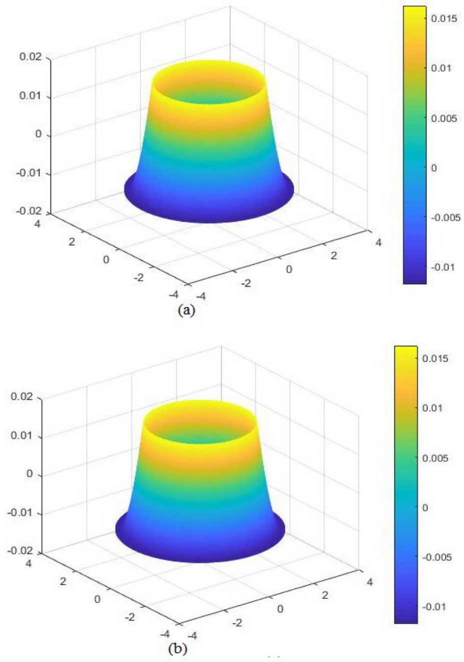


**Figure 4.** (a) The analytical solution $u(t)$ when $t = 0.8$. (b) The NBE solution $u_h^k$ when $t = 0.8$.

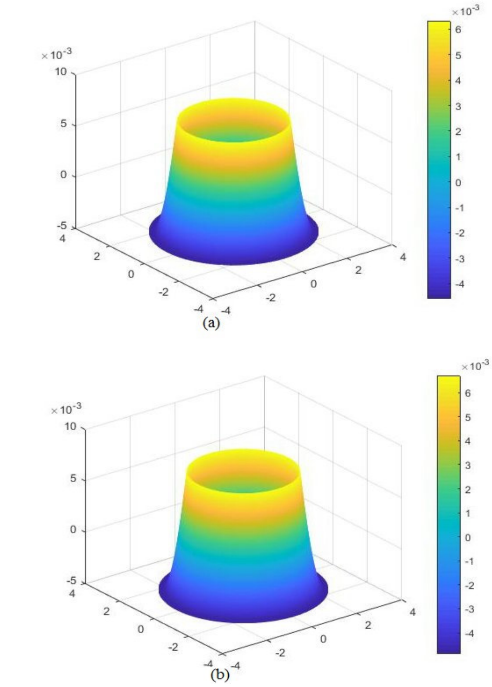


**Figure 5.** (a) The analytical solution $u(t)$ when $t = 1$. (b) The NBE solution $u_h^k$ when $t = 1$.

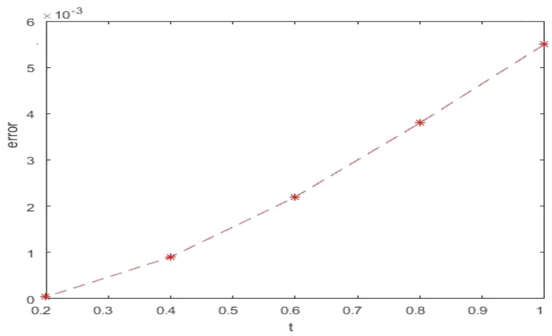


**Figure 6.** The error between the exact solution $u(t)$ and the NBE solution $u_h^k$ when $t \in [0.2, 1]$.

## 5 Conclusions

This article presents the construction of a time-stepping difference (TSD) scheme for the viscoelastic wave equation in a 2D unbounded domain, along with theoretical analysis on the existence, stability, and error estimates of the TSD solutions. In particular, a fully discretized numerical boundary element (NBE) method is developed. The existence and stability of the fully discretized NBE solutions are established, and error estimates between the analytical and numerical solutions are derived. Numerical simulations are also provided to verify the accuracy and effectiveness of the proposed NBE method. Notably, this study is the first to propose the application of the NBE method to the viscoelastic wave equation in a 2D unbounded domain, representing a novel and original contribution. Furthermore, the NBE method has the potential to be applied to a variety of real-world problems.

## Data Availability Statement

Data will be made available on request.


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## Conflicts of Interest

The author declares no conflicts of interest.

## Ethical Approval and Consent to Participate

Not applicable.

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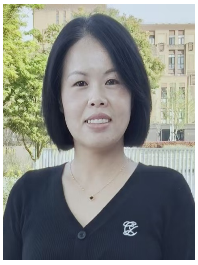

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